Experimental Computation and Visual Theorems: Part III. Random Walks

Jonathan Borwein FRSC FAAAS FAA FBAS FAMS
(With Aragón, Bailey, P. Borwein, Skerritt, Straub, Tam, Wan, Zudilin, ...)

Centre for Computer Assisted Research Mathematics and its Applications
The University of Newcastle, Australia

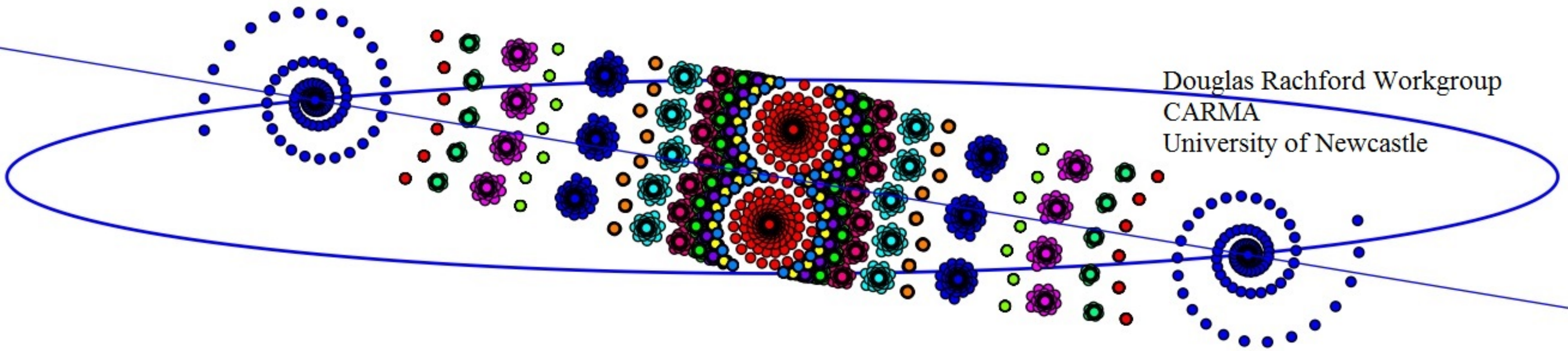
http://carma.newcastle.edu.au/meetings/evims/

http://www.carma.newcastle.edu.au/jon/visuals-ext-abst.pdf

For 2016 Presentations

Revised 25–02–16

Reflect-Reflect-Average for a line and ellipse



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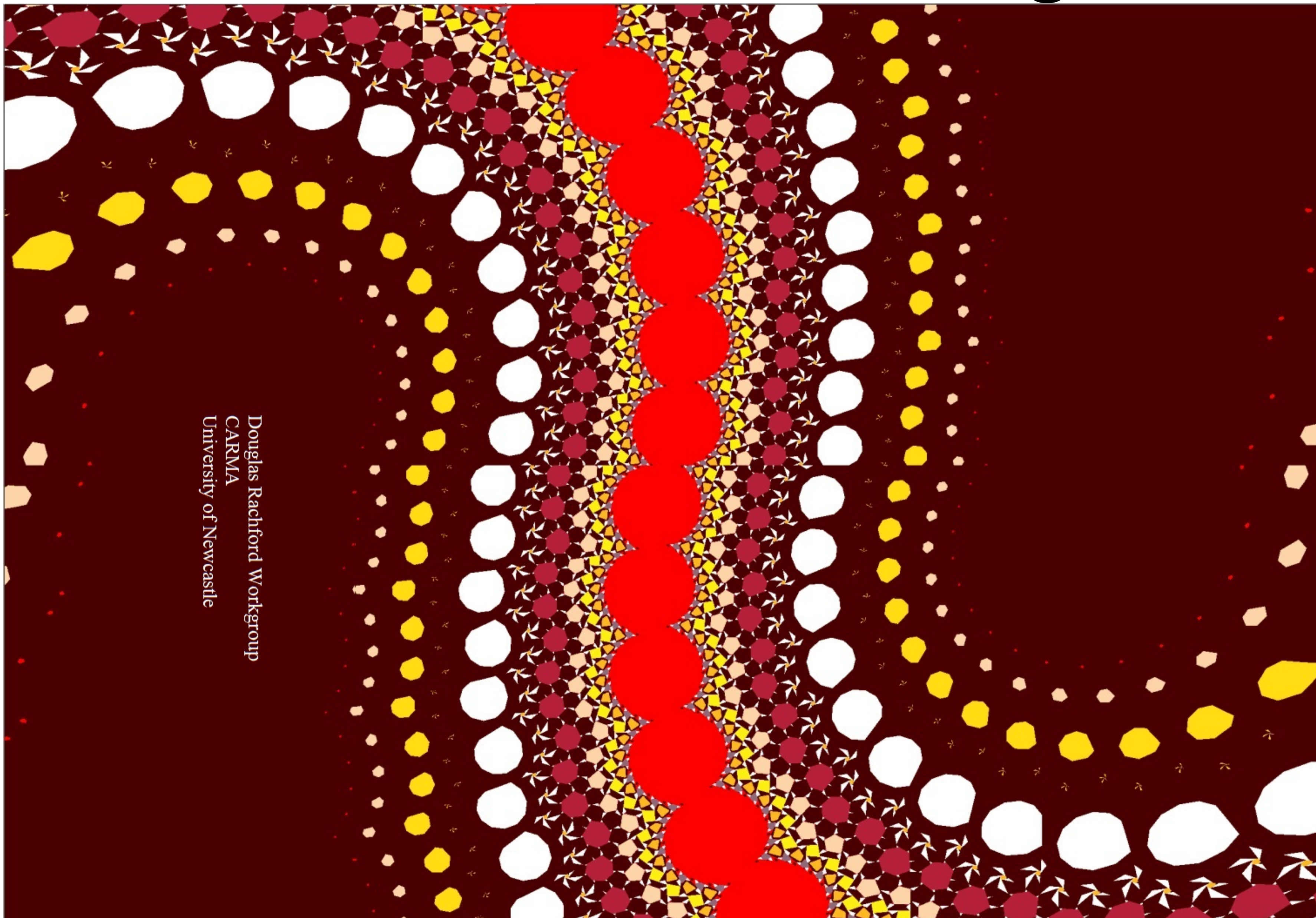

2016 Presentations

Jonathan Borwein FRSC FAA FAAA
Laureate Professor
and Director CARMA
www.carma.newcastle.edu.au

2016 Presentations as Distinguished Scholar in Residence Western University, London Ontario

Western UNIVERSITY · CANADA

April 12-13 : Owens Lectures
Wayne State University
1. Lambert W in Optimization
2. Walking on Numbers

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"I never run for trains." Nasim Nicholas Taleb (The Black Swan)

Revised 4-02-2016

EXTENDED ABSTRACT

Long before current graphic, visualisation and geometric tools were available, John E. Littlewood (1885-1977) wrote in his delightful *Miscellany*[1]:

> *A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself). [p. 53]*

[1] J.E. Littlewood, *A mathematician's miscellany*, London: Methuen (1953); Littlewood, J. E. and Bollobás, Béla, ed., *Littlewood's miscellany*, Cambridge University Press, 1986.

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Over the past decade, the role of visual computing in my own research has expanded dramatically.

In part this was made possible by the increasing speed and storage capabilities—and the growing ease of programming—of modern multi-core computing environments [BMC].

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But, at least as much, it has been driven by my group's paying more active attention to the possibilities for graphing, animating or simulating most mathematical research activities.

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- *I first briefly discuss both visual theorems and experimental computation.*
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- *After an algorithmic interlude (Case studies II), I end with description of work from my group in probability (behaviour of short random walks [BS, BSWZ]) and transcendental number theory (normality of real numbers [AB3]). (Case studies III)*

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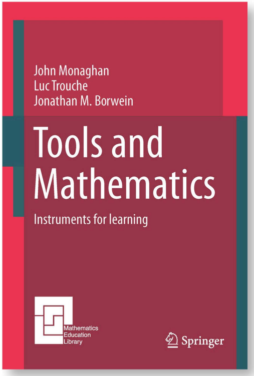
> *JMB was among roughly 60 new 2015 Fellows of the American Mathematical Society. He was cited "For contributions to nonsmooth analysis and classical analysis as well as experimental mathematics and visualization of mathematics."*

Tools and Mathematics 2016



1st ed. 2016, XXI, 481 p. 133 illus., 92 illus. in color.

Printed book

J. **Monaghan**, L. **Trouche**, J.M. **Borwein**

Tools and Mathematics

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- **The only book on the topic of tools and mathematics education**
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- **Content divided equally among the areas of curriculum, assessment, and policy design**

This book is an exploration of tools and mathematics and issues in mathematics education related to tool use. The book has four parts. The first part sets the scene with a reflection on doing a mathematical task with different tools, a mathematician's account of tool use in his work and historical considerations of tool use. The second part opens with a broad review of technology and intellectual trends, circa 1970, and continues with three case studies of approaches in mathematics education and the place of tools in these approaches. The third part considers issues related to mathematics instructions: curriculum, assessment and policy; the calculator debate; mathematics in the real world; and teachers' use of technology. The final part looks to the future and digital tools: task design; the importance of artefacts in gameplay; and new forms of activity via connectivity.

Key References and URLS

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- J.M. Borwein, A. Straub, J. Wan and W. Zudilin (with an Appendix by Don Zagier), "Densities of short uniform random walks." *Can. J. Math.* **64**(5), (2012), 961-990. `http://dx.doi.org/10.4153/CJM-2011-079-2`.

Contents

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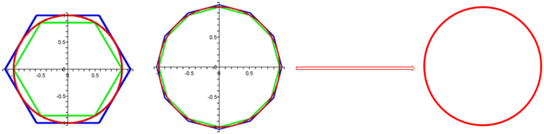

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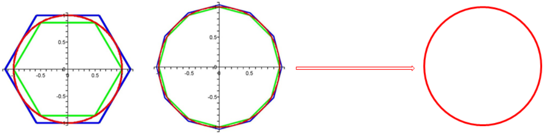

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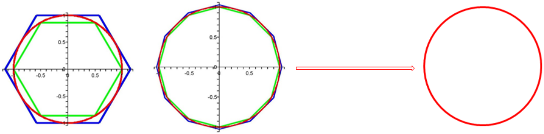
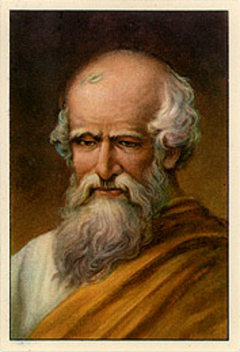


$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$ to obtain the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

Archimedes' "Method of Mechanical Theorems"

Pi movie below



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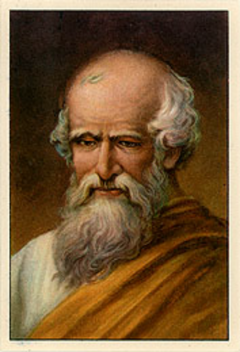

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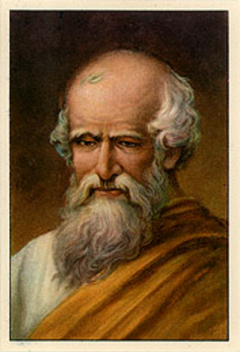

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- Only recently rediscovered and even more recently *reconstructed* ...

Proving π is not $\frac{22}{7}$

Even *Maple* or *Mathematica* 'knows' this since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \qquad (1)$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

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- Accidentally, $22/7$ is one of the early continued fraction approximation to π. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

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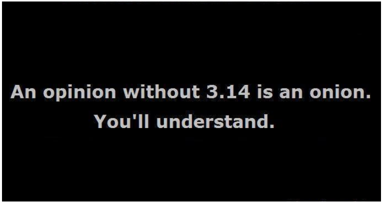
$$\int_0^{t} \frac{x^4(1-x)^4}{1+x^2}\,dx \quad = \quad \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus **proves** (1). **QED**



Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be "random":

$$\pi = 3.1415926535897932384626433832795028841971693993751\ldots$$

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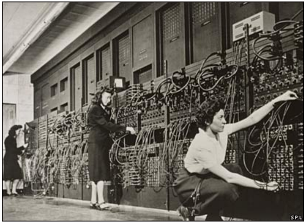
$$\pi = 3.14159265358979323846264338327950288419716939937 5\ldots$$

$$e = 2.7182818284590452353602874713526624977572470936 99\ldots$$

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753 76\ldots$$

Are they really?

- **1949 ENIAC** (*Electronic Numerical Integrator and Calculator*) computed of π to **2,037** decimals (in **70** hours)—proposed by polymath John von Neumann (**1903-1957**) to shed light on distribution of π (and of e).



Contents

Two continued fractions

Change representations often

Gauss map. Remove the integer, invert the fraction and repeat: for 3.1415926 and 2.7182818 to get the fractions below.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}$$

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$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Leonhard Euler (**1707-1783**) named e and π.

"Lisez Euler, lisez Euler, c'est notre maître à tous." Simon Laplace (**1749-1827**)

Contents

Are the digits of π random?

Digit	Ocurrences
0	99,993,942
1	99,997,334
2	100,002,410
3	99,986,911
4	**100,011**,958
5	**99,998**,885
6	100,010,387
7	99,996,061
8	100,001,839
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Total	**1,000,000,000**

Table: Counts of first billion digits of π. Second half is 'right' for law of large numbers.

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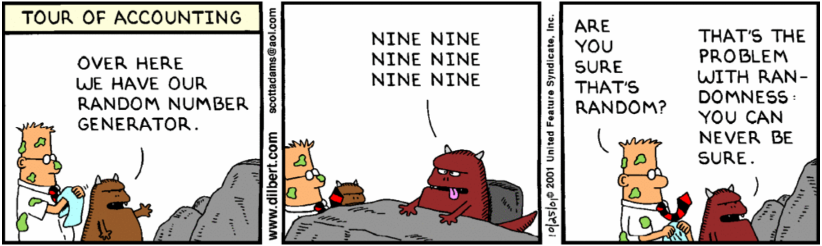
- Or pretty much anything else...

What is "random"? A hard question



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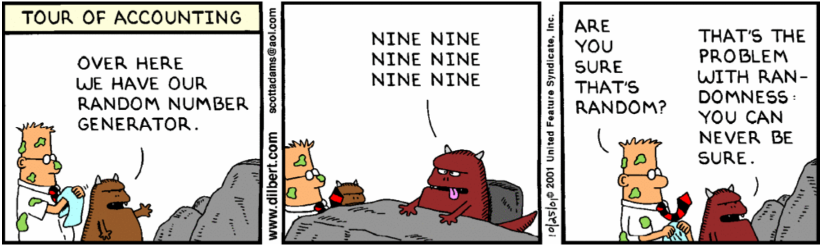

It might be:

- Unpredictable (fair dice or coin-flips)?
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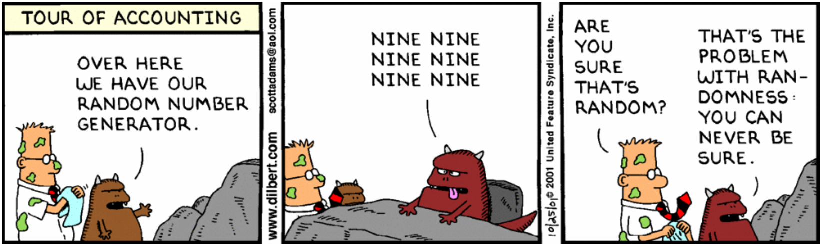

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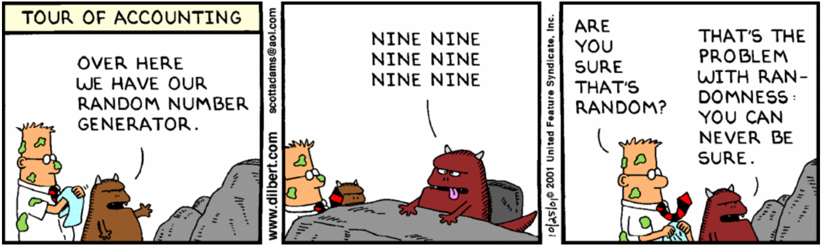

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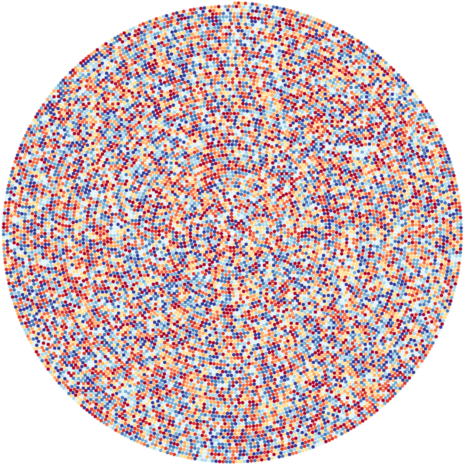
Best Theorem [BBCP, 04] (Feeble but hard) Asymptotically all degree d algebraics have at least $n^{1/d}$ ones in binary (should be $n/2$)

Randomness in Pi?

http://mkweb.bcgsc.ca/pi/art/



Randomness in Pi?

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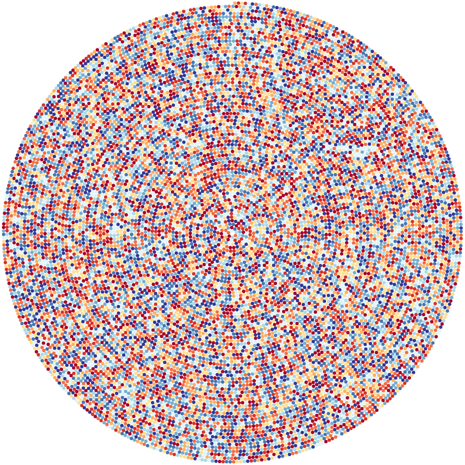

- a better color palette for art if not for science

Contents

Normality

A property random numbers must possess

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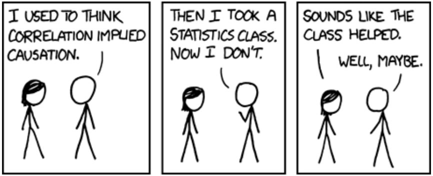

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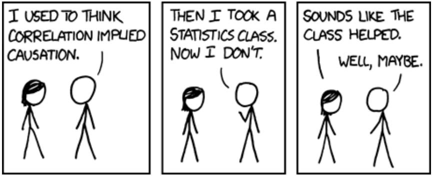
- Indeed, almost all real numbers are b-normal simultaneously for all positive integer bases ("absolute normality").



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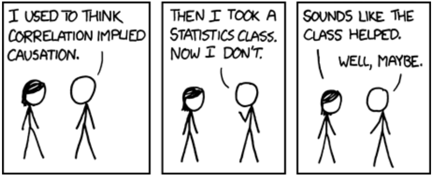
- Given an integer $b \geq 2$, almost all real numbers, with probability one, are b-normal (Borel).
- Indeed, almost all real numbers are b-normal simultaneously for all positive integer bases ("absolute normality").
- Unfortunately, it has been very difficult to prove normality for any number in a given base b, much less all bases simultaneously.



Normal numbers *concatenation* numbers

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$$C_{10} := 0.123456789101112131415161718\ldots$$

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$$CE(10) := 0.23571113171923293137414347\ldots$$

is 10-normal (concatenation works in all bases).

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- Normality proofs are not known for $\pi, e, \log 2, \sqrt{2}$ etc.

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Let p be any polynomial positive on the natural numbers. Then the concatenation number

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- Includes Champernowne's number and $0.1491625\dots$ (Besicovich)
- See H. Davenport and P. Erdös, "Note on normal numbers." *Can. J. Math.*, **4** (1952), 58–63.

Contents

Is π 10-normal?

String	Occurrences	String	Occurrences	String	Occurrences
0	99,993,942	00	10,004,524	000	1,000,897
1	99,997,334	01	9,998,250	001	1,000,758
2	100,002,410	02	9,999,222	002	1,000,447
3	99,986,911	03	10,000,290	003	1,001,566
4	100,011,958	04	10,000,613	004	1,000,741
5	**99,998,885**	05	**10,002,048**	005	**1,002,881**
6	100,010,387	06	9,995,451	006	999,294
7	99,996,061	07	9,993,703	007	998,919
8	100,001,839	08	10,000,565	008	999,962
9	100,000,273	09	9,999,276	009	999,059
		10	9,997,289	010	998,884
		11	9,997,964	011	1,001,188
		⋮	⋮	⋮	⋮
		99	10,003,709	099	999,201
				⋮	⋮
				999	1,000,905
TOTAL	1,000,000,000	**TOTAL**	1,000,000,000	**TOTAL**	1,000,000,000

Table: Counts for the first billion digits of π.

Is π 16-normal

That is, in Hex?

$\hookleftarrow$ Counts of first trillion hex digits

0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
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7	62499878794
8	**62500**216752
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- They are $353CB3F7F0C9ACCFA9AA215F2$

See www.karrels.org/pi/index.html

Modern π Calculation Records: and IBM Blue Gene/L at LBL

Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000
Takahashi	Jan. 2009	1,649,000,000,000
Takahashi	April 2009	2,576,980,377,524
Bellard	Dec. 2009	2,699,999,990,000
Kondo and Yee	Aug. 2010	**5,000,000,000,000**
Kondo and Yee	Oct. 2011	**10,000,000,000,000**
Kondo and Yee	Dec. 2013	**12,100,000,000,000**

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- An algorithm found by computer—now used to check record π computations and in some compilers.

What BBP Is?

Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \qquad (2)$$

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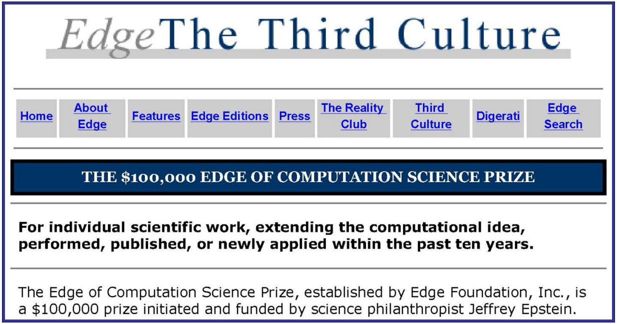
- Bailey-Crandall (220) link BBP and normality.

Edge of Computation Prize Finalist (2005)



Edge The Third Culture

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THE $100,000 EDGE OF COMPUTATION SCIENCE PRIZE

For individual scientific work, extending the computational idea, performed, published, or newly applied within the past ten years.

The Edge of Computation Science Prize, established by Edge Foundation, Inc., is a $100,000 prize initiated and funded by science philanthropist Jeffrey Epstein.

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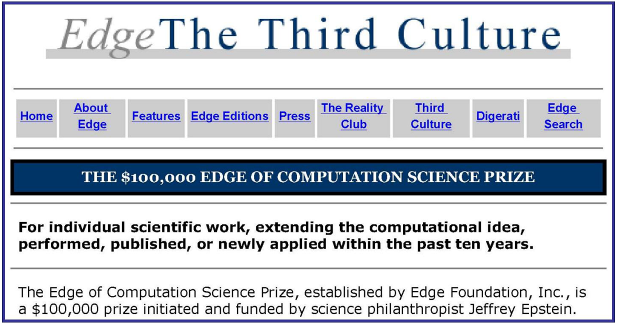

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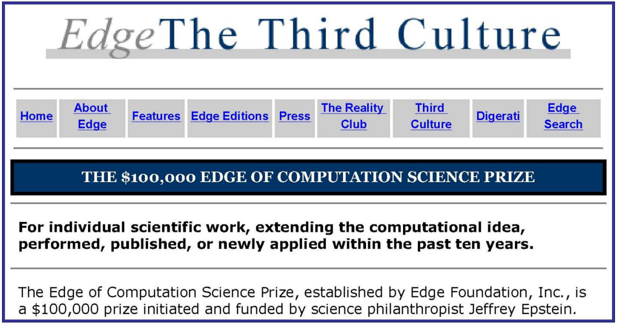

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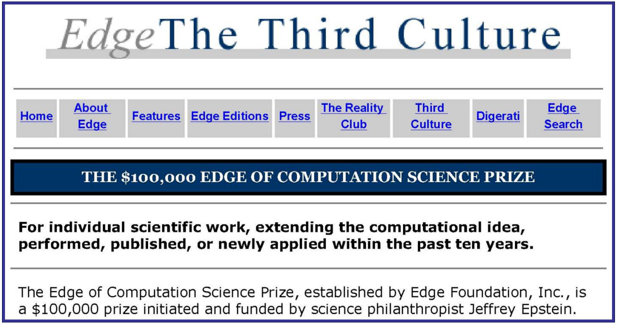

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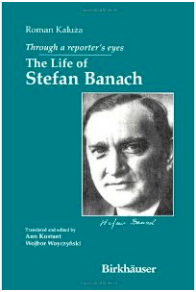
- Won by David Deutsch — discoverer of Quantum Computing.

Stefan Banach (1892-1945) Another Nazi casualty

A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. [3]

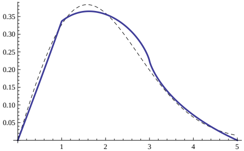


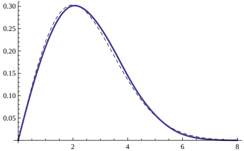
[3] Only the best get stamps. Quoted in `www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html`.

Contents

A Little History:

From a vast literature





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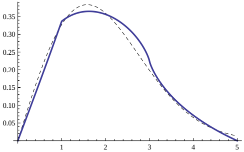

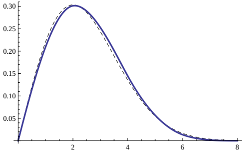

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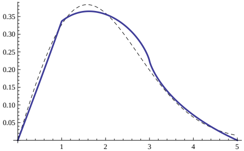

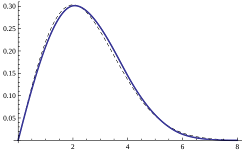

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Karl Pearson (1857-1936): founded statistics, eugenicist & socialist, changed name ($C \mapsto K$), declined knighthood.

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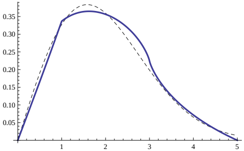

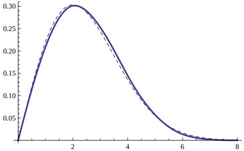

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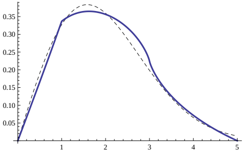
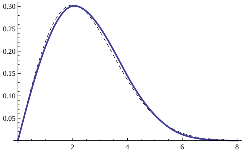
A Little History:

From a vast literature





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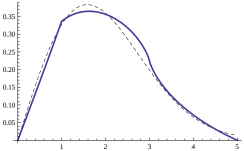

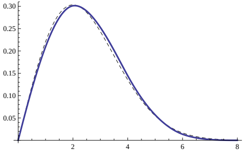

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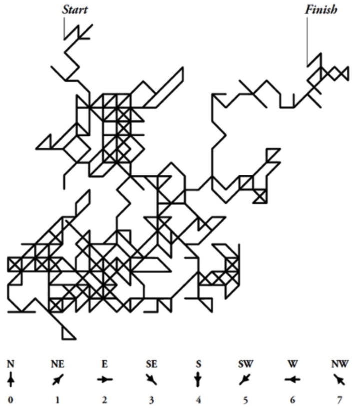
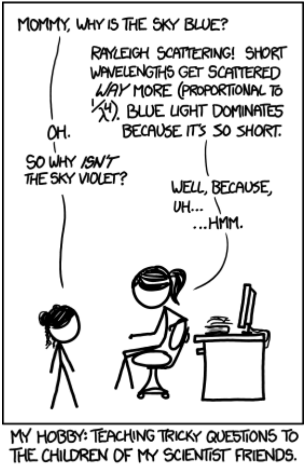
- appear in graph theory, quantum chemistry, in quantum physics as hexagonal and diamond lattice integers, etc ...

The first walk (Venn)

Why is the sky blue?

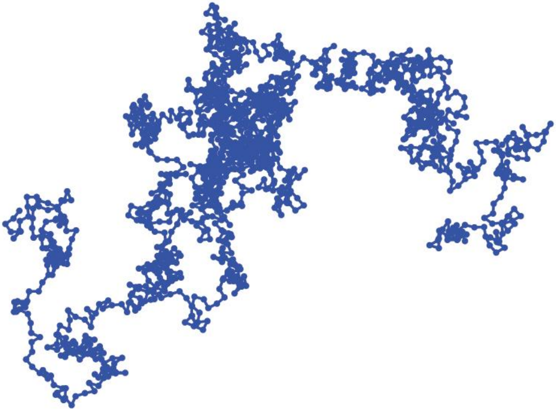


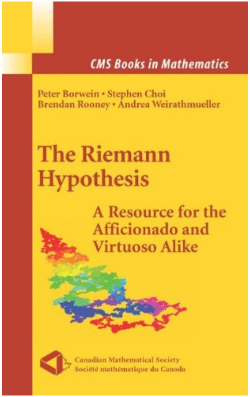
he first person to visualize the random nature of pi’s decimal digits was the Victorian nathematician John Venn. In *The Logic of Chance* (1888), he suggested that the digits 0 to 7 epresent eight compass directions, and he followed the path tracked by these digits in pi. He nisses out the initial 3, and starts 14159. Venn’s image was the first “random walk”, an idea now ised frequently in probability and statistics. (The illustration is taken from my book, Alex's dventures in Numberland)



One 1500-step ramble: a familiar picture

Liouville function





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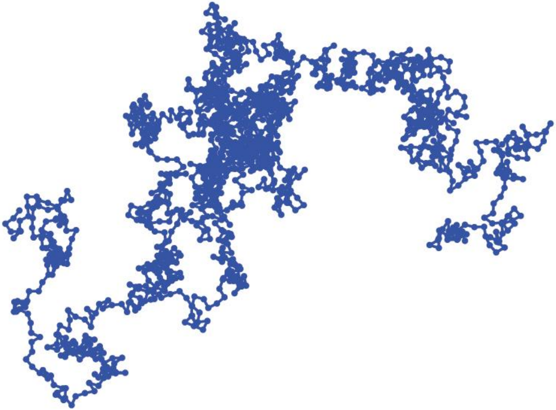

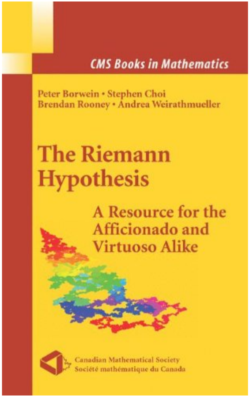

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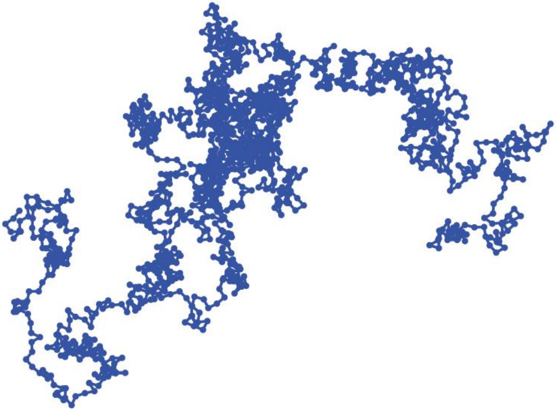
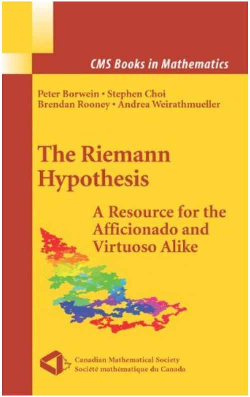


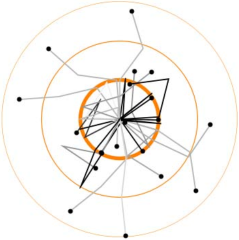


- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
- 1D or 2D *lattice*: probability one of returning to the origin.

Contents

Case study II: short rambles

a less familiar picture?



1000 three-step uniform planar walks

The moments of an n-step planar walk: $W_n := W_n(1)$

- Second simplest case:

$$W_2 = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = \ ?$$

[4] Quadrature was our first interest

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- There is always a 1-dimension reduction[4]

$$\begin{aligned} W_n(s) &= \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1}, x_n) \\ &= \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1}) \end{aligned}$$

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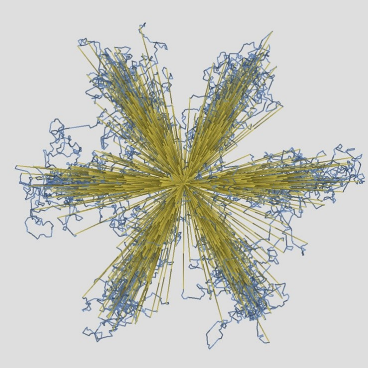
- So $W_2 = 4 \int_0^{1/4} \cos(\pi x) \mathrm{d}x \ = \ \frac{4}{\pi}$.

[4]Quadrature was our first interest

Art meets science

AAAS & Bridges conference



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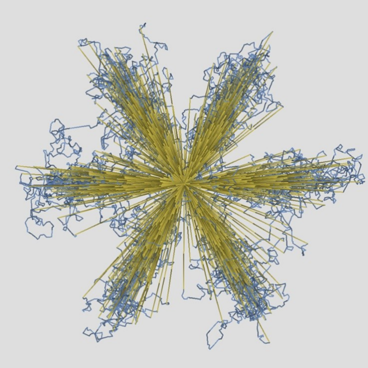

A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds.

(Nadia Whitehead 2014-03-25 16:15)

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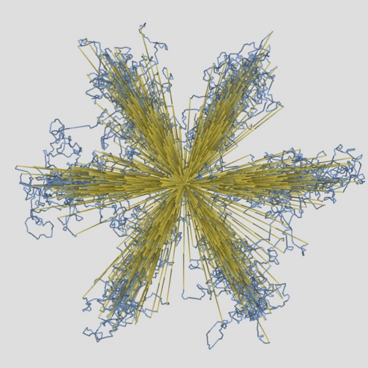
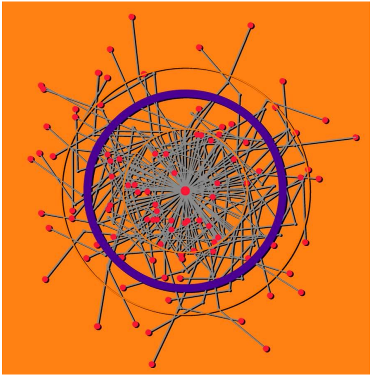


A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds.

(Nadia Whitehead 2014-03-25 16:15)

(JonFest 2011 Logo) *Three-step random walks.*
*The (purple) expected distance travelled is **1.57459** ...*

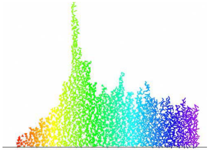
The closed form W_3 is given below.

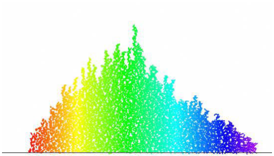


$$W_3 = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4}$$

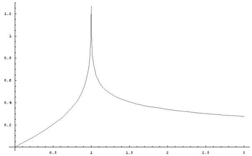
Contents

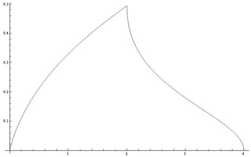
Simulating the densities for $n = 3, 4$ ANIMATION





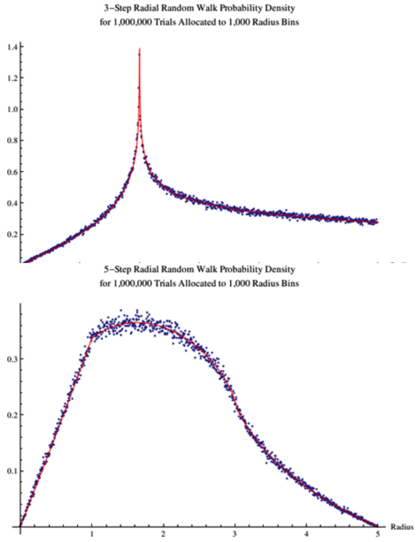
The densities p_3 (L) and p_4 (R)





Simulation thanks to Cam Rogers

The radial densities for $3 \leq n \leq 6$ (simulations by A. Mattingly)



Pearson's original full question

and comment on p_5

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

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- **JMAA 2016.** Our analysis of short walks extends interestingly to arbitrary dimensions (JMB, Straub, Vignat).

The radial densities for $n = 3, 4$ are modular functions

Let $\sigma(x) := \frac{3-x}{1+x}$. Then σ is an involution on $[0,3]$ sending $[0,1]$ to $[1,3]$:

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)). \tag{3}$$

So $\frac{3}{4} p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$, $p(1) = \infty$.

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So $\frac{3}{4}p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$, $p(1) = \infty$. We found:

$$p_3(\alpha) = \frac{2\sqrt{3}\alpha}{\pi\,(3+\alpha^2)}\, {}_2F_1\left(\left.{\frac{1}{3}, \frac{2}{3} \atop 1}\right| \frac{\alpha^2\left(9-\alpha^2\right)^2}{\left(3+\alpha^2\right)^3}\right) = \frac{2\sqrt{3}}{\pi} \frac{\alpha}{\mathrm{AG}_3\left(3+\alpha^2, 3\left(1-\alpha^2\right)^{2/3}\right)}$$

where AG_3 is the *cubically convergent* mean iteration (1991):

$$\mathrm{AG}_3(a,b) := \frac{a+2b}{3} \bigotimes \left(b \cdot \frac{a^2+ab+b^2}{3}\right)^{1/3}.$$

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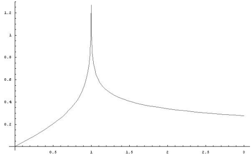

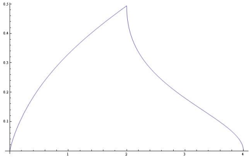

Formula for the 'shark-fin' p_4

▸ SKIP

We ultimately deduce on $2 \le \alpha \le 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16-\alpha^2}}{\alpha} \, {}_3F_2\left({\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{5}{6}, \frac{7}{6}} \middle| \frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4} \right). \tag{5}$$

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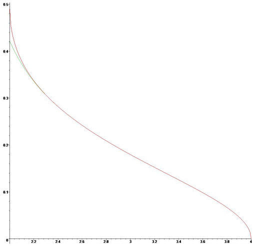


$\leftarrow$ p_4 from (5) vs 18-terms of empirical power series

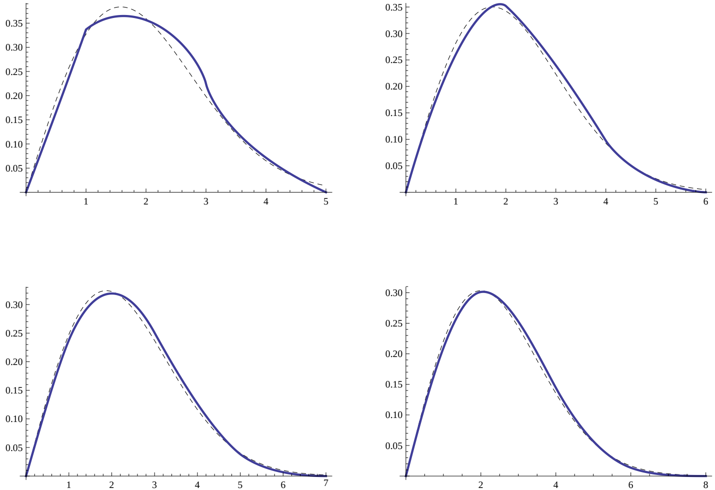
✓ Proves $p_4(2) = \frac{2^{7/3}\pi}{3\sqrt{3}}\,\Gamma\left(\frac{2}{3}\right)^{-6} = \frac{\sqrt{3}}{\pi}\,W_3(-1) \approx 0.494233 < \frac{1}{2}$

- Empirically, quite marvelously, we found — and proved by a subtle use of distributional Mellin transforms — that on $[0,2]$ as well:

$$p_4(\alpha) \overset{?}{=} \frac{2}{\pi^2}\frac{\sqrt{16-\alpha^2}}{\alpha}\,\Re\,{}_3F_2\left(\left.{\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{5}{6},\frac{7}{6}}\right| \frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4}\right) \tag{6}$$

(Discovering this $\Re$ brought us full circle.)

The radial densities for $5 \leq n \leq 8$ (and large n approximation)



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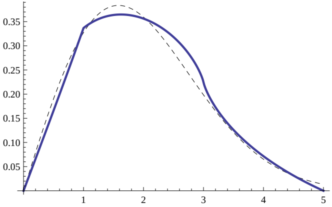
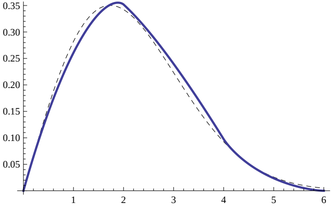
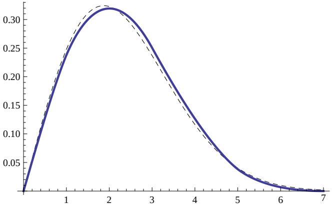
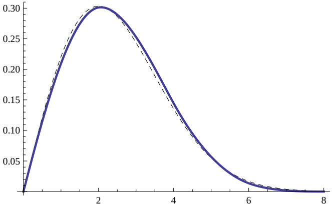




Both p_{2n+4}, p_{2n+5} are n-times continuously differentiable for $x > 0$ with $p_n(x) \sim \frac{2x}{n} e^{-x^2/n}$.

So "four is small" *but* "eight is large."





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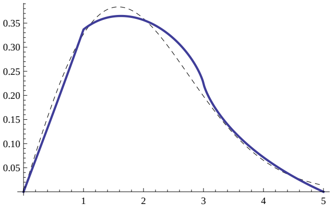

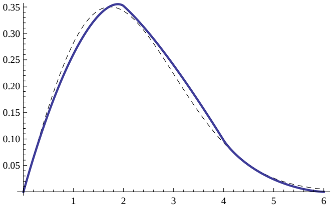

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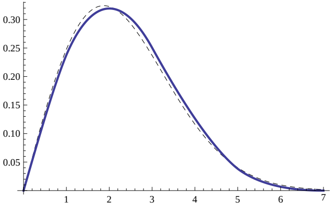
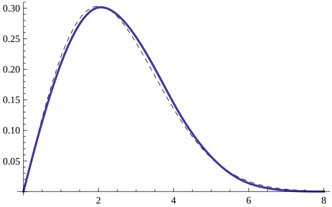




- Pearson wondered if p_5 was linear on $[0, 1]$. Only disproven in sixties.

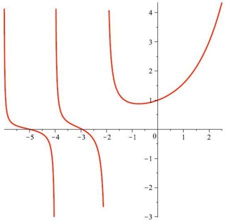
Contents

Meijer-G (1936) form for W_3 and graph on real line

Theorem (Meijer-G form for W_3)

For s not an odd integer

$$W_3(s) = \frac{\Gamma(1+\frac{s}{2})}{\sqrt{\pi}\,\Gamma(-\frac{s}{2})}\, G^{21}_{33}\left(\begin{matrix} 1,1,1 \\ \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \,\middle|\, \frac{1}{4}\right).$$



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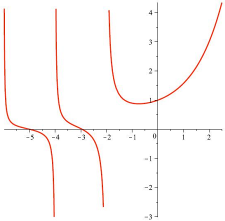
- Proved using residue calculus methods.



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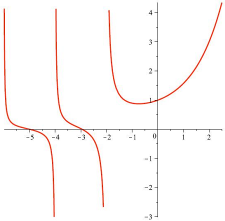
- First found by Crandall via CAS.
- Proved using residue calculus methods.
- $W_3(s)$ is among the first non-trivial higher order Meijer-G function to be placed in closed form.



Meijer-G (1936) form for W_4

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Theorem (Meijer form for W_4)

For $\Re s > -2$ and s not an odd integer

$$W_4(s) = \frac{2^s}{\pi} \frac{\Gamma(1+\frac{s}{2})}{\Gamma(-\frac{s}{2})} G_{44}^{22}\left(\begin{matrix} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \middle| 1 \right). \tag{7}$$

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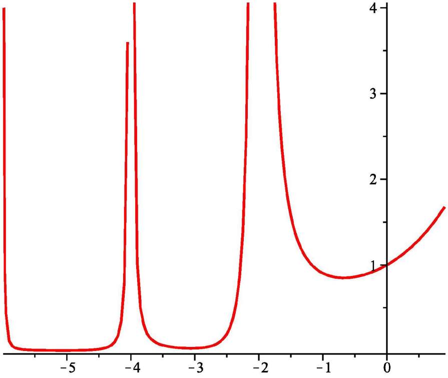
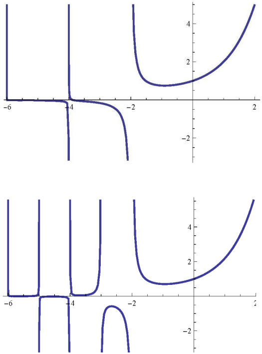
But we really need a formula with $s = 1$, that is an **integer**.

Visualizing W_4, W_5, and W_6 on the real line





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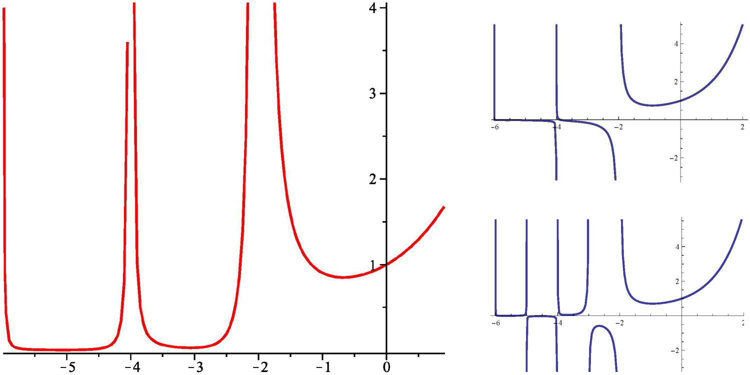

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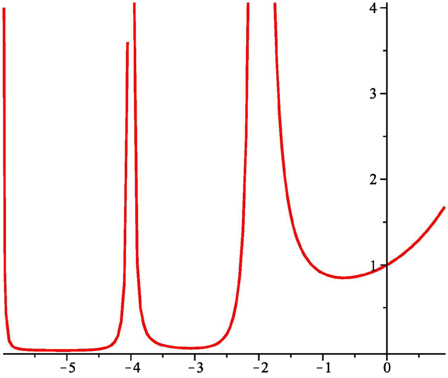
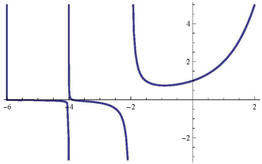
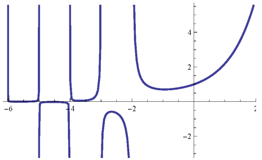
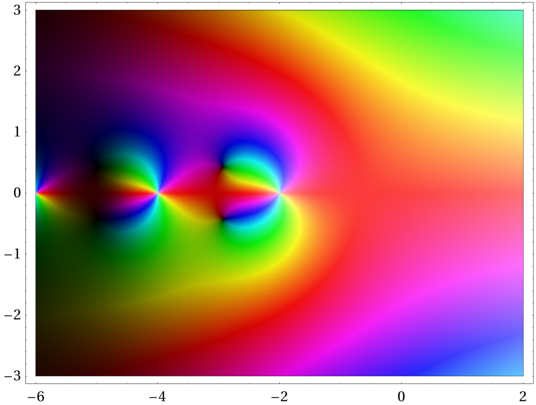






- Use recursion from $s > 1$
- Nonnegativity of W_4 was hard to prove (Wan)

Visualizing W_4 in the complex plane



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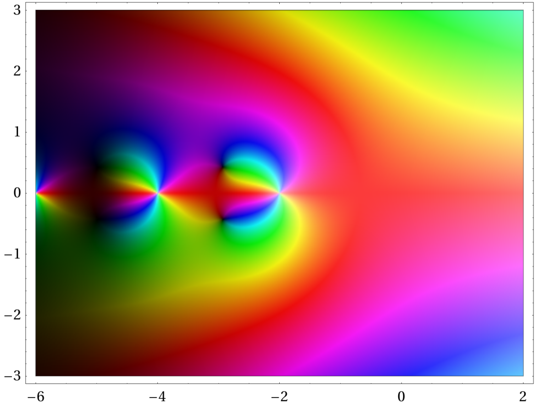

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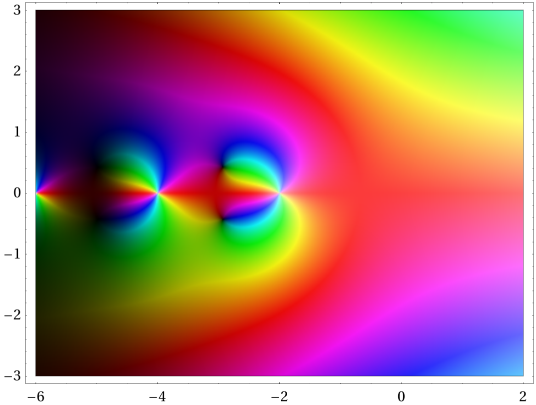

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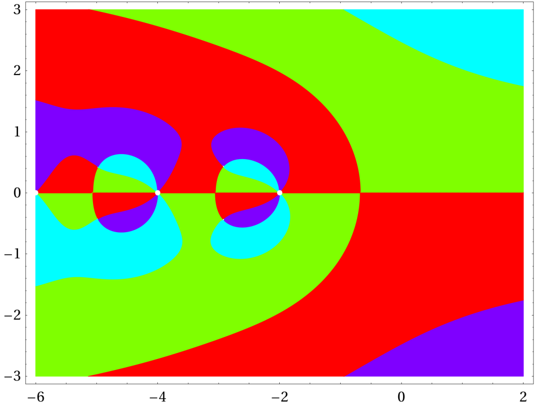

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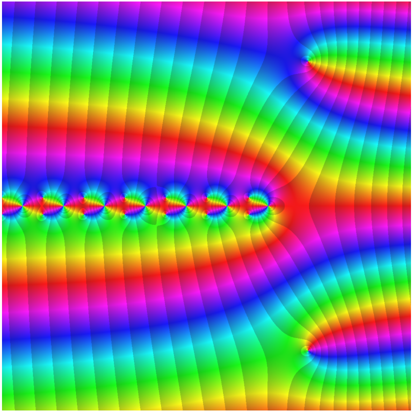

- Less easily drawn now from the Meijer-G representation.
- As prepared for Springer's Mathematical Beauties (2016).

Simplifying the Meijer integrals for W_3 and W_4

- We (humans and/or computers) now obtained:

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Corollary (Hypergeometric forms for non-integer $s > -2$)

$$W_3(s) = \frac{\tan\left(\frac{\pi s}{2}\right)}{2^{2s+1}} \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left({\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{s+3}{2}, \frac{s+3}{2}} \middle| \frac{1}{4} \right) + \binom{s}{\frac{s}{2}} {}_3F_2\left({-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, -\frac{s-1}{2}} \middle| \frac{1}{4} \right),$$

and

$$W_4(s) = \frac{\tan\left(\frac{\pi s}{2}\right)}{2^{2s}} \binom{s}{\frac{s-1}{2}}^3 {}_4F_3\left({\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s}{2}+1 \atop \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2}} \middle| 1 \right) + \binom{s}{\frac{s}{2}} {}_4F_3\left({\frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, 1, -\frac{s-1}{2}} \middle| 1 \right).$$

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- We (humans) were able to provably take the limit at ± 1: e.g.,

$$W_4(-1) = \frac{\pi}{4} {}_7F_6\left(\begin{matrix}\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1\end{matrix} \middle| 1\right) = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}$$

$$= \frac{\pi}{4} {}_6F_5\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1\end{matrix} \middle| 1\right) + \frac{\pi}{64} {}_6F_5\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2\end{matrix} \middle| 1\right).$$

Contents

Hypergeometric values of W_3: from Meijer-G values.

With much work involving moments of elliptic integrals we obtain:

Theorem (Tractable hypergeometric form for W_3)

(a) For $s \neq -3, -5, -7, \ldots$, we have

$$W_3(s) = \frac{3^{s+3/2}}{2\pi}\beta\left(s+\frac{1}{2}, s+\frac{1}{2}\right) {}_3F_2\left(\begin{matrix}\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2}\\ 1, \frac{s+3}{2}\end{matrix}\middle|\frac{1}{4}\right). \qquad (8)$$

(b) For every natural number $k = 1, 2, \ldots$,

$$W_3(-2k-1) = \frac{\sqrt{3}\binom{2k}{k}^2}{2^{4k+1}3^{2k}} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ k+1, k+1\end{matrix}\middle|\frac{1}{4}\right).$$

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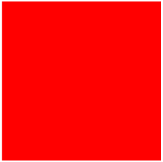
- The following formula hints at role played by Bessel functions (Kluywer 1906 and `http://www.carma.newcastle.edu.au/jon/walks-anu.pdf`):

$$W_n = n \int_0^\infty J_1(x) J_0(x)^{n-1} \frac{\mathrm{d}x}{x} \approx \frac{\sqrt{n\pi}}{2}.$$

Contents

What is a (base four) random walk ?

Pick a random number in $\{0, 1, 2, 3\}$ and move according to $0 = \rightarrow$, $1 = \uparrow$, $2 = \leftarrow$, $3 = \downarrow$



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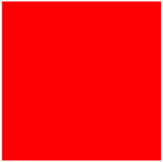

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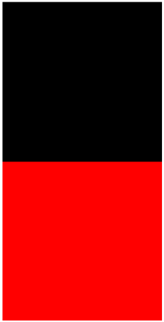

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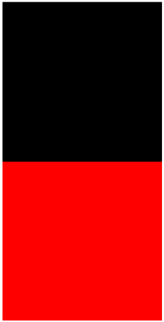

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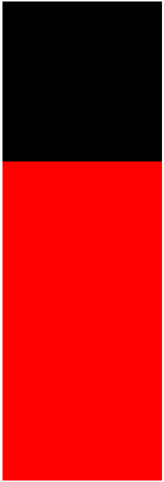

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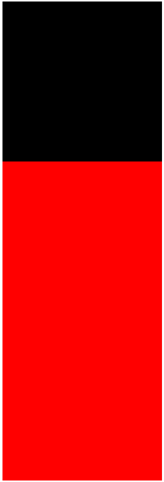

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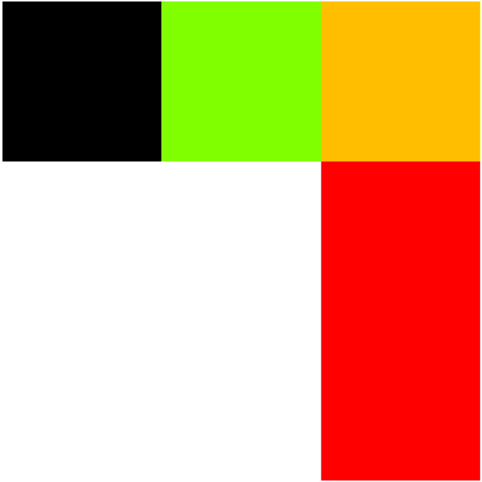

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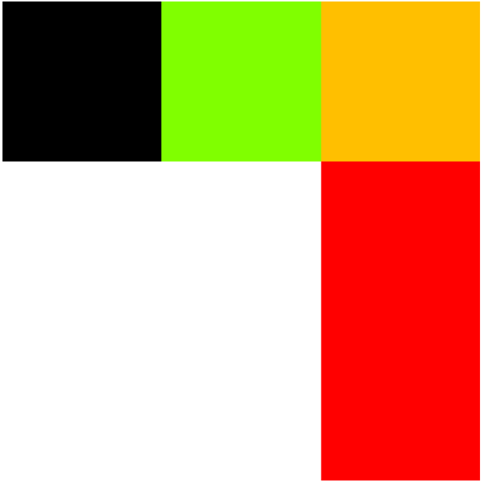

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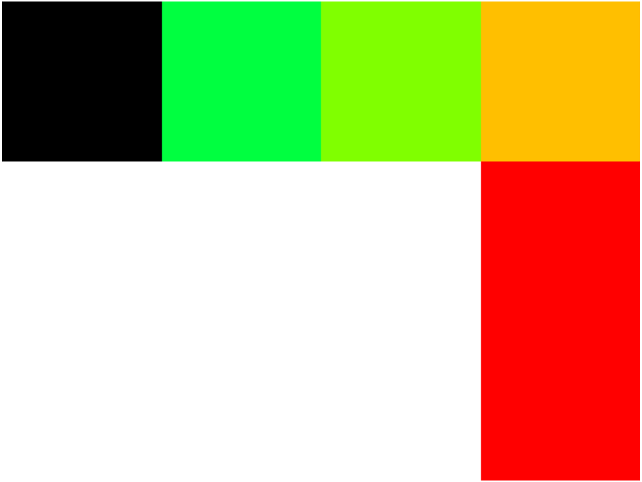

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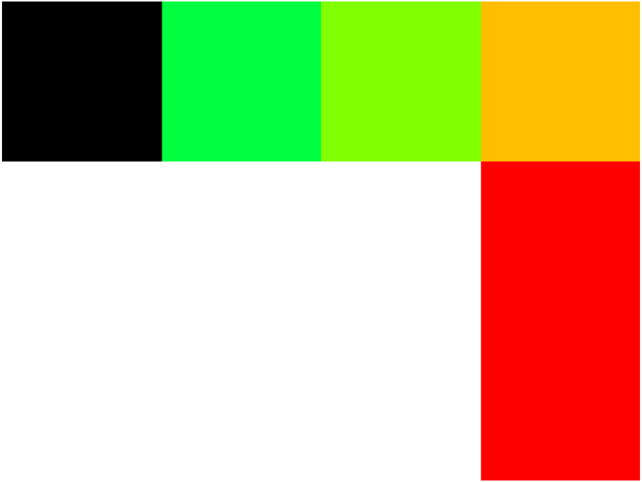

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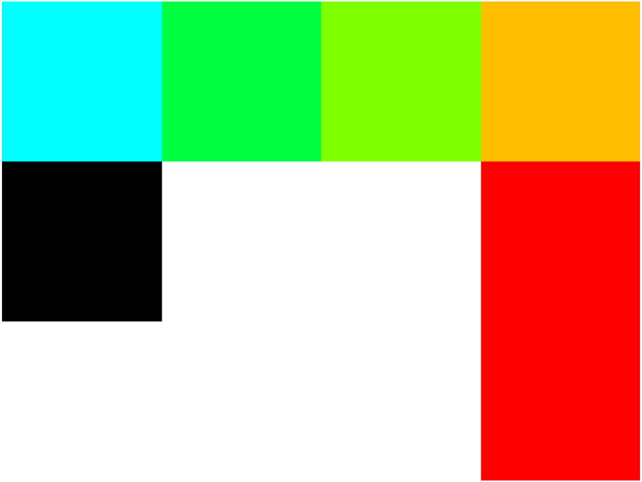

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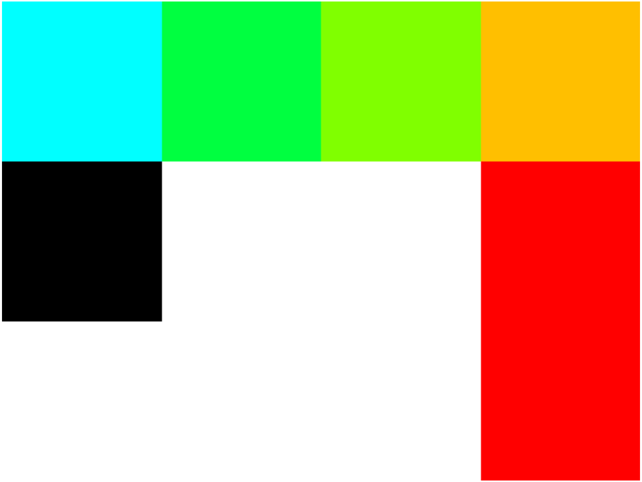

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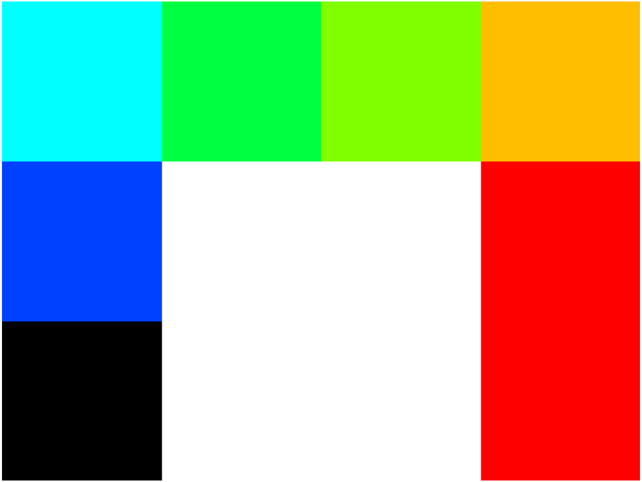

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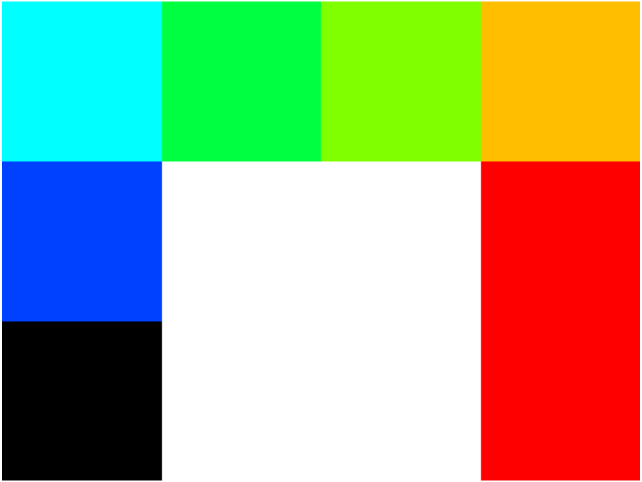

$$0 = \rightarrow$$

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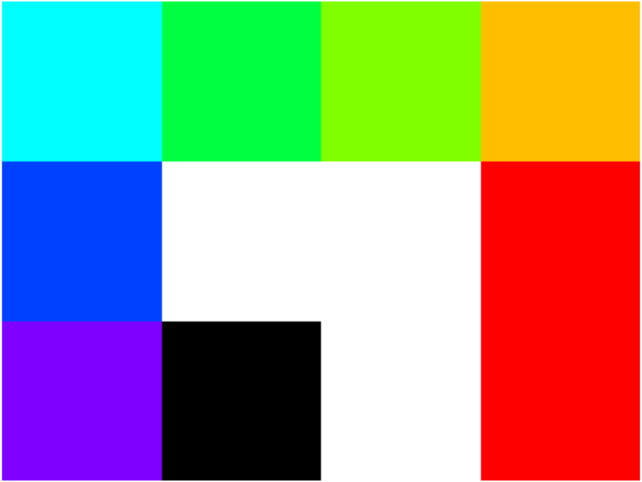

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What is a random walk (base 4)?

Pick a random number in $\{0,1,2,3\}$ and move $0 = \rightarrow,\ 1 = \uparrow,\ 2 = \leftarrow,\ 3 = \downarrow$ ANIMATION

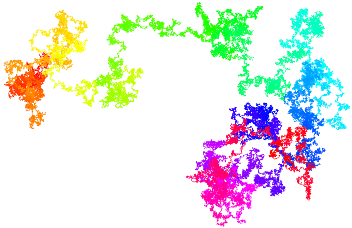

Figure: A million step base-4 pseudorandom walk. We use the spectrum to show when we visited each point (ROYGBIV and R).

Random walks look similarish

Chaos theory (order in disorder)

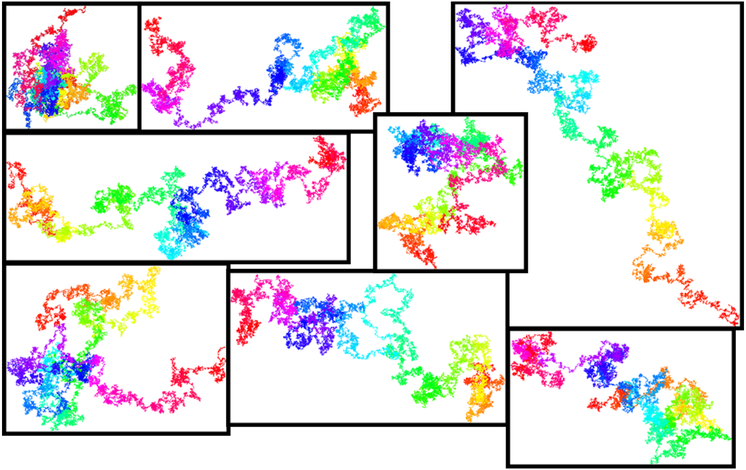

Figure: Eight different base-4 (pseudo)random[5] walks of one million steps.

[5] Python uses the *Mersenne Twister* as the core generator. It has a period of $2^{19937} - 1 \approx 10^{6002}$.

Base-b random walks: Our direction choice

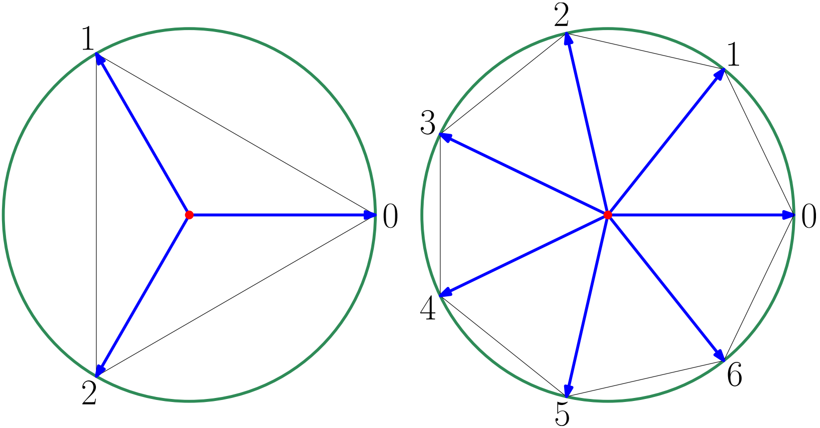

Figure: Directions for base-3 and base-7 random walks.

Contents

III: Two rational numbers

ANIMATION

The base-4 digit expansion of $Q1$ and $Q2$:

```
Q1=
0.221221012232121200122101223121001222100011232123121000122210001222
10001222100012221000012221000122201103010122010012010311033333333333
33333333333333330111111111111111111111111111100100000000300300320032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232223000322230032221330023321233023213232112112121222323233
33303000001000323003230032203032030110333011103301103101111011332333
3232322321221211211121122322222122...
```

```
Q2=
0.221221012232121200122101223121001222100011232123121000122210001222
10001222100012221000012221000122201103010122010012010311033333333333
33333333333333330111111111111111111111111111100100000000300300320032
00320030223000322203000322230003022220300032223000322230003222300032
22320000232223000322230032221330023321233023213232112112121222323233
33303000001000323003230032203032030110333011103301103101111011000000
000000...
```

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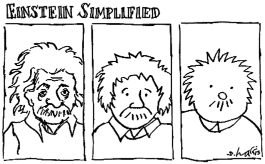

Figure: Self-referent walks on the rational numbers $Q1$ (top) and $Q2$ (bottom).

Two more rationals

Hard to tell from their decimal expansions

The following relatively small rational numbers [G. Marsaglia, 2010]

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{1000000000061},$$

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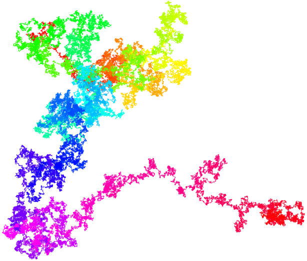
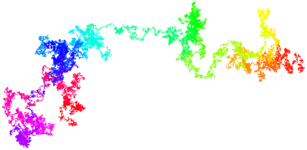


(a) $Q3$



(b) $Q4$

Figure: Walks on the first million base-10 digits of the rationals $Q3$ and $Q4$.

Walks on the digits of numbers

ANIMATION

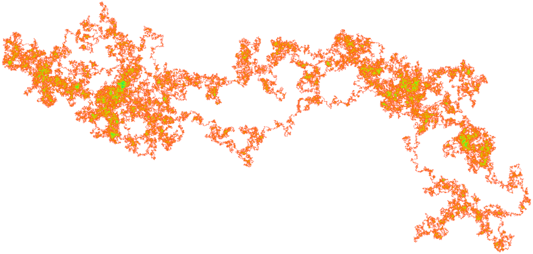


Figure: A walk on the first 10 million base-4 digits of π.

See also D. Bailey, J. Borwein, R. Brent and M. Reisi, "Reproducibility in computational science a case study: randomness of the digits of Pi." Preprint 2016.

https://www.carma.newcastle.edu.au/jon/pi-repro.pdf

Walks on the digits of numbers

Coloured by hits (more pink is more hits)

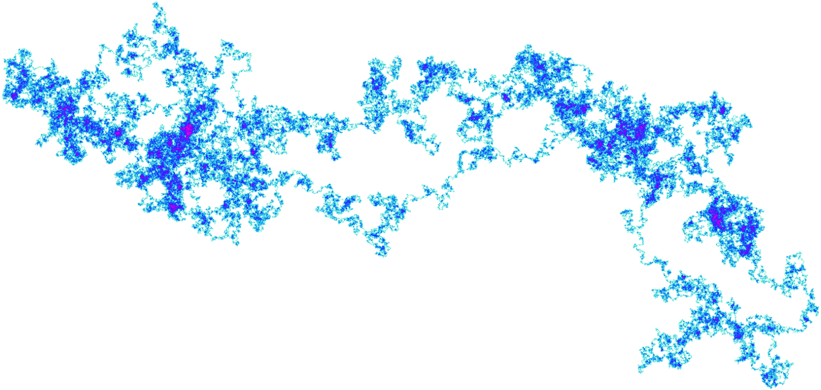


Figure: 100 million base-4 digits of π coloured by number of returns to points.

Contents

The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

1973 Richard Stoneham proved some of the following (nearly 'natural') constants are b-normal for relatively prime integers b, c:

$$\alpha_{b,c} := \frac{1}{cb^c} + \frac{1}{c^2 b^{c^2}} + \frac{1}{c^3 b^{c^3}} + \ldots.$$

Such super-geometric sums are Stoneham constants. To 10 places

$$\alpha_{2,3} = \frac{1}{24} + \frac{1}{3608} + \frac{1}{3623878656} + \ldots$$

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- Since $3 < 2^{3-1} = 4$, $\alpha_{2,3}$ is 2-normal and 6-nonnormal !

The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

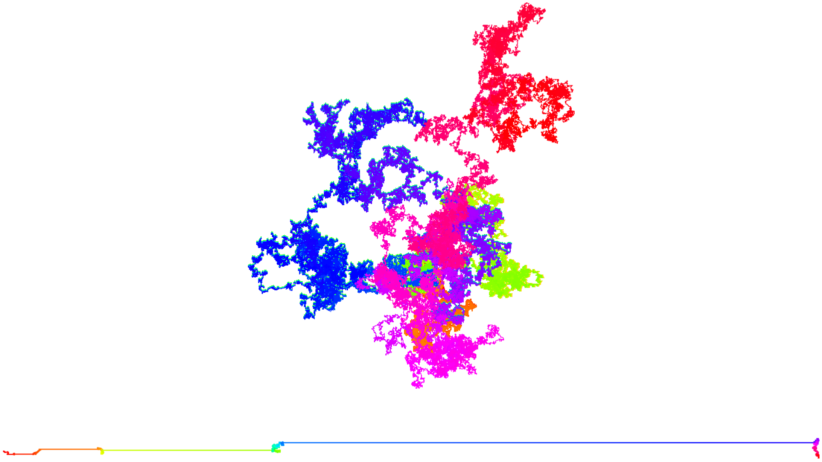

Figure: $\alpha_{2,3}$ is 2-normal (top) but 6-nonnormal (bottom). Is seeing believing?

The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

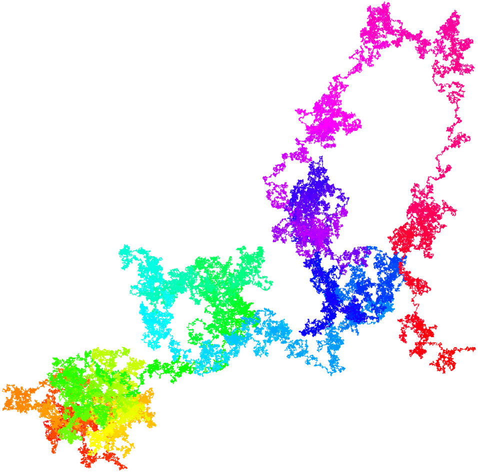

Figure: Is $\alpha_{2,3}$ 3-normal or not?

Contents

The expected distance to the origin

$\frac{\sqrt{\pi N}}{2\,d_N} \to 1$

Theorem

The expected distance d_N to the origin of a base-b *random walk* of N steps behaves like to $\sqrt{\pi N}/2$.

The expected distance to the origin

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Theorem

The expected distance d_N to the origin of a base-b *random walk* of N steps behaves like to $\sqrt{\pi N}/2$.

Number	Base	Steps	Average normalized dist. to the origin: $\frac{1}{\text{Steps}} \sum_{N=2}^{\text{Steps}} \frac{\text{dist}_N}{\frac{\sqrt{\pi N}}{2}}$	Normal
Mean of 10,000 random walks	4	1,000,000	**1.00315**	Yes
Mean of 10,000 walks on the digits of π	4	1,000,000	**1.00083**	?
$\alpha_{2,3}$	3	1,000,000	0.89275	?
$\alpha_{2,3}$	4	1,000,000	0.25901	Yes
$\alpha_{2,3}$	6	1,000,000	108.02218	No
π	4	1,000,000	0.84366	?
π	6	1,000,000	0.96458	?
π	10	1,000,000	0.82167	?
π	10	1,000,000,000	0.59824	?
$\sqrt{2}$	4	1,000,000	0.72260	?
Champernowne C_{10}	10	1,000,000	59.91143	Yes

Contents

Number of points visited

For a 2D lattice

- The expected number of distinct points visited by an N-step random walk on a two-dimensional lattice behaves for large N like $\pi N/\log(N)$ (Dvoretzky–Erdős, **1951**).

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$$\left(\frac{\pi(N+0.84)}{1.16\pi-1-\log 2+\log(N+2)}, \frac{\pi(N+1)}{1.066\pi-1-\log 2+\log(N+1)}\right).$$

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- For example, for $N = 10^6$ these bounds are $(199256.1, 203059.5)$, while $\pi N/\log(N) = 227396$, which overestimates the expectation.

Catalan's constant

$G = 1 - 1/4 + 1/9 - 1/16 + \cdots$

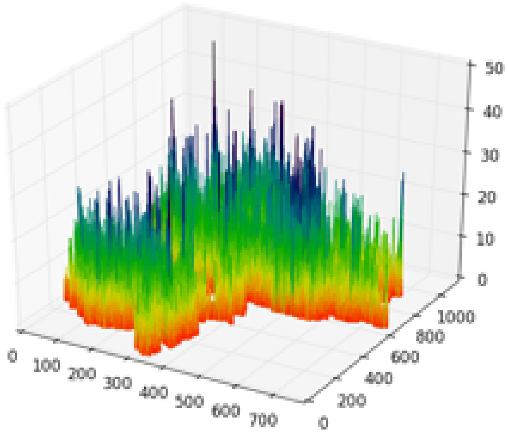
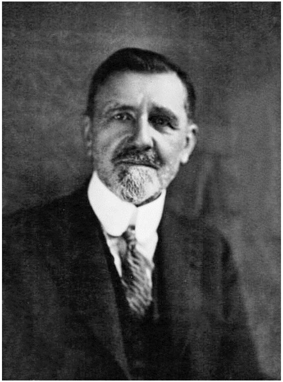


Figure: A walk on one million quad-bits of G with height showing frequency

Paul Erdős (1913-1996) "My brain is open"

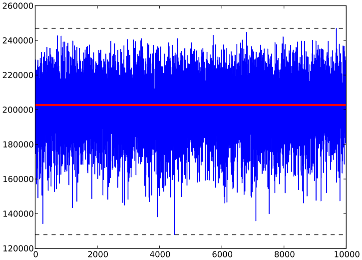
(a) Paul Erdős (Banff 1981. I was there)



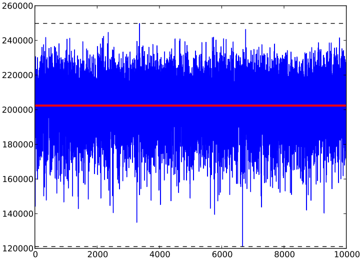
(b) Émile Borel (1871–1956)

Figure: Two of my favourites. Consult MacTutor.

Number of points visited: Again π looks random



(a) (Pseudo)random walks.



(b) Walks built by chopping up 10 billion digits of π.

Figure: Number of points visited by $10,000$ million-steps base-4 walks.

Points visited by various base-4 walks

Number	Steps	Sites visited	Bounds on the expectation of sites visited by a random walk	
			Lower bound	Upper bound
Mean of 10,000 random walks	1,000,000	**202,684**	199,256	203,060
Mean of 10,000 walks on the digits of π	1,000,000	**202,385**	199,256	203,060
$\alpha_{2,3}$	1,000,000	95,817	199,256	203,060
$\alpha_{3,2}$	1,000,000	195,585	199,256	203,060
π	1,000,000	**204,148**	199,256	203,060
π	10,000,000	1,933,903	1,738,645	1,767,533
π	100,000,000	16,109,429	15,421,296	15,648,132
π	1,000,000,000	138,107,050	138,552,612	140,380,926
e	1,000,000	176,350	199,256	203,060
$\sqrt{2}$	1,000,000	**200,733**	199,256	203,060
$\log 2$	1,000,000	**214,508**	199,256	203,060
Champernowne C_4	1,000,000	548,746	199,256	203,060
Rational number Q_1	1,000,000	378	199,256	203,060
Rational number Q_2	1,000,000	939,322	199,256	203,060

Normal numbers need not be so "random" ...

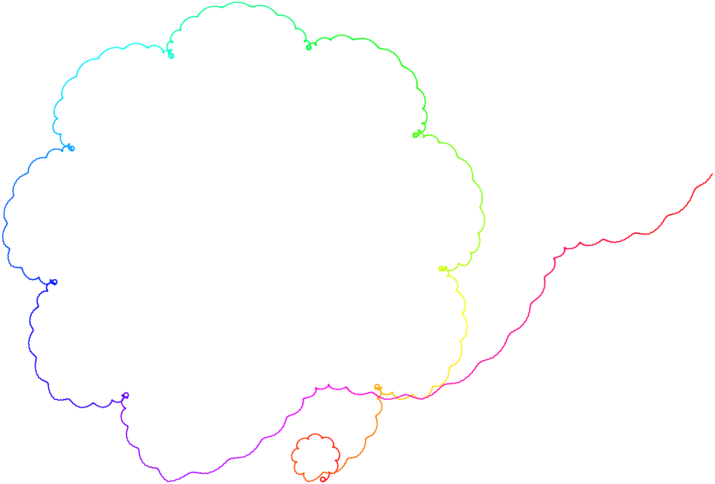


Figure: Champernowne $C_{10} = 0.123456789101112\ldots$ (normal).
Normalized distance to the origin: 15.9 (50,000 steps).

Normal numbers need not be so "random" ...

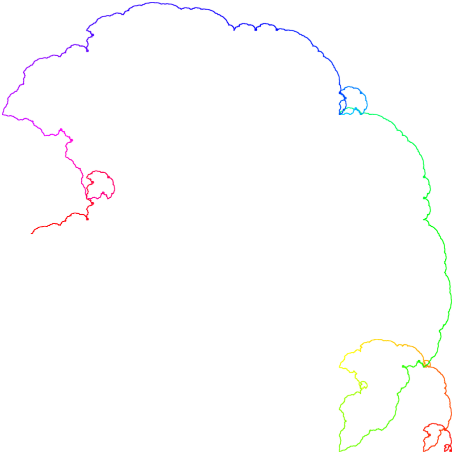


Figure: Champernowne $C_4 = 0.123101112132021\ldots$ (normal).
Normalized distance to the origin: 18.1 (100,000 steps).
Points visited: 52760. Expectation: (23333, 23857).

Normal numbers need not be so "random" ...

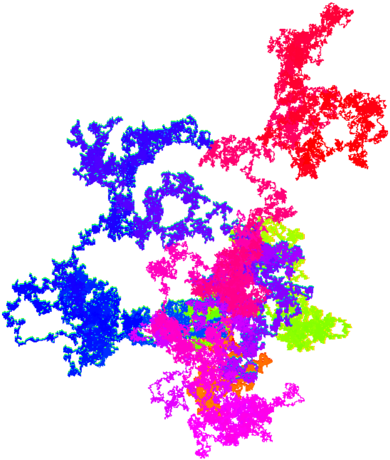
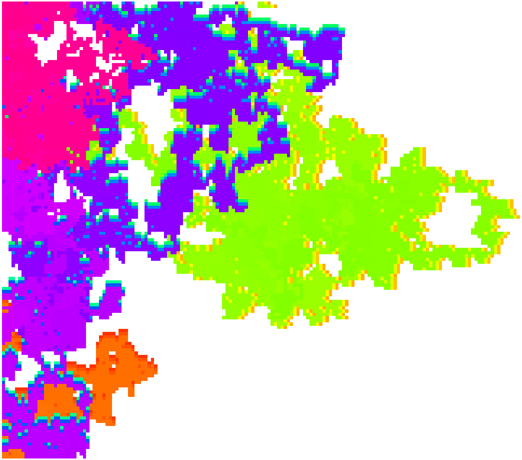


Figure: Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

Normal numbers need not be so "random" ...



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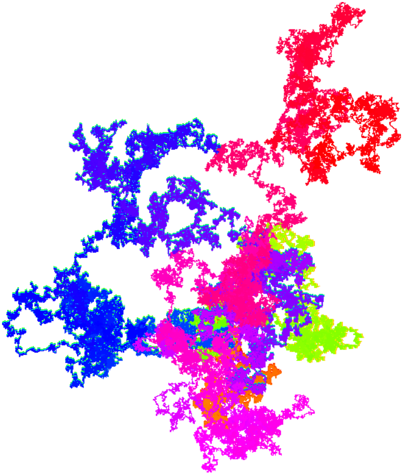
$\alpha_{2,3}$ is 4-normal but not so "random" ANIMATION



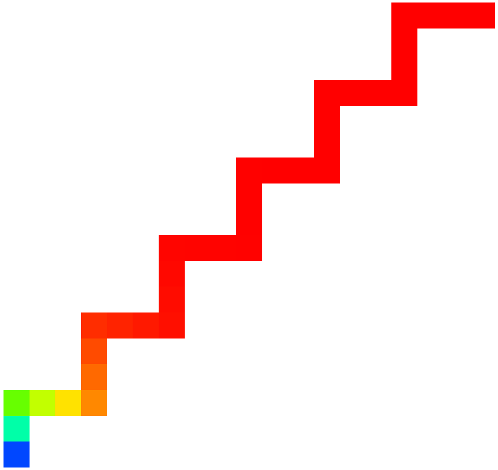


Figure: A pattern in the digits of $\alpha_{2,3}$ base 4. We show only positions of the walk after $\frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2\cdot 3^n$ steps, $n=0,1,\ldots,11$.

Experimental conjecture

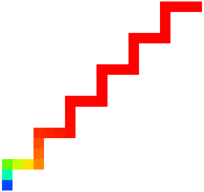
Proven 12-12-12 by Coons

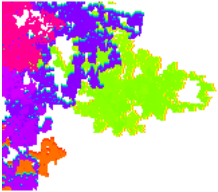
Theorem (Base-4 structure of Stoneham $\alpha_{2,3}$)

Denote by a_k the k^{th} digit of $\alpha_{2,3}$ in its base 4 expansion: $\alpha_{2,3} = \sum_{k=1}^{\infty} a_k/4^k$, with $a_k \in \{0,1,2,3\}$ for all k. Then, for all $n = 0,1,2,\ldots$ one has:

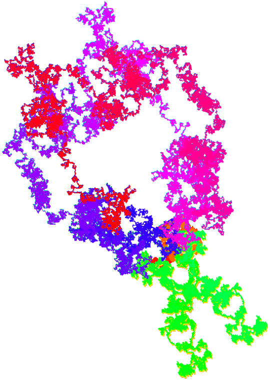
(i) $$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n} e^{a_k \pi i/2} = \begin{cases} -i, & \text{n odd} \\ -1, & \text{n even} \end{cases};$$

(ii) $a_k = a_{k+3^n} = a_{k+2\cdot 3^n}$ if $k = \frac{3(3^n+1)}{2}, \frac{3(3^n+1)}{2}+1, \ldots, \frac{3(3^n+1)}{2}+3^n-1$.





Likewise, $\alpha_{3,5}$ is 3-normal ... but not very "random" ANIMATION



Contents

Box-dimension:

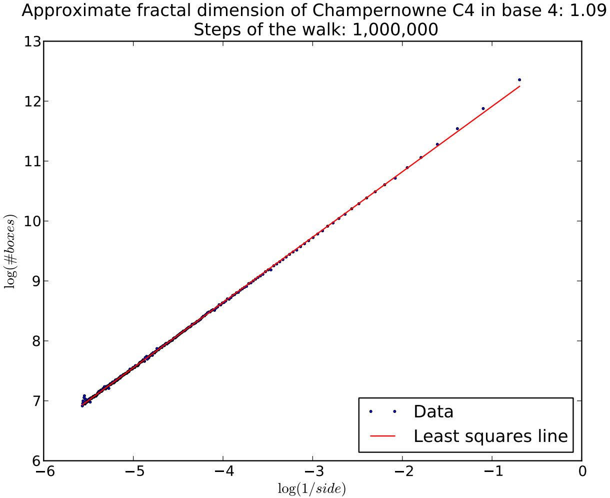
Tends to '2' for a planar random walk ▸ SKIP

$$\text{Box-dimension} = \lim_{\text{side}\to 0} \frac{\log(\#\ \text{boxes})}{\log(1/\text{side})}$$

Norway is "frillier" — *Hitchhiker's Guide to the Galaxy*

Box-dimension:

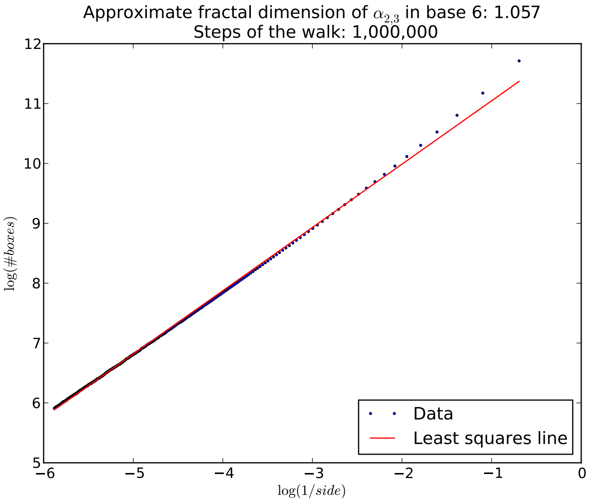
Tends to '2' for a planar random walk ▸ SKIP



Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

Box-dimension:

Tends to '2' for a planar random walk ▸ SKIP



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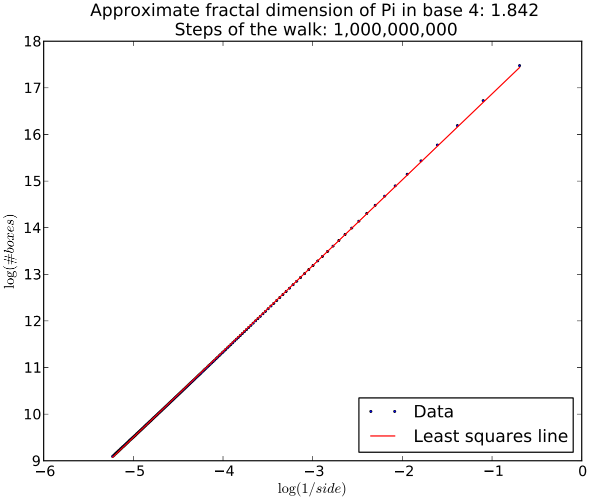

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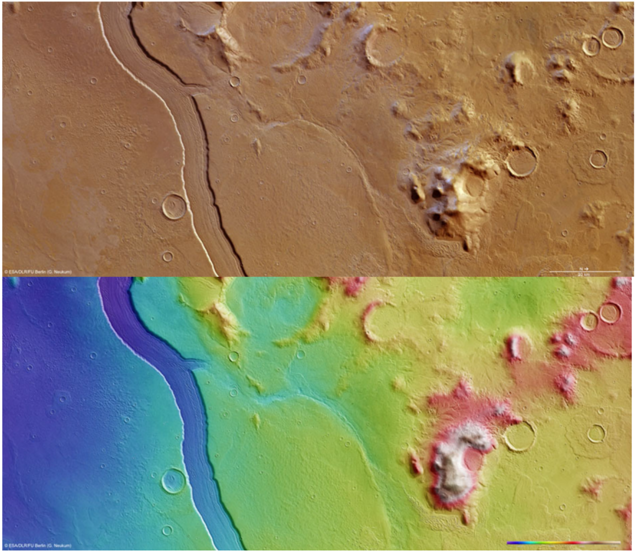
Contents

Fractals everywhere

From Mars

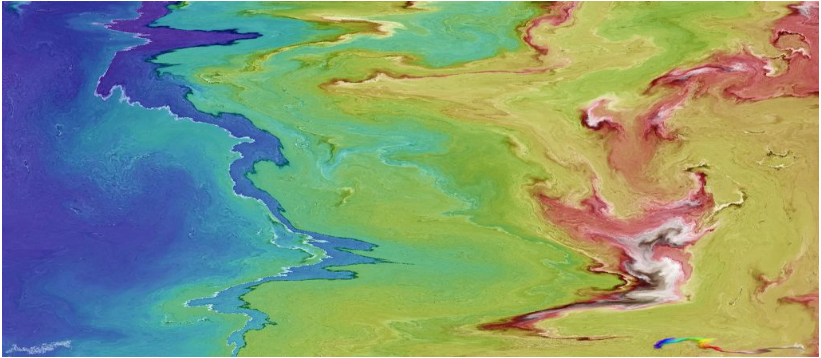
SKIP



Fractals everywhere

From Mars

▸ SKIP

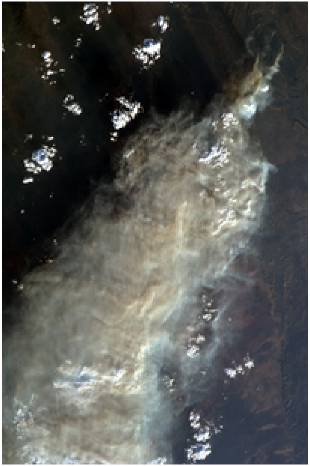


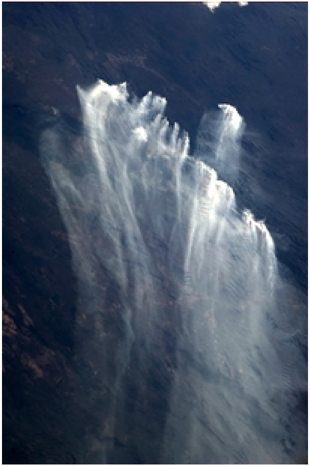
The picture fractalized by the Barnsley's

`http://frangostudio.com/frangocamera.html`

Fractals everywhere

From Space





Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

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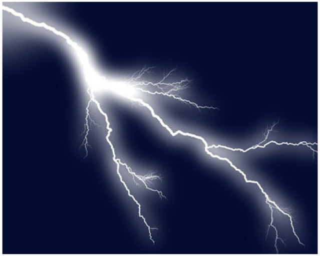
$1 \mapsto 3$ or $1 \mapsto 8$ or ...



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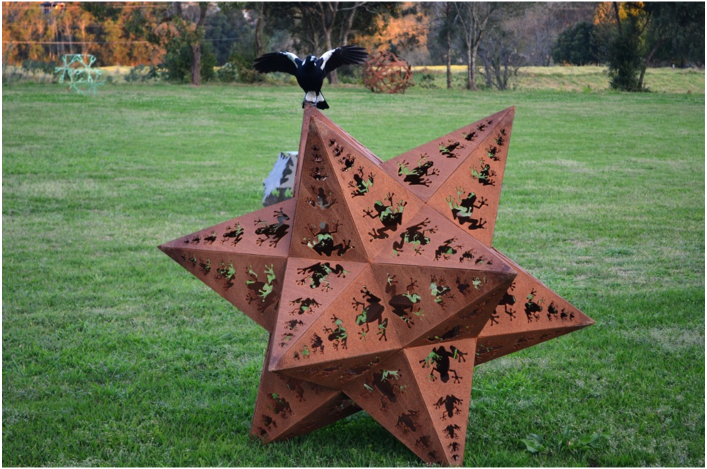


Fractals everywhere

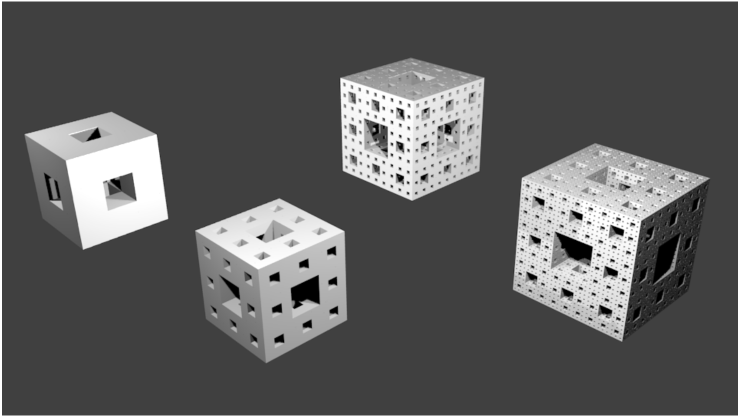
$1 \mapsto 3$ or $1 \mapsto 8$ or ...

Pascal triangle modulo two

[1] [1,1] [1,2,1] [1,3,3,1,] [1,4,6,4,1] [1,510,10,5,1] ...

Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

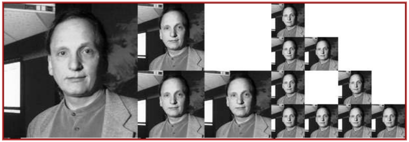


Steps to construction of a Sierpinski cube

Fractals everywhere

The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$



Fractals everywhere — The Sierpinski Triangle

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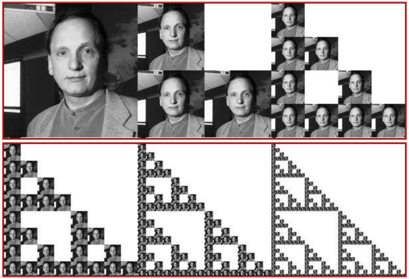

Fractals everywhere
The Sierpinski Triangle

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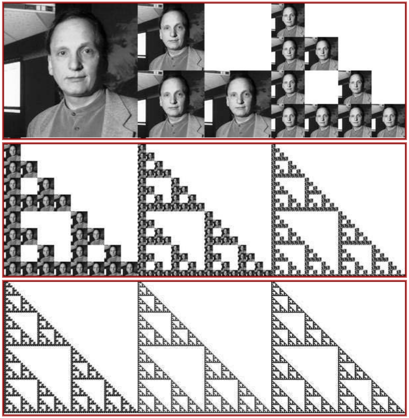
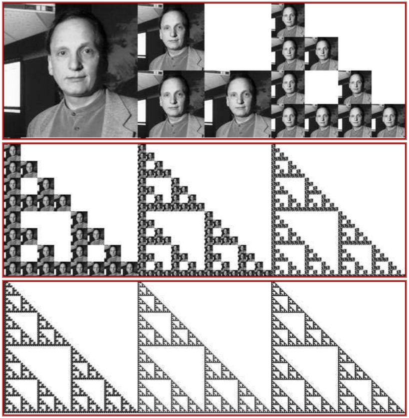
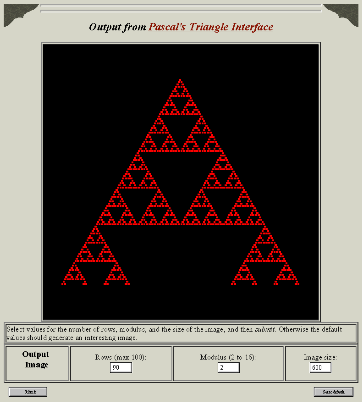


Fractals everywhere

The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$





http://oldweb.cecm.sfu.ca/cgi-bin/organics/pascalform

Contents

Three dimensional walks: Using base six — soon on 3D screen

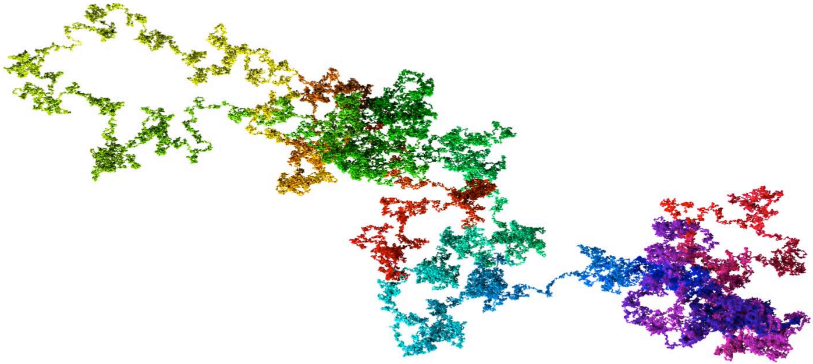


Figure: Matt Skerritt's 3D walk on π (base 6), showing one million steps. But 3D random walks are not recurrent.

Three dimensional walks:

Using base six — soon on 3D screen

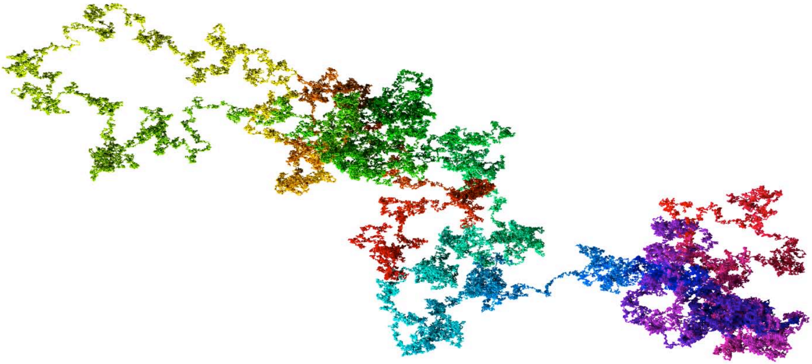
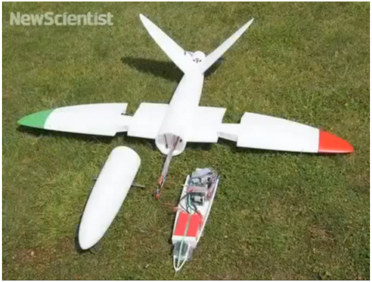


Figure: Matt Skerritt's 3D walk on π (base 6), showing one million steps. But 3D random walks are not recurrent.

"A drunken man will find his way home, a drunken bird will get lost forever." (Kakutani)

Three dimensional printing:

3D everywhere



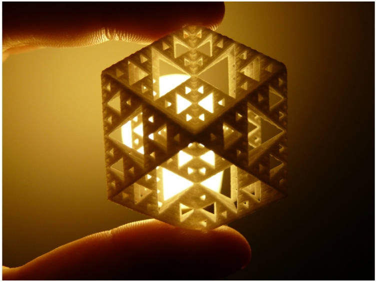


Figure: The future is here ...

www.digitaltrends.com/cool-tech/the-worlds-first-plane-created-entirely-by-3d-printing-takes-flight/

www.shapeways.com/shops/3Dfractals

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Chaos games: Move half-way to a (random) corner

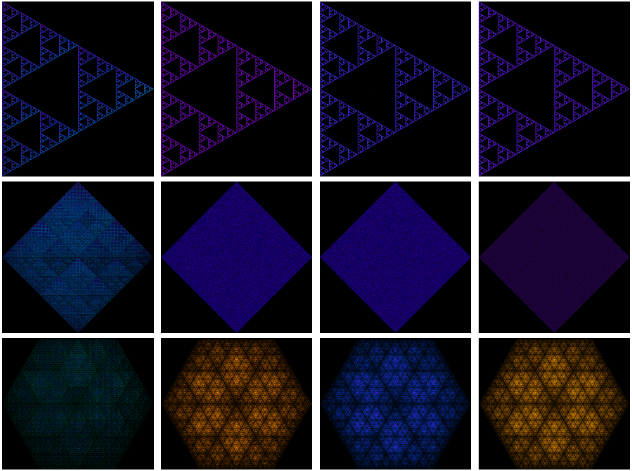

Figure: Coloured by frequency — leads to random fractals.
Row 1: Champernowne C_3, $\alpha_{3,5}$, random, $\alpha_{2,3}$. Row 2: Champernowne C_4, π, random, $\alpha_{2,3}$. Row 3: Champernowne C_6, $\alpha_{3,2}$, random, $\alpha_{2,3}$.

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Automatic numbers: Thue–Morse and Paper-folding

Automatic numbers are never normal. They are given by simple but fascinating rules...giving structured/boring walks:

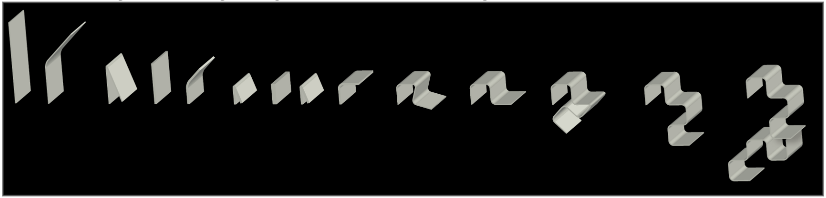

Figure: **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

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Thue–Morse constant (transcendental; 2-automatic, hence nonnormal):

$$TM_2 = \sum_{n=1}^{\infty} \frac{1}{2^{t(n)}} \text{ where } t(0) = 0, \text{ while } t(2n) = t(n) \text{ and } t(2n+1) = 1 - t(n)$$

$$0.01101001100101101001011001101001\ldots$$

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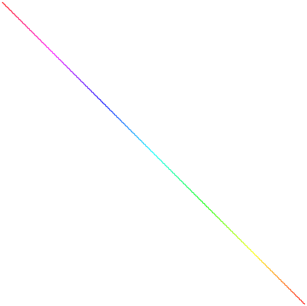
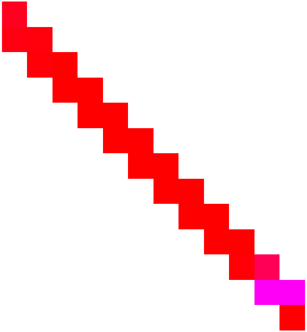


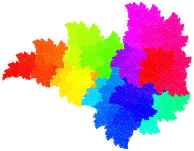
(a) 1,000 bits of Thue–Morse sequence.



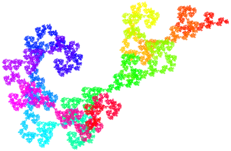
(b) 10 million bits of paper-folding sequence.

Figure: Walks on two automatic and so nonnormal numbers.

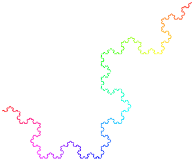
Automatic numbers: Turtle plots look great!



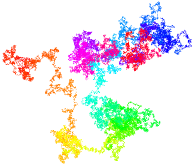
(a) Ten million digits of the paper-folding sequence, rotating 60°.



(b) One million digits of the paper-folding sequence, rotating 120° (a dragon curve).



(c) 100,000 digits of the Thue–Morse sequence, rotating 60° (a Koch snowflake).



(d) One million digits of π, rotating 60°.

Figure: Turtle plots on various constants with different rotating angles in base 2—where '**0**' yields forward motion and '**1**' rotation by a fixed angle.

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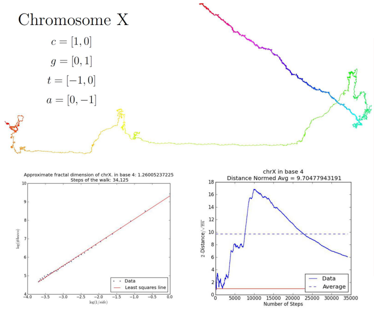
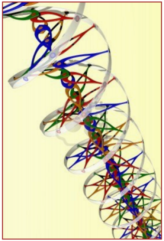
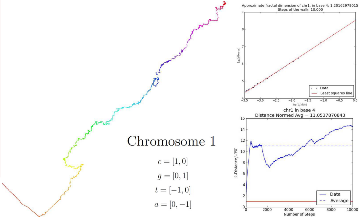
Genomes as walks: We are all base 4 numbers (ACGT/U)







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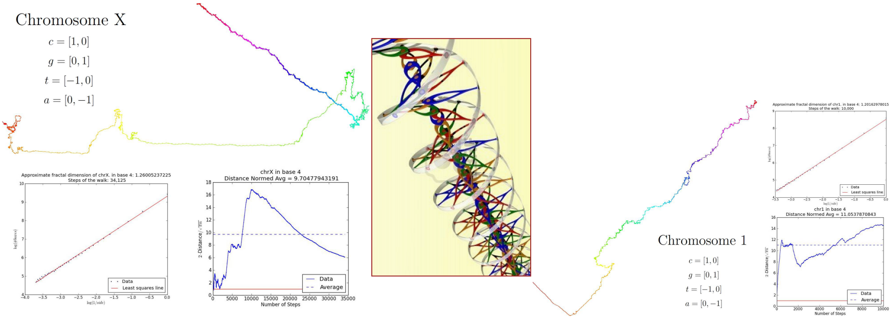

The X Chromosome (34K) and Chromosome One (10K).

Genomes as walks:

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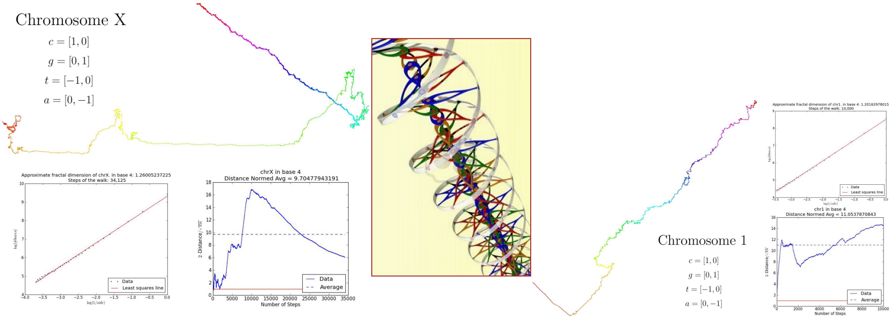

The X Chromosome (34K) and Chromosome One (10K).

- Ⓡ Chromosomes look less like π and more like concatenation numbers?

DNA for Storage:

We are all base 4 numbers (ACGT/U)

News 〉 Science 〉 Biochemistry and molecular biology

Shakespeare and Martin Luther King demonstrate potential of DNA storage

All 154 Shakespeare sonnets have been spelled out in DNA to demonstrate the vast potential of genetic data storage

Ian Sample, science correspondent
The Guardian, Thursday 24 January 2013
Jump to comments (...)

When written in DNA, one of Shakespeare's sonnets weighs 0.3 millionths of a millionth of a gram. Photograph: Oli Scarff/Getty

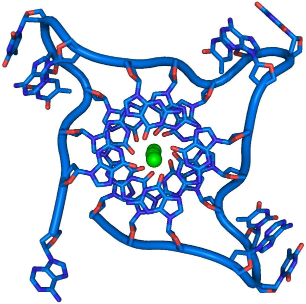

Figure: The potential for DNA storage (L) and the quadruple helix (R)

The end with some fractal dessert

The end with some fractal dessert

Thank you

Other References

http://carma.newcastle.edu.au/walks/

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