of finding a place in the standard treatises have had to be rediscovered is perhaps due partly to the fact that the journal to which they were communicated was not primarily mathematical. Hermite's better known formulae, *Journal für Math.*, 81 (1876), 220; *Oeuvres* III, 236; Krause, *Theorie der Doppeltperiodischen Functionen* (1895), 129, may have diverted attention, and they have a different basis. Reference should be made to an excellent paper by André, *Ann. Ec. Normale* (2), 8 (1879), 151, who does not however anticipate the use of recurrence formulae involving $I_{2m}(u)$.]

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ON A SPECIAL FUNCTIONAL EQUATION

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In this note I consider real functions of a variable $z \geqslant 0$, which satisfy the functional equation

$$\frac{f(z+\omega)-f(z)}{\omega} = f(qz),$$

where $\omega \neq 0$ and q $(0 < q < 1)$ are real constants. I prove that, for large z,

$$f(z) = e^{O(1)} \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{z^n}{n!},$$

if $f(z)$ is greater than a positive constant and is bounded in every finite interval.

From this result, an approximate formula for the number of solutions of

$$h = h_0 + rh_1 + r^2 h_2 + \dots \quad (r \geqslant 2 \text{ a constant integer})$$

in non-negative integers h_0, h_1, h_2, ... is derived, when h is large.

1. *Construction of a special solution.*

Let ω be an arbitrary real number different from zero, and let q be a number such that $0 < q < 1$. In this note we study those solutions of

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the functional equation

$$(1) \qquad \frac{f(z+\omega)-f(z)}{\omega}=f(qz)$$

which are positive for all sufficiently large positive values of z.

We first find a special analytic solution $f(z)$ of (1) and suppose that it can be written as

$$f(z)=\int_{-\infty}^{+\infty} u(x)\, e^{zq^{xi}}\, dx,$$

where $u(x)$ is an unknown function; under the integral sign

$$q^{xi}=e^{xi \log q}$$

with the real value of $\log q$. Then obviously

$$\frac{f(z+\omega)-f(z)}{\omega}=\int_{-\infty}^{+\infty} u(x)\,\frac{e^{\omega q^{xi}}-1}{\omega}\, e^{zq^{xi}}\, dx$$

and

$$f(qz)=\int_{-\infty}^{+\infty} u(x)\, e^{zq^{(x-i)i}}\, dx=\int_{-\infty}^{+\infty} u(x+i)\, e^{zq^{xi}}\, dx,$$

assuming that the path of integration can be moved parallel to itself a distance 1.

Hence we must determine $u(x)$ so that it satisfies the functional equation

$$(2) \qquad u(x+i)=u(x)\,\frac{e^{\omega q^{xi}}-1}{\omega},$$

i.e. the equation

$$u(x)=u(x-i)\, q^{(x-i)i}\,\frac{e^{\omega q^{(x-i)i}}-1}{\omega q^{(x-i)i}}.$$

Put
$$U(t)=\prod_{h=0}^{\infty}\frac{e^{q^h t}-1}{q^h t},$$

and
$$u(x)=U(\omega q^{(x-i)i})\, v(x).$$

Obviously $U(t)$ is an integral function of t and satisfies the equation

$$U(t)=\frac{e^t-1}{t}\, U(qt).$$

Hence $v(x)$ must satisfy the equation

$$v(x)=v(x-i)\, q^{(x-i)i}.$$

Now the special function

$$v(x) = c e^{\frac{1}{2}(\log q)(x-\frac{1}{2}i)^2},$$

where c is an arbitrary constant, has this property; therefore

$$u(x) = c e^{\frac{1}{2}(\log q)(x-\frac{1}{2}i)^2}\, U(\omega q^{(x-i)i})$$

satisfies (2). We replace x by $x+\frac{1}{2}i$ and put $c = \left(\frac{\log(1/q)}{2\pi}\right)^{\frac{1}{2}}$; then we obtain the special solution of the functional equation (1), namely

$$\Phi(z\,|\,\omega) = \sqrt{\left(\frac{\log(1/q)}{2\pi}\right)} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(\log q)x^2 + zq^{xi-\frac{1}{2}}}\, U(\omega q^{xi+\frac{1}{2}})\, dx. \tag{3}$$

Evidently $\Phi(z\,|\,\omega)$ is an integral function of z. It can now be easily verified that the transformations under the integral sign made in this construction of Φ are legitimate.

2. *The infinite series for* $\Phi(z\,|\,\omega)$.

For $\omega = 0$, $\Phi(z\,|\,\omega)$ becomes

$$F(z) = \sqrt{\left(\frac{\log(1/q)}{2\pi}\right)} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(\log q)x^2 + zq^{xi-\frac{1}{2}}}\, dx, \tag{4}$$

and it is easily proved that this new function satisfies the equation

$$F'(z) = F(qz). \tag{5}$$

From the integral,

$$F(0) = \sqrt{\left(\frac{\log(1/q)}{2\pi}\right)} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(\log q)x^2}\, dx = 1.$$

$F(z)$ is an integral function and can therefore be expanded as a power series

$$F(z) = \sum_{h=0}^{\infty} a_h z^h \quad (a_0 = 1).$$

By means of (5), the coefficients a_n are easily determined; the result is

$$F(z) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{z^n}{n!}. \tag{6}$$

We can express $\Phi(z\,|\,\omega)$ as a series in terms of $F(z)$. For write $U(t)$ as an infinite power series

$$U(t) = \sum_{h=0}^{\infty} A_h t^h,$$

where the first coefficients are

$$A_0 = 1,$$

$$A_1 = \frac{1}{2(1-q)},$$

$$A_2 = \frac{q}{4(1-q)(1-q^2)} + \frac{1}{6(1-q^2)},$$

$$A_3 = \frac{q^3}{8(1-q)(1-q^2)(1-q^3)} + \frac{q^2}{12(1-q^2)(1-q^3)} + \frac{q}{12(1-q)(1-q^3)} + \frac{1}{24(1-q^3)}.$$

On majorizing $U(t)$ by using $(e^t-1)/t \leqslant e^t$, we find

$$0 \leqslant A_h \leqslant \frac{1}{h!(1-q)^h}.$$

Substituting the series

$$U(\omega q^{xi+\frac{1}{2}}) = \sum_{h=0}^{\infty} A_h (\omega \sqrt{q})^h e^{hxi \log q},$$

which converges uniformly for all real values of x, under the integral sign in (3), we get

$$\Phi(z|\omega) = \sqrt{\left(\frac{\log(1/q)}{2\pi}\right)} \sum_{h=0}^{\infty} A_h(\omega\sqrt{q})^h \int_{-\infty}^{+\infty} {}^{\frac{1}{2}(\log q)\, x^2 + hxi \log q + zq^{xi-\frac{1}{2}}} dx.$$

Now

$$\frac{\log q}{2} x^2 + hxi \log q = \frac{\log q}{2} \{(x+hi)^2 + h^2\}.$$

Hence, if we replace x by $x-hi$ in the integral of index h,

$$\Phi(z|\omega) = \sqrt{\left(\frac{\log(1/q)}{2\pi}\right)} \sum_{h=0}^{\infty} A_h(\omega\sqrt{q})^h q^{\frac{1}{2}h^2} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(\log q)\, x^2 + zq^{ri+h+\frac{1}{2}}} dx,$$

and therefore

$$\Phi(z|\omega) = \sum_{h=0}^{\infty} A_h \omega^h q^{\frac{1}{2}\{h(h+1)\}} F(q^h z). \tag{7}$$

From this formula, we easily derive the power series

$$\Phi(z|\omega) = \sum_{n=0}^{\infty} \left(\sum_{h=0}^{\infty} A_h \omega^h q^{\frac{1}{2}h(h+1) + hn + \frac{1}{2}n(n-1)} \right) \frac{z^n}{n!}.$$

3. *The asymptotic behaviour of* $F(z)$ *and* $\Phi(z|\omega)$.

Suppose now that z is real and positive and tends to $+\infty$. Then, by means of (7), an asymptotic formula for $\Phi(z|\omega)$ can be obtained if one for $F(z)$ is already known. The latter we find in the following way:

Put

$$\vartheta(t)=\sum_{h=-\infty}^{+\infty} q^{\frac{1}{2}h^2}t^h, \tag{8}$$

and let n be the positive integer for which

$$q^{-(n-1)}n \leqslant z < q^{-n}(n+1); \tag{9}$$

thus
$$q^{\frac{1}{2}} \leqslant \frac{q^{n-\frac{1}{2}}z}{n} < q^{-\frac{1}{2}}\left(1+\frac{1}{n}\right) \leqslant 2q^{-\frac{1}{2}}.$$

Then $F(z)$ can be written as

$$F(z)=\frac{q^{\frac{1}{2}n(n-1)}z^n}{n!}\left\{\vartheta\left(\frac{q^{n-\frac{1}{2}}z}{n}\right)+r_n(z)\right\}, \tag{10}$$

with the error term

$$r_n(z)=-\sum_{k=n+1}^{\infty} q^{\frac{1}{2}k^2}\left(\frac{q^{n-\frac{1}{2}}z}{n}\right)^{-k}+\sum_{k=-n}^{\infty} q^{\frac{1}{2}k^2}\left(\frac{q^{n-\frac{1}{2}}z}{n}\right)^{k}\left(\frac{n!\,n^k}{(n+k)!}-1\right),$$

$$=r_n'(z)+r_n''(z),$$

say. Here

$$r_n'(z)=O\left\{\sum_{k=n+1}^{\infty} q^{\frac{1}{2}n^2+n(k-n)+\frac{1}{2}(k-n)^2}q^{-\frac{1}{2}n-\frac{1}{2}(k-n)}\right\}=O(q^{\frac{1}{2}(n^2-n)}).$$

Further

$$r_n''(z)=O\left\{\sum_{k=-n}^{\infty} q^{\frac{1}{2}k^2}(2q^{-\frac{1}{2}})^{|k|}\left(\frac{n!\,n^k}{(n+k)!}-1\right)\right\}.$$

Put

$$\lambda=\frac{n!\,n^k}{(n+k)!}=\begin{cases}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{k}{n}\right)\right\}^{-1} & \text{for } k>0,\\[2mm] \left\{\left(1+\frac{1}{n-1}\right)\left(1+\frac{2}{n-2}\right)\cdots\left(1+\frac{|k|-1}{n-|k|+1}\right)\right\}^{-1} & \text{for } k<0,\end{cases}$$

and, therefore, in both cases and also for $k=0$,

$$0<\lambda\leqslant 1,\quad |\lambda-1|\leqslant 1.$$

Also, since $e^x \geqslant 1+x$ for all real x,

$$\lambda \geqslant \begin{cases} e^{-\{(1/n)+(2/n)+\ldots+(k/n)\}} \geqslant e^{-k^2/n} \geqslant 1-\dfrac{k^2}{n} \geqslant 1-\dfrac{k^2}{n-|k|} & \text{for } k>0, \\ e^{-\{1/(n-1)+2/(n-2)+\ldots+(|k|-1)/(n-|k|+1)\}} \geqslant e^{-(|k|-1)^2/(n-|k|+1)} \geqslant 1-\dfrac{k^2}{n-|k|} & \text{for } k<0, \end{cases}$$

and $\lambda = 1$ for $k=0$. Hence for $|k| \leqslant n^{\frac13}$ and all sufficiently large z, and so n,

$$\left| \frac{n!\,n^k}{(n+k)!} - 1 \right| \leqslant 2n^{-\frac13}.$$

Write

$$\sum_{k=-n}^{\infty} q^{\frac12 k^2}(2q^{-\frac12})^{|k|}\left(\frac{n!\,n^k}{(n+k)!}-1\right) = \sum_{|k|\leqslant n^{\frac13}} + \sum_{|k|>n^{\frac13}} = \Sigma_1+\Sigma_2.$$

Then $$\Sigma_1 = O\left\{n^{-\frac13}\sum_{-\infty}^{+\infty} q^{\frac12 k^2}(2q^{-\frac12})^{|k|}\right\} = O\left(\frac{1}{n^{\frac13}}\right),$$

and

$$\Sigma_2 = O\left\{\sum_{k<n^{\frac13}} q^{\frac12 k^2}(2q^{-\frac12})^k\right\} = O\left\{q^{\frac12 n}\,(2q^{-\frac12})^n \sum_{l=0}^{\infty} q^{n^{\frac13} l}(2q^{-\frac12})^l\right\} = O(q^{\frac13 n^{\frac23}}).$$

Therefore, finally,

$$r_n'(z) = O(q^{\frac12 n(n-1)}) = o(1), \qquad r_n''(z) = O\left(\frac{1}{n^{\frac13}}\right)+O(q^{\frac13 n^{\frac23}}) = o(1),$$

and $$r_n(y) = r_n'(z)+r_n''(z) = o(1).$$

Hence, from (10),

(11) $$F(z) \sim \frac{q^{\frac12 n(n-1)}z^n}{n!}\,\vartheta\left(\frac{q^{n-\frac12}z}{n}\right),$$

where n is given by (9). Hence for large z

$$F(q^h z) = o\big(F(z)\big) \quad (h = 1, 2, 3, \ldots),$$

and therefore, by (7),

$$\Phi(z\,|\,\omega) \sim F(z),$$

so that also

(12) $$\Phi(z\,|\,\omega) \sim \frac{q^{\frac12 n(n-1)}z^n}{n!}\,\vartheta\left(\frac{q^{n-\frac12}z}{n}\right).$$

The last two formulae give as a first approximation

(13) $$\log F(z) \sim \log \Phi(z\,|\,\omega) \sim \frac{(\log z)^2}{2\log(1/q)}.$$

4. *The general solution of* $\{f(z+\omega)-f(z)\}/\omega = f(qz)$.

THEOREM. *Let $f(z)$ be a real function of the real variable $z \geqslant 0$ which in every finite interval is bounded, but not necessarily continuous, and which satisfies the equation*

$$\frac{f(z+\omega)-f(z)}{\omega} = f(qz).$$

If, as $z \to \infty$, n is the integer for which

$$q^{-(n-1)}n \leqslant z < q^{-n}(n+1),$$

then

$$f(z) = O\left(\frac{q^{\frac{1}{2}n(n-1)}z^n}{n!}\right).$$

This inequality can be improved to

$$f(z) = \frac{q^{\frac{1}{2}n(n-1)}z^n}{n!}e^{O(1)},$$

if $f(z)$ is greater than a positive constant C for all sufficiently large $z \geqslant 0$.

Proof. By (12), there exists a positive number ζ such that

$$\Phi(z\,|\,\omega) > 0 \quad \text{for} \quad z \geqslant \zeta.$$

Put

$$\zeta' = \max\left(\zeta + |\omega|, \frac{\zeta}{q}\right).$$

Then there is a positive constant c such that

$$|f(z)| \leqslant c\Phi(z\,|\,\omega) \quad \text{for} \quad \zeta \leqslant z \leqslant \zeta';$$

and if $f(z) \geqslant C$ in this whole interval, there is a second positive constant c' such that

$$f(z) \geqslant c'\Phi(z\,|\,\omega) \quad \text{for} \quad \zeta \leqslant z \leqslant \zeta'.$$

Put

$$g_\eta(z) = c\Phi(z\,|\,\omega) - \eta f(z) \quad (\eta = \mp 1),$$

and in the special case when $f(z) \geqslant C$ for large $z \geqslant 0$, put

$$g^*(z) = f(z) - c'\Phi(z\,|\,\omega).$$

Both functions $h(z) = g_\eta(z)$ and $h(z) = g^*(z)$ are non-negative in $\zeta \leqslant z \leqslant \zeta'$ and satisfy the functional equation

$$\frac{h(z+\omega)-h(z)}{\omega} = h(qz),$$

i.e.
$$h(z+\omega) = h(z)+\omega h(qz) \qquad (\omega > 0),$$
$$h(z) = h(z-|\omega|)+|\omega|h(qz) \quad (\omega < 0).$$

Hence, given any number $z \geqslant \zeta'$, there exist an index r, and r numbers $z_1, z_2, \ldots, z_r$ in the closed interval (ζ, ζ') and r positive numbers $a_1, a_2, \ldots, a_r$, such that
$$h(z) = a_1 h(z_1)+a_2 h(z_2)+\ldots+a_r h(z_r).$$

Therefore
$$h(z) \geqslant 0 \quad \text{for} \quad z \geqslant \zeta,$$
and the theorem follows at once, since in (12)
$$\vartheta\left(\frac{q^{n-\frac{1}{2}}z}{n}\right) = e^{O(1)}.$$

The special function
$$f(z) = \Phi(z|\omega)$$
shows that the error factor $e^{O(1)}$ does not necessarily tend to a limit.

5. *An application to a special partition problem.*

It is easily verified that, if $r \geqslant 2$ is an integer, and
$$\prod_{h=0}^{\infty} (1-z^{r^h})^{-1} = \sum_{h=0}^{\infty} C_h t^h,$$
then
$$C_{hr} = C_{hr+1} = \ldots = C_{hr+r-1}, \qquad C_{hr} = C_{(h-1)r}+C_{[h/r]r};$$
obviously C_h is the number of solutions of
$$h = h_0+rh_1+r^2 h_2+r^3 h_3+\ldots$$
in non-negative integers $h_0, h_1, h_2, \ldots$.

Let us define a function of the variable $z \geqslant 0$ by
$$f(z) = C_{hr} \quad \text{for} \quad h \leqslant z < h+1.$$

Then $f(z) > 0$ for all $z \geqslant 0$, and
$$\frac{f(z-1)-f(z)}{-1} = f(z)-f(z-1) = f\left(\frac{z}{r}\right).$$

Hence, by the theorem of § 4 for large z,
$$f(z) = \frac{r^{-\frac{1}{2}n(n-1)} z^n}{n!} e^{O(1)},$$

where n is the integer for which

$$r^{n-1}n \leqslant z < r^n(n+1).$$

This gives for integral z an asymptotic formula for C_h; the first term of this formula is

$$\log C_h \sim \frac{(\log h)^2}{2\log r}.$$

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A CHARACTERIZATION OF SCHOUTEN'S AND HAYDEN'S DEFORMATION METHODS

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1. *Introduction.*

In this note we give a geometric interpretation of both Schouten's method of studying deformations (known as the absolute variation, **1**) and Hayden's method (known as either the direction deformation, **2**, or the apparent differential, **3**). We use the terms "variation" and "deformation" interchangeably. Our discussion reveals that both Schouten's and Hayden's deformation methods can be characterized in a very simple manner.

We consider a manifold L_n of n dimensions with a general connection. In this manifold, we imbed a submanifold X_m of m dimensions. By use of the projection factors, **4**, a connection is induced in this submanifold and hence it can be denoted by L_m. Furthermore, attached to each point of L_n is a local affine space E_n. In each such E_n, there exist three types of vectors which can be distinguished under deformation: (1) the measure vectors; (2) the coordinate differential vectors or linear elements; (3) all other vectors.

Our *fundamental idea* is that the *deformation operator* $(\overset{v}{\delta})$, which has all the operator properties of the covariant differential operator (δ), *acts only on E_n quantities.* From this idea, we are easily led to the characterization

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