The intersection formulae for a Grassmannian variety: W. V. D. Hodge.

Dirac's equation and Einstein's geometry of distant parallelism: H. W. Haskey.

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(1) Lattice points in two dimensional star domains; (2) Note on lattice points in star domains: K. Mahler.

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NOTE ON LATTICE POINTS IN STAR DOMAINS

K. MAHLER*.

About a year ago, in a paper not yet published, Prof. Mordell proved a number of very general theorems on lattice points in finite and infinite regions bounded by concave curves. His results opened up a new domain of research, not dealt with by Minkowski's theories. They were also the more important because they could be applied to concrete cases. I refer the reader to his note, *Journal London Math. Soc.*, 16 (1941), 149–151, for an enumeration of some of his results.

Prof. Mordell used an entirely new method, different from that which Minkowski applied to analogous questions concerning convex domains. I therefore asked myself whether Minkowski's original ideas could not be so generalized as to be applicable to non-convex domains. In a rather long paper submitted for publication in the *Proceedings* of the Society, I show now that this is indeed so.

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I treat the general *star domain* K, that is, a closed bounded point set of the following kind:

(*a*) K contains the origin O of the coordinate system (x, y) as an inner point;

(*b*) the boundary L of K is a Jordan curve consisting of a finite number of analytical arcs;

(*c*) every radius vector from O intersects L in one, and only one, point.

I assume, further, that the domain is symmetrical about O, *i.e.* that if it contains a point (x, y) it contains also the point $(-x, -y)$. The general unsymmetrical case is reduced to this symmetrical one by a trivial transformation.

A lattice Λ of points P

$$(x, y) = (\alpha h + \beta k, \gamma h + \delta k) \quad (h, k = 0, \pm 1, \pm 2, \ldots)$$

is called *K-admissible* if the origin O is the only point of Λ which is an inner point of K. Let

$$d(\Lambda) = |\alpha\delta - \beta\gamma|$$

be the determinant of Λ, and $\Delta(K)$ the lower limit of $d(\Lambda)$ for all K-admissible lattices. It is easily proved that $\Delta(K) > 0$. I show that there always exists at least one K-admissible lattice Λ such that

$$d(\Lambda) = \Delta(K),$$

a *critical lattice* in Prof. Mordell's notation.

I have developed, in my paper referred to above, a method by which *all critical lattices of K can be determined in a finite number of steps*; hence $\Delta(K)$ can also be found. While this method is theoretically perfect, it may require in practice a formidable amount of work in solving systems of a finite number of equations in a finite number of unknowns.

My method, as presented, is restricted to bounded domains. I think, however, that this restriction can be removed by a simple limiting process. It seems also probable that the method can be extended to problems in three or more dimensions.

So far, I have applied the method only to a few special cases. These simple results seem to be new.

(1) *The excentric ellipse.* Let K be an ellipse of area $J\pi$ which contains O as an inner point. Let the concentric, similar, and similarly situated ellipse through O be of area $J_0\pi$. Then

$$\Delta(K) = \frac{\sqrt{(J-J_0)}}{2}\{2\sqrt{(J_0)}+\sqrt{(3J+J_0)}\}.$$

I am much indebted to Mrs. W. R. Lord for solving a problem in Euclidean geometry from which I derived this value of $\Delta(K)$.

(2) *The excentric parallelogram.* Let K be a parallelogram which contains O as an inner point. Let the lines through O parallel to its sides divide K into four parallelograms of areas J_1, J_2, J_3, J_4, where the indices are chosen such that $J_1 \leqslant J_2 \leqslant J_3 \leqslant J_4$. Then

$$\Delta(K) = J_2+J_3-J_1.$$

(3) *The excentric triangle.* Let K be a triangle which contains O as an inner point. Let the lines through O parallel to two of its sides, together with the third side, form triangles of areas J_1, J_2, J_3, where the notation is such that $J_1 \leqslant J_2 \leqslant J_3$. Then

$$\Delta(K) = 2\sqrt{(J_2 J_3)}.$$

(4) *The domain K obtained by combining two concentric ellipses.* Let K be the set of all points (x, y) such that either

$$a_1x^2+2b_1xy+c_1y^2 \leqslant 1 \quad \text{or} \quad a_2x^2+2b_2xy+c_2y^2 \leqslant 1.$$

Here the two quadratic forms on the left-hand sides are assumed to be positive definite and of determinants 1; *i.e.*,

$$a_1c_1-b_1^2 = a_2c_2-b_2^2 = 1.$$

Their simultaneous invariant is

$$J = a_1c_2-2b_1b_2+c_1a_2.$$

Excluding the case when the forms are identical, we have

$$J > 2,$$

and it is easily seen that $\Delta(K) = D(J)$ is a function of J only.

I develop a simple algorithm for obtaining $D(J)$ for every $J > 2$; in particular, I give the explicit value of $D(J)$ for $2 < J \leqslant 25$. Further, a table of the critical lattices for every J in this interval is given. Both $D(J)$ and these critical lattices depend in a rather complicated way on arithmetical functions of J. There are an infinity of values of J for which $D(J)=\frac{1}{2}\sqrt{3}$. For all J,

$$\frac{\sqrt{3}}{2}\leqslant D(J)\leqslant\frac{\sqrt{15}}{2},$$

and

$$\lim_{J\to\infty} D(J)=\frac{\sqrt{3}}{2}.$$

It may be remarked that $1/D(J)$ is not less than the minimum of the smaller of the two numbers

$$a_1x^2+2b_1xy+c_1y^2,\quad a_2x^2+2b_2xy+c_2y^2$$

for integral values of x and y not both zero.

The method used in (4) can also be applied to other domains obtained by combining two convex domains, *e.g.*, to Prof. Mordell's star-shaped octagon (*loc. cit.*, 149), or to that obtained from two rectangles with centres at the origin and sides parallel to the axes.

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NOTE ON THE ABSOLUTE SUMMABILITY OF TRIGONOMETRICAL SERIES

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A series ΣA_n is said† to be summable $|A|$ if $F(r)=\Sigma A_n r^n$ is of bounded variation in the interval $0<r<1$. A series which is summable $|C|$ is also‡ summable $|A|$, but one which is summable $|A|$ need not be summable (C), as is shown by the well-known example $F(r)=e^{(1+r)^{-1}}$, while a convergent series need not§ be summable $|A|$.

Necessary and sufficient conditions for the summability $|C|$ of a Fourier series have been given by Bosanquet‖. On the other hand, the author has proved the following result¶.

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* Received 1 June, 1942; read 18 June, 1942.

† J. M. Whittaker, *Proc. Edinburgh Math. Soc.* (2), 2 (1930), 1–5.

‡ M. Fekete, *Proc Edinburgh Math. Soc.* (2), 3 (1933), 132–134.

§ Whittaker, *loc. cit.*

‖ L. S. Bosanquet [1], [2], *Journal London Math. Soc.*, 11 (1936), 11–15, and *Proc. London Math. Soc.* (2), 41 (1936), 517–528.

¶ F. T. Wang [1], *Journal London Math. Soc.*, 16 (1941), 174–176,