ON A PROPERTY OF POSITIVE DEFINITE TERNARY QUADRATIC FORMS.

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Recently, H. Davenport gave a very elegant new proof§ of Remak's theorem on the product of three linear polynomials||. Davenport's proof (like Remak's original proof) is based on the following lemma, which presents the chief difficulty:

LEMMA A. *Let $f(x, y, z)$ be a positive definite ternary quadratic form of determinant 1 which assumes its minimum in three linearly independent lattice points. Then, given any three real numbers x_0, y_0, z_0, there are three integers x^1, y^1, z^1 such that*

$$f(x_0+x^1,\ y_0+y^1,\ z_0+z^1) \leqslant \tfrac{3}{4},$$

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§ *Journal London Math. Soc.*, 14 (1939), 47–51.

|| *Math. Zeitschrift*, 17 (1923), 1–34 and 18 (1923), 173–200.

with equality if and only if $f(x, y, z) = x^2+y^2+z^2$ *and* $2x_0$, $2y_0$, $2z_0$ *are odd integers.*

Davenport derived this lemma from the theorem of Korkine and Zolotareff on the minimum of quaternary quadratic forms†. While this method is extremely elegant, it does not seem to lend itself to generalizations. I therefore give in this paper another proof of the lemma by a method which I have already applied to Hermitian forms‡ and which may also be applied to other problems. The method is geometrical and very simple in its idea. It easily leads to an upper bound for the expression $f(x_0+x^1, y_0+y^1, z_0+z^1)$ in the form of a certain algebraic function of two parameters. That this expression is not larger than $\frac{3}{4}$ can be proved by elementary calculations.

I am indebted to Dr. Davenport for the proof given in § 7, which he supplied at my request while I was interned.

1. *The problem.*

As usual, we describe two real numbers x_0 and x_1 as congruent modulo 1, in symbols

$$x_0 \equiv x_1,$$

when the difference $x_0 - x_1$ is an integer.

In the following paragraphs we prove

LEMMA B. *Let*

$$f(x, y, z) = a(x^2+y^2+z^2)+2bxy+2cxz+2dyz \tag{1}$$

be a positive definite ternary quadratic form of determinant 1 *with coefficients satisfying the inequalities*

$$1 \leqslant a \leqslant \sqrt[3]{2}, \quad 0 \leqslant b \leqslant \tfrac{1}{2}a. \tag{2}$$

Then, given any three real numbers x_0, y_0, z_0, *there are three real numbers* x_1, y_1, z_1, *such that*

$$x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad z_1 \equiv z_0, \quad f(x_1, y_1, z_1) \leqslant \tfrac{3}{4}, \tag{3}$$

with equality if and only if

$$x_0 \equiv y_0 \equiv z_0 \equiv \tfrac{1}{2}, \quad f(x, y, z) = x^2+y^2+z^2.$$

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† See, for example, Bachmann, *Die Arithmetik der quadratischen Formen*, zweite Abteilung (Leipzig, 1923), 270.

‡ *Journal London Math. Soc.*, 15 (1940), 213–236.

Lemma A is contained in this theorem. For the form

$$f(x,\ y,\ z) = \sum_{h,\ k=1}^{3} a_{hk} x_h x_k \quad (a_{hk} = a_{kh};\ x_1 = x,\ x_2 = y,\ x_3 = z)$$

in Lemma A may evidently be replaced by any equivalent form, and therefore may be assumed to be reduced in the sense of Minkowski†. Its coefficients then satisfy the inequalities

$$0 < a_{11} \leqslant a_{22} \leqslant a_{33}, \quad 0 \leqslant a_{23} \leqslant \tfrac{1}{2} a_{22}, \quad |a_{13}| \leqslant \tfrac{1}{2} a_{11}, \quad 0 \leqslant a_{12} \leqslant \tfrac{1}{2} a_{11},$$

$$a_{23} - a_{13} + a_{12} \leqslant \tfrac{1}{2}(a_{11} + a_{22}), \quad 1 \leqslant a_{11} a_{22} a_{33} \leqslant 2.$$

By hypothesis, $f(x, y, z)$ assumes its minimum a_{11} in three linearly independent lattice points. Therefore

$$a_{11} = a_{22} = a_{33}, \quad i.e., \quad 1 \leqslant a_{11}^3 \leqslant 2,$$

for a_{11}, a_{22}, a_{33} are the first three consecutive minima of $f(x, y, z)$ in all sets of three linearly independent lattice points†. Hence the inequalities for the coefficients of $f(x, y, z)$ include the conditions

$$a_{11} = a_{22} = a_{33}, \quad 1 \leqslant a_{11} \leqslant \sqrt[3]{2}, \quad 0 \leqslant a_{12} \leqslant \tfrac{1}{2} a_{11},$$

and so $f(x, y, z)$ is of the form required in Lemma B. Lemma A therefore follows at once when Lemma B has been proved.

For the proof of Lemma B, we put

(4) $$F(x, y) = f(x, y, 0) = a(x^2 + y^2) + 2bxy,$$

and write $f(x, y, z)$ as

(5) $$f(x, y, z) = F(x + \lambda z,\ y + \mu z) + \frac{z^2}{a^2 - b^2},$$

where $$\lambda = \frac{ac - bd}{a^2 - b^2}, \quad \mu = \frac{ad - bc}{a^2 - b^2}.$$

By comparing the coefficients of z^2 in (1) and (5), we obtain the equation

(6) $$F(\lambda, \mu) = a - \frac{1}{a^2 - b^2}.$$

† Bachmann, *loc. cit.*, 279.

It is this equation and not the actual value of λ or μ which enters into the proof of Lemma B.

Obviously, the three numbers x_0, y_0, z_0 may be replaced by any three numbers x_0^1, y_0^1, z_0^1 congruent to them or congruent to $-x_0$, $-y_0$, $-z_0$. Hence it is permissible to assume that

(7) $$-\tfrac{1}{2} \leqslant z_0 \leqslant 0,$$

and that, with this fixed value of z_0,

$$F(x_0+\lambda z_0,\ y_0+\mu z_0) \leqslant F(x_1+\lambda z_0,\ y_1+\mu z_0) \text{ for all } x_1 \equiv x_0 \text{ and all } y_1 \equiv y_0.$$

Put

(8) $$\xi_0 = x_0+\lambda z_0, \quad \eta_0 = y_0+\mu z_0;$$

then the last inequality may be written as

(9) $$F(\xi_0, \eta_0) \leqslant F(\xi_1, \eta_1) \text{ for all } \xi_1 \equiv \xi_0 \text{ and } \eta_1 \equiv \eta_0.$$

2. *The hexagon net.*

We use a geometrical representation of $F(x, y)$ which goes back to Gauss.

Draw two lines inclined at an angle ϕ through an arbitrary origin O in the plane Π such that

$$\cos\phi = b/a, \quad 0 < \phi \leqslant \tfrac{1}{2}\pi.$$

Take these lines as oblique coordinate axes, and represent the pair of real numbers x, y by the point P with coordinates (x, y) measured on such a scale that both the point $(1, 0)$ on the x-axis and the point $(0, 1)$ on the y-axis are at distance $\sqrt{a}$ from the origin O. Then, as is easily verified, the distance of P from O is

$$\sqrt{\{F(x, y)\}} = \sqrt{\{a(x^2+y^2)+2bxy\}}.$$

The lattice points P_{gh} with integral coordinates $x = g$, $y = h$ form a parallelogram lattice in the plane Π. Let H_{gh} be the set of all points (x, y) in Π whose distance from the lattice point P_{gh} is not greater than that from any other lattice point. All sets H_{gh} are congruent to the special one H_{00} which is defined by the inequalities

$$F(x, y) \leqslant \min\Big(F(1-x, -y),\ F(-x, 1-y),\ F(-1-x, 1-y),$$
$$F(-1-x, -y),\ F(-x, -1-y),\ F(1-x, -1-y)\Big),$$

i.e., by (4),

(10) $$\left|x+\frac{b}{a}y\right|\leqslant\tfrac{1}{2},\quad \left|\frac{b}{a}x+y\right|\leqslant\tfrac{1}{2},\quad |x-y|\leqslant 1.$$

Hence H_{00} is a hexagon with its vertices at the points

$$\mp\left(\frac{a}{2(a+b)},\ \frac{a}{2(a+b)}\right),\qquad \mp\left(\frac{-a}{2(a+b)},\ \frac{a+2b}{2(a+b)}\right),\qquad \mp\left(\frac{-a-2b}{2(a+b)},\ \frac{a}{2(a+b)}\right);$$

it degenerates into the square $|x|\leqslant\frac{1}{2}$, $|y|\leqslant\frac{1}{2}$ for $b=0$. All six (or four) vertices of H_{00} are at the same distance

$$\rho=\sqrt{\left(\frac{a^2}{2(a+b)}\right)}$$

from the origin; hence, for all points (x, y) in H_{00},

(11) $$F(x, y)\leqslant\rho^2=\frac{a^2}{2(a+b)}.$$

By the translation

$$x\to x+g,\quad y\to y+h,$$

H_{00} is changed into H_{gh}. No two different hexagons H_{gh} have inner points in common, and the set of all hexagons H_{gh} therefore covers the whole plane Π without overlapping. Hence it follows that to every point (x, y) there is a point (x^1, y^1) in H_{00} for which $x^1\equiv x$, $y^1\equiv y$; there is only one such point if (x, y) is an inner point of one of the hexagons. Let Γ be the graph consisting of all points in the plane Π which lie on the boundary of at least one hexagon H_{gh}.

3. *The case of large b.*

It is evident from the inequalities (9) for ξ_0 and η_0, that (ξ_0, η_0) is a point of the hexagon H_{00}. Hence, by (11),

(12) $$F(\xi_0, \eta_0)\leqslant\frac{a^2}{2(a+b)},$$

and therefore, by (8) and the identity (5),

(13) $$f(x_0, y_0, z_0)\leqslant\frac{a^2}{2(a+b)}+\frac{1}{4(a^2-b^2)}.$$

If in this inequality the right-hand side is *less than* $\frac{3}{4}$, then Lemma B is already proved. Now

$$\frac{a^2}{2(a+b)}+\frac{1}{4(a^2-b^2)}<\tfrac{3}{4}$$

if
$$3(a^2-b^2) > 2a^2(a-b)+1,$$
i.e.
$$(3b-a^2)^2 < a^4-3(a-1)^2(2a+1),$$
and therefore

(14) $\frac{1}{3}[a^2-\sqrt{\{a^4-3(a-1)^2(2a+1)\}}] < b < \frac{1}{3}[a^2+\sqrt{\{a^4-3(a-1)^2(2a+1)\}}].$

In this inequality the right-hand side is greater than $\frac{1}{2}a$. For $1 \leqslant a \leqslant \sqrt[3]{2}$, and therefore
$$3(a-1)^2\,(2a+1) \leqslant 3(\sqrt[3]{2}-1)^2(2\sqrt[3]{2}+1) = 15-9\sqrt[3]{4} < \tfrac{3}{4} \leqslant \tfrac{3}{4}a^4;$$
hence
$$\tfrac{1}{3}[a^2+\sqrt{\{a^4-3(a-1)^2\,(2a+1)\}}] > \tfrac{1}{3}\{a^2+\sqrt{(a^4-\tfrac{3}{4}a^4)}\} = \tfrac{1}{2}a^2 \geqslant \tfrac{1}{2}a.$$
By the second formula (2), the inequality (14) is therefore identical with
$$\tfrac{1}{3}[a^2-\sqrt{\{a^4-3(a-1)^2\,(2a+1)\}}] < b \leqslant \tfrac{1}{2}a,$$
and, when these inequalities hold, Lemma B is proved with the sign "$<$" instead of "$\leqslant$".

Hence we may suppose from now onwards that

(15) $$0 \leqslant b \leqslant \tfrac{1}{3}(a^2-\sqrt{\{a^4-3(a-1)^2\,(2a+1)\}}).$$

It is useful to remark that the expression on the right-hand side is
$$\tfrac{1}{3}[a^2-\sqrt{\{a^4-3(a-1)^2\,(2a+1)\}}]$$
$$= \frac{(a-1)^2(2a+1)}{a^2+\sqrt{\{a^4-3(a-1)^2\,(2a+1)\}}} \leqslant \frac{(a-1)^2(2a+1)}{\frac{3}{2}a^2} \leqslant 2(a-1)^2,$$
and so b also satisfies the weaker conditions

(16) $$0 \leqslant b \leqslant 2(a-1)^2.$$

As a consequence of (15) or (16), the right-hand side of (13) is not less than $\frac{3}{4}$. We may assume that even the stronger inequality
$$f(x_0, y_0, z_0) = F(\xi_0, \eta_0)+\frac{{z_0}^2}{a^2-b^2} \geqslant \tfrac{3}{4}$$
holds, since otherwise Lemma B would again be true with the sign "$<$" instead of "$\leqslant$". From (7) and (12), we get the further inequalities

(17) $$F(\xi_0, \eta_0) \geqslant \tfrac{3}{4}-\frac{1}{4(a^2-b^2)}$$

and

$$z_0^2 \geqslant (a^2-b^2)\left(\tfrac{3}{4}-\frac{a^2}{2(a+b)}\right). \tag{18}$$

The formulae (15)–(18) form the basis of the following considerations.

4. *A geometrical maximum problem.*

Let us solve the following extremal problem:

Let $P(x, y)$ and $P'(x', y')$ be any two points in the plane Π such that the vector of components $x'-x$, $y'-y$ connecting P with P' has length

$$l=\sqrt{\left(a-\frac{1}{a^2-b^2}\right)},$$

and that the distance of P from the nearest lattice point is not less than

$$r=\tfrac{1}{2}\sqrt{\left(3-\frac{1}{a^2-b^2}\right)}.$$

To find the maximum $\Delta(a, b)$ of the distance δ of P' from the nearest lattice point.

If $a=1$ then, by (19) below, $b=0$, $l=0$, and $r=\sqrt{\tfrac{1}{2}}$. The hexagon H_{gh} is now a square of side 1 and diagonal $\sqrt{2}$, so that both P and P' fall at the centre of one of the squares. Hence

$$\Delta(1, 0)=\sqrt{\tfrac{1}{2}}.$$

From now on, we suppose that $1<a\leqslant\sqrt[3]{2}$. Then, first, the distance r is not greater than the radius

$$\rho=\sqrt{\left(\frac{a^2}{2(a+b)}\right)}$$

of the circumscribed circle of H_{gh} (§ 2), for, by (15),

$$\rho^2-r^2=\frac{(a^2-3b)^2-\{a^4-3(a-1)^2(2a+1)\}}{12(a^2-b^2)}\geqslant 0.$$

Secondly, the difference

$$[\sqrt{\{\tfrac{1}{2}(a-b)\}}]^2-r^2=\frac{(a^2-3b)^2-\{a^4-3(a-1)^2(2a+1)\}-2(a-b)\,b^2}{12(a^2-b^2)}$$

steadily decreases from positive to negative values, if b runs from the left to the right of the interval (15). Thirdly,

$$r^2-\tfrac{1}{4}a\geqslant\tfrac{1}{4}>0,$$

since $a \leqslant \sqrt[3]{2} < \frac{4}{3}$; hence, by (16),

$$r^2 - \frac{a+1}{4} = \frac{(a-1)(1+a-a^2)-(2-a)b^2}{4(a^2-b^2)}$$

$$\geqslant \frac{(a-1)\{1+a-a^2-4(2-a)(a-1)^3\}}{4(a^2-b^2)},$$

and therefore

$$r^2 - \frac{a+1}{4} \geqslant \frac{a-1}{4(a^2-b^2)}\left(1+\tfrac{4}{3}-(\tfrac{4}{3})^2-4\,.\,1\,.\,(\tfrac{1}{3})^3\right) > 0.$$

Consider the hexagon H_{gh} with centre at the lattice point (g, h). It has four long sides L of length $\sqrt{\left(a\,\frac{a-b}{a+b}\right)}$ and distance $\frac{\sqrt{a}}{2}$ from (g, h), and two short sides S of length $b\sqrt{\left(\frac{2}{a+b}\right)}$ and distance $\sqrt{\left(\frac{a-b}{2}\right)}$ from (g, h). These sides follow each other in the sequence $L_1 L_2 S_3 L_1 L_2 S_3$ on the boundary of H_{gh}, where parallel sides have the same index. In the graph Γ consisting of the sides of all H_{gh} (§ 2), there are two long sides and one short side radiating from each of its vertices (the short sides degenerate into points for $b = 0$; it seems unnecessary to mention this trivial case in what follows).

Let us now draw with each lattice point (g, h) as centre a circle of radius r. By the last inequalities, this circle C_{gh} intersects the long sides of H_{gh} (not their end points, unless b has its largest value

$$\tfrac{1}{3}[a^2-\sqrt{\{a^4-3(a-1)^2(2a+1)\}}]),$$

since

$$\tfrac{1}{2}\sqrt{a} < r \leqslant \rho.$$

It depends, however, on b whether the circle has or has not points in common with the short sides of this hexagon.

We denote by Π^* the set of all points of the plane Π which are not inner points of any of the circles C_{gh}. This set consists of an infinity of separate parts, one belonging to each parallelogram of the lattice and all lying in congruent positions. It suffices therefore to consider only that part which lies in the parallelogram Ω with vertices

$$(0,\ 0),\quad (1,\ 0),\quad (1,\ 1),\quad (0,\ 1).$$

There is exactly one short side S in Ω with vertices at

$$Q:\left(\frac{a}{2(a+b)},\ \frac{a}{2(a+b)}\right) \quad \text{and} \quad Q':\left(\frac{a-2b}{2(a+b)},\ \frac{a-2b}{2(a+b)}\right)$$

and satisfying the equation

$$y-x=0.$$

The equations of the long sides radiating from Q and Q' are

$$L_1: \qquad x+\frac{b}{a}\,y=\tfrac{1}{2},$$

$$L_2: \ \frac{b}{a}\,(x-1)+y=\tfrac{1}{2},$$

$$L_3: \ x+\frac{b}{a}\,(y-1)=\tfrac{1}{2},$$

$$L_4: \qquad \frac{b}{a}\,x+y=\tfrac{1}{2};$$

exactly one half of each side lies in Ω.

Put $$\tau=\frac{\sqrt{[a\{(3-a)(a^2-b^2)-1\}]}}{2(a^2-b^2)}.$$

Then the four points

$$p_1:\ \left(\tfrac{1}{2}-\frac{b}{a}\,\tau,\ \tau\right); \qquad p_2:\ \left(1-\tau,\ \tfrac{1}{2}+\frac{b}{a}\,\tau\right);$$

$$p_3:\ \left(\tfrac{1}{2}+\frac{b}{a}\,\tau,\ 1-\tau\right), \qquad p_4:\ \left(\tau,\ \tfrac{1}{2}-\frac{b}{a}\,\tau\right),$$

belong to Ω, and each point p_k lies on the line L_k with the same index. The only points of intersection in Ω of

$$\text{the circle}\ \begin{Bmatrix} C_{00} \\ C_{10} \\ C_{11} \\ C_{01} \end{Bmatrix}\ \text{with the long sides of}\ \begin{Bmatrix} H_{00} \\ H_{10} \\ H_{11} \\ H_{01} \end{Bmatrix}\ \text{are}\ \begin{Bmatrix} p_4 \text{ and } p_1, \\ p_1 \text{ and } p_2, \\ p_2 \text{ and } p_3, \\ p_3 \text{ and } p_4. \end{Bmatrix}$$

The circles C_{10} and C_{01} may or may not intersect the short side S which is common to H_{10} and H_{01}; this depends on b.

The part $\Pi_0{}^*$ or Π^* lying in Ω consists now of all those points in Ω which lie outside or on the boundary of the four circles. Therefore $\Pi_0{}^*$ is a curvilinear parallelogram with vertices at p_1, p_2, p_3, p_4 if the circles C_{10} and C_{01} do not meet the short side S. If, however, they intersect this line at, say the two points q_1 and q_2 (which may coincide), then $\Pi_0{}^*$ consists of two identical curvilinear triangles, one with its vertices at p_1, p_4, q_1, and the other one at p_2, p_3, q_2. In both cases the greatest distance between any two points in Π_0^* is evidently that between two opposite vertices like p_1 and p_3. Now, by § 2, the distance between two arbitrary points (x_1, y_1) and (x_2, y_2) is equal to

$$\sqrt{\{F(x_1-x_2,\ y_1-y_2)\}}.$$

Hence the maximum distance between two points in Π_0^* (namely that from p_1 to p_3) has the value

$$(19)\qquad \sigma = \sqrt{F\left(-2\frac{b}{a}\tau,\ 2\tau-1\right)} = \sqrt{\left(\frac{4(a^2-b^2)(\tau^2-\tau)+a^2}{a}\right)}.$$

I now prove that

$$(20)\qquad \sigma < l;$$

it is therefore impossible for both P and P' to lie in Π_0^*. In order to prove this inequality, we start from the formula

$$(21)\qquad (a-1)(-5a^2+13a+4)-8b^2 > 0,$$

which is true since $1 < a < \frac{4}{3}$, $0 \leqslant b \leqslant 2(a-1)^2$, $b^2 \leqslant 4\,.\,(\frac{1}{3})^3\,(a-1)$ and therefore

$$(a-1)(-5a^2+13a+4) \geqslant \left(-5\,.\,(\tfrac{4}{3})^2+13+4\right)(a-1)$$

$$> 7(a-1) > 8\,.\,4\,.\,(\tfrac{1}{3})^3(a-1) \geqslant 8b^2.$$

The inequality (21) implies

$$-\tau < -\frac{a(3-a)}{4(a^2-b^2)}.$$

The last inequality changes into the square of (20), if τ^2 is added on the left-hand side, its expression in a and b on the right-hand side, both sides are then multiplied by $4(a^2-b^2)/a$, and finally a is added.

We next determine the shortest distance between a point in Π_0^* and a point in any other part of Π^*. For reasons of symmetry, this minimum distance is equal to that of the point p_1 from the point

$$p_3':\ \left(\tfrac{1}{2}+\frac{b}{a}\tau,\ -\tau\right),$$

which lies symmetrically to p_1 with respect to the x-axis, and can be derived from p_3 by the lattice translation

$$x \to x,\quad y \to y-1.$$

Hence this distance

$$\sigma^* = \sqrt{\left\{F\left(-2\frac{b}{a}\tau,\ 2\tau\right)\right\}} = \sqrt{\left(\frac{(3-a)(a^2-b^2)-1}{a^2-b^2}\right)},$$

so that

$$\sigma^{*2}-1 = \frac{(a-1)(1+a-a^2)-(2-a)b^2}{a^2-b^2} = 4r^2-a-1 \geqslant 0, \quad \sigma^* \geqslant 1,$$

by the inequality proved at the beginning of this paragraph.

On the other hand

$$l = \sqrt{\left(a-\frac{1}{a^2-b^2}\right)} \leqslant \sqrt{\left(a-\frac{1}{a^2}\right)} \leqslant \sqrt{\left(\sqrt[3]{2}-\frac{1}{(\sqrt[3]{2})^2}\right)} = \frac{1}{\sqrt[3]{2}} < 1.$$

Hence we have proved that, *under the conditions of the problem, the second point P' cannot lie in the set Π^*, if $a > 1$.*

5. *Solution of the geometrical maximum problem.*

It is now easy to determine Δ (a, b) for $a > 1$. Let P again be a point in Π_0^*. Then, since $l \leqslant 1/\sqrt[3]{2}$, the second point P' can lie only in one of the four hexagons H_{00}, H_{10}, H_{11}, H_{01} which contribute to Π_0^*. Fix P for the moment anywhere in Π_0^*, and assume that under this restriction P' has such a position that its distance from the nearest lattice point, which is necessarily one of the four points

(22) $$(0,\ 0),\quad (1,\ 0),\quad (1,\ 1),\quad (0,\ 1),$$

is a maximum. *Then P' lies on one of the long sides L_1, L_2, L_3, L_4* which radiate from the vertices p_1, p_2, p_3, p_4 of Π_0^*; for otherwise P' is nearer to one of the four points (22) than to the other three, and it is therefore possible to move it a little on the circle of radius l and centre P so that its distance from this nearest lattice-point increases while still remaining smaller than its distances from the other three lattice points.

For reasons of symmetry, *we may assume that P' lies on the long side L_1*; P' has therefore the same distance $\delta = \delta(P')$ from the two lattice points (0, 0) and (1, 0) which are nearest to it. We have to find that position of P in Π_0^* for which $\delta(P')$ becomes a maximum. Now $\delta(P')$ increases when the distance of P' from the x-axis increases. Let P' fall into the point P_1' on L_1 if P in Π_0^* coincides with p_1, and into the point P_3' if P coincides with p_3. Then, for variable P in Π_0^*, P' can be any point in the closed interval from P_1' to P_3', but it cannot lie outside this interval.

Hence finally

$$\Delta(a, b) = \max\big(\delta(P_1'), \delta(P_3')\big).$$

The values of $\delta(P_1')$ and $\delta(P_3')$ are easily determined. Both points P_1' and P_3' lie on the line perpendicular to the x-axis,

$$L_1:\ x+\frac{b}{a}\,y = \tfrac{1}{2},$$

and belong to the hexagon H_{00}; the first one has the distance l from p_1, and the second one has this distance from p_3. The point p_1 has the distance

$$\tfrac{1}{2}\sigma^* = \sqrt{(r^2-\tfrac{1}{4}a)}$$

from the x-axis, the point P_1', therefore, the distance

$$|\sqrt{(r^2-\tfrac{1}{4}a-l)}|,$$

since both lie on L_1. Since the distance of this line from the origin (0, 0) is $\frac{1}{2}\sqrt{a}$, we find that

$$\delta(P_1')^2 = \{\sqrt{(r^2-\tfrac{1}{4}a)}-l\}^2+(\tfrac{1}{2}\sqrt{a})^2. \tag{23}$$

The point p_3 has the same distance $\frac{1}{2}\sigma^*$ from the line $y=1$ as p_1 has from $y=0$. Now the distance between the two lines $y=0$ and $y=1$ is equal to $d = \sqrt{\left(\frac{a^2-b^2}{a}\right)}$; for L_1 is perpendicular to both, intersects them in the points $(\frac{1}{2}, 0)$ and $(\frac{1}{2}-b/a, 1)$, and the distance between these points is

$$\sqrt{\left\{F\left(\frac{b}{a}, -1\right)\right\}} = \sqrt{\left(\frac{a^2-b^2}{a}\right)}.$$

Hence the perpendicular from p_3 to the x-axis has length

$$d-\tfrac{1}{2}\sigma^* = \sqrt{\left(\frac{a^2-b^2}{a}\right)}-\sqrt{\left(r^2-\frac{a}{4}\right)}.$$

Now p_3 lies on the line L_3, $x+(b/a)(y-1) = \frac{1}{2}$, and P_3' on L_1. The distance apart of these two parallel lines is $b/\sqrt{a}$; for both are perpendicular to the x-axis and intersect it in the two points $(\frac{1}{2}, 0)$ and $\{\frac{1}{2}+(b/a), 0\}$, whose distance apart is

$$\sqrt{\left\{F\left(-\frac{b}{a}, 0\right)\right\}} = \frac{b}{\sqrt{a}}.$$

The point P_3' has the distance l from p_3, has a smaller ordinate, and lies on L_1. Its distance from the x-axis has therefore the value

$$\left|\sqrt{\left(\frac{a^2-b^2}{a}\right)}-\sqrt{\left(r^2-\frac{a}{4}\right)}-\sqrt{\left(l^2-\frac{b^2}{a}\right)}\right|,$$

and we find that

$$\delta(P_3')^2=\left\{\sqrt{\left(\frac{a^2-b^2}{a}\right)}-\sqrt{\left(r^2-\frac{a}{4}\right)}-\sqrt{\left(l^2-\frac{b^2}{a}\right)}\right\}^2+\left(\frac{\sqrt{a}}{2}\right)^2. \tag{24}$$

We substitute the values of r and l as functions of a and b in (23) and (24), and so get finally

$$\Delta(a,\, b)=\max(\delta_1,\, \delta_2), \tag{25}$$

where

$$\begin{cases}\delta_1^2=\left\{\tfrac{1}{2}\sqrt{\left(3-a-\frac{1}{a^2-b^2}\right)}-\sqrt{\left(a-\frac{1}{a^2-b^2}\right)}\right\}^2+\frac{a}{4}, \\ \delta_2^2=\left\{\sqrt{\left(\frac{a^2-b^2}{a}\right)}-\tfrac{1}{2}\sqrt{\left(3-a-\frac{1}{a^2-b^2}\right)}-\sqrt{\left(\frac{a^2-b^2}{a}-\frac{1}{a^2-b^2}\right)}\right\}^2+\frac{a}{4}.\end{cases}$$

Which of the two numbers δ_1 and δ_2 is the larger depends on a and b. The formula (25) holds also in the limiting case $a=1$, $b=0$, for then it gives $\Delta(1,\, 0)=\sqrt{\tfrac{1}{2}}$, as we found before.

6. *An upper bound for $f(x_1,\, y_1,\, z_1)$.*

By means of the results in § 3 and in the last paragraph, we now easily find an upper bound for $f(x_1,\, y_1,\, z_1)$, if $x_1,\, y_1,\, z_1$ are suitably chosen.

Put
$$z_1=z_0+1,$$

so that $z_1\equiv z_0$. By (7) and (18),

$$0\leqslant z_1\leqslant 1-\sqrt{\left\{(a^2-b^2)\left(\tfrac{3}{4}-\frac{a^2}{2(a+b)}\right)\right\}},$$

and therefore

$$\frac{z_1^2}{a^2-b^2}\leqslant\frac{1}{a^2-b^2}\left[1-\sqrt{\left\{(a^2-b^2)\left(\tfrac{3}{4}-\frac{a^2}{2(a+b)}\right)\right\}}\right]^2. \tag{26}$$

If x_0 and y_0 remain unaltered, while z_0 is replaced by z_1, then the numbers $\xi_0 = x_0 + \lambda z_0$ and $\eta_0 = y_0 + \mu z_0$ change into

$$\xi_0{}^1 = x_0 + \lambda z_1 = \xi_0 + \lambda \quad \text{and} \quad \eta_0{}^1 = y_0 + \mu z_1 = \eta_0 + \mu.$$

By (6), the distance between the two points (ξ_0, η_0) and $(\xi_0{}^1, \eta_0{}^1)$ in Π is

$$\sqrt{\{F(\xi_0{}^1 - \xi_0, \; \eta_0{}^1 - \eta_0)\}} = \sqrt{\{F(\lambda, \; \mu)\}} = l.$$

The point (ξ_0, η_0) lies in H_{00} and satisfies (17). Its distance from the nearest lattice point (namely the origin) is therefore not less than

$$r = \tfrac{1}{2}\sqrt{\left(3 - \frac{1}{a^2 - b^2}\right)},$$

and it follows that (ξ_0, η_0) belongs to the set Π^*. Hence, by the last two paragraphs, there exists a lattice point (g, h) such that the distance between $(\xi_0{}^1, \eta_0{}^1)$ and (g, h) does not exceed the number $\Delta\,(a, b)$ defined in (25); in symbols,

$$\sqrt{\{F(\xi_0{}^1 - g, \; \eta_0{}^1 - h)\}} \leqslant \Delta\,(a, \; b).$$

If

$$x_1 = x_0 - g, \quad y_1 = y_0 - h,$$

and

$$\xi_1 = x_1 + \lambda z_1 = \xi_0{}^1 - g, \quad \eta_1 = y_1 + \mu z_1 = \eta_0{}^1 - h,$$

we have

$$F(\xi_1, \; \eta_1) \leqslant \{\Delta(a, \; b)\}^2. \tag{27}$$

We now combine the identity (5) with the inequalities (25), (26) and (27), and obtain the following result: *if the conditions* (15)–(18) *for* a *and* b, x_0, y_0, *and* z_0 *are satisfied, then there exist three numbers* x_1, y_1, *and* z_1, *such that*

$$x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad z_1 \equiv z_0$$

and

$$f(x_1, \; y_1, \; z_1) = F(\xi_1, \; \eta_1) + \frac{z_1{}^2}{a^2 - b^2} \leqslant \tfrac{3}{4} + \max\,(A_1, \; A_2), \tag{28}$$

where

$$A_1 = -\tfrac{3}{4} + \left\{\tfrac{1}{2}\sqrt{\left(3 - a - \frac{1}{a^2 - b^2}\right)} - \sqrt{\left(a - \frac{1}{a^2 - b^2}\right)}\right\}^2 + \frac{a}{4} + \frac{1}{a^2 - b^2}\left[1 - \sqrt{\left\{(a^2 - b^2)\left(\tfrac{3}{4} - \frac{a^2}{2(a+b)}\right)\right\}}\right]^2, \tag{29}$$

$$(30)\quad A_2=-\tfrac{3}{4}$$
$$+\left\{\sqrt{\left(\frac{a^2-b^2}{a}\right)}-\tfrac{1}{2}\sqrt{\left(3-a-\frac{1}{a^2-b^2}\right)}-\sqrt{\left(\frac{a^2-b^2}{a}-\frac{1}{a^2-b^2}\right)}\right\}^2$$
$$+\frac{a}{4}+\frac{1}{a^2-b^2}\left[1-\sqrt{\left\{(a^2-b^2)\left(\tfrac{3}{4}-\frac{a^2}{2(a+b)}\right)\right\}}\right]^2.$$

We prove in the next paragraph that neither A_1 nor A_2 is positive if, as we have assumed,

$$(31)\quad 1\leqslant a\leqslant \sqrt[3]{2},\quad 0\leqslant b\leqslant \tfrac{1}{3}[a^2-\sqrt{\{a^4-3(a-1)^2(2a+1)\}}]\leqslant 2(a-1)^2,$$

and that both numbers are in fact negative except for $a=1$, $b=0$. Hence the first assertion of Lemma B is true. The second assertion is also true; for if $f(x_1, y_1, z_1)\geqslant\frac{3}{4}$ for all $x_1\equiv x_0$, $y_1\equiv y_0$, $z_1\equiv z_0$, then, by the last remark, $a=1$, $b=0$, and therefore also $c=d=0$, since the determinant $1-c^2-d^2$ of $f(x, y, z)$ is 1 by hypothesis. Hence

$$f(x,\ y,\ z)=x^2+y^2+z^2,$$

and therefore $x_0\equiv y_0\equiv z_0\equiv\frac{1}{2}$.

7. *Proof that A_1 and A_2 are not positive.*

The inequality $A_1\leqslant 0$ may be written

$$[\sqrt{\{(3-a)(a^2-b^2)-1\}}-2\sqrt{\{a(a^2-b^2)-1\}}]^2$$
$$+[2-\sqrt{\{3(a^2-b^2)-2a^2(a-b)\}}]^2\leqslant(3-a)(a^2-b^2),$$

or say
$$R_1{}^2+R_2{}^2\leqslant(3-a)(a^2-b^2).$$

If we put $a=1+\alpha$, the right-hand side is

$$2+3\alpha-\alpha^3-(2-\alpha)b^2,$$

which is greater than or equal to 2, since $0\leqslant b\leqslant 2\alpha^2$ and $0\leqslant\alpha\leqslant 0{\cdot}26$. Hence it suffices to prove that

$$R_1+R_2\leqslant 2.$$

The expressions for R_1, R_2, are

$$R_1=\sqrt{\{1+3\alpha-\alpha^3-(2-\alpha)b^2\}}-2\sqrt{\{3\alpha+3\alpha^2+\alpha^3-(1+\alpha)b^2\}},$$
$$R_2=2-\sqrt{(1-3\alpha^2-2\alpha^3-3b^2+2\alpha^2b)}.$$

We have

$$R_1 \leqslant \sqrt{(1+3a-a^3)}-2\sqrt{(3a+3a^2+a^3-8a^4)}$$

$$\leqslant \sqrt{(1+3a+\tfrac{9}{4}a^2)}-2\sqrt{\left(\frac{3}{0\cdot 26}a^2+3a^2-8(0\cdot 26)^2a^2\right)}$$

$$\leqslant 1+\tfrac{3}{2}a-7a;$$

and $$R_2 \leqslant 2-\sqrt{(1-3a^2-2a^3)} \leqslant 1+3a^2+2a^3 \leqslant 1+2a;$$

whence the result.

The inequality $A_2 \leqslant 0$ may be written

$$\frac{1}{a}\left\{2(a^2-b^2)-\sqrt{[a\{(3-a)(a^2-b^2)-1\}]}-2\sqrt{[(a^2-b^2)^2-a]}\right\}^2$$
$$+[2-\sqrt{\{3(a^2-b^2)-2a^2(a-b)\}}]^2 \leqslant (3-a)(a^2-b^2),$$

or say $$\frac{1}{a}R_3^2+R_2^2 \leqslant (3-a)(a^2-b^2).$$

Again (*a fortiori* since $a \geqslant 1$) it suffices to prove that

$$R_3+R_2 \leqslant 2.$$

We have

$$R_3 \leqslant 2(1+a)^2-\sqrt{\{(1+a)(1+3a-a^3-8a^4)\}}$$
$$-2\sqrt{(3a+6a^2+4a^3+a^4-16a^4)}.$$

Now $$a^3+8a^4 \leqslant 2a,$$

$$4a^3-15a^4 \geqslant 0,$$

$$\sqrt{(3a+6a^2)} \geqslant \sqrt{(17a^2)} \geqslant 4a.$$

Hence $$R_3 \leqslant 2+4a+2a^2-(1+a)-8a$$

$$\leqslant 1-4a.$$

Combining this with the previous inequality for R_2, we have the result. It is clear that A_1 and A_2 are negative except when $a=0$, in which case $a=1$, $b=0$.

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