fying the canonical equation (11) identically, on substituting in (15), (16), (22) for ξ, η, ζ from (26) and (25). The unicursal curve represented by this parametric solution of (11) is clearly of *order* 18. From a previous general result*, it is *the complete intersection of the cubic surface* (11) *and another algebraic surface of order six.*

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ON LATTICE POINTS IN AN INFINITE STAR DOMAIN

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In my paper "On lattice points in star domains", which is to appear in the *Proceedings of the London Mathematical Society*, I defined a *finite star domain* by the following properties:

(1) The domain K is a bounded closed point set in the (x, y)-plane.

(2) K contains the origin $O = (0, 0)$ as an inner point.

(3) The boundary C of K is a Jordan curve.

(4) Every radius vector from O intersects C in just one point.

(5) If K contains the point $P = (x, y)$, then it also contains the point $-P = (-x, -y)$ symmetrical to P in O.

I called a lattice

$$(\Lambda) \quad (x, y) = (\alpha h + \beta k, \gamma h + \delta k) \quad (\alpha, \beta, \gamma, \delta \text{ real numbers}; \ h, k = 0, \pm 1, \ldots)$$

K-admissible, if O is the only *inner* point of K belonging to Λ. Then

$$d(\Lambda) = |\alpha\delta - \beta\gamma|$$

is called the determinant of Λ, and $\Delta(K)$ denotes the lower bound of $d(\Lambda)$ for all K-admissible lattices. It was shown that $\Delta(K) > 0$, and that there exists at least one *critical lattice*, *i.e.* a K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$. It was further proved trivially that if the finite star

* Cf. B. Segre, "A note on arithmetical properties of cubic surfaces", *loc. cit.*, Theorem VII.

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domain K is contained in the finite star domain K', then

$$\Delta(K) \leqslant \Delta(K').$$

In this note, I consider *infinite star domains, i.e.* point sets K in the (x, y)-plane such that

"*if K_r is, for every positive number r, the set of all those points of K which have a distance not greater than r from O, then K_r is a finite star domain*".

If $r < r'$, then K_r is contained in $K_{r'}$; hence

$$\Delta(K_r) \leqslant \Delta(K_{r'}).$$

Therefore $\Delta(K_r)$ is an *increasing* function of r. Put

$$\Delta(K) = \lim_{r \to \infty} \Delta(K_r).$$

If $\Delta(K) = \infty$, then every lattice contains an infinity of points of K; an example of a domain of this kind is given by

$$|x^2 y| \leqslant 1,$$

since $|xy| \leqslant d(\Lambda)/\sqrt{5}$, $|x| < \epsilon$ is solvable for every $\epsilon > 0$. *In this note, I assume from now on that $\Delta(K)$ is finite.*

THEOREM 1. *There exists at least one critical lattice Λ of K, i.e. a lattice with the following properties*:

(1) *O is the only inner point of K belonging to Λ.*

(2) $d(\Lambda) = \Delta(K)$, *i.e.* $= \lim_{r \to \infty} \Delta(K_r)$. [This differs from the definition of $\Delta(K)$ for finite domains.]

(3) *There is no K-admissible lattice of determinant less than $\Delta(K)$.* [*I.e.* the definition of $\Delta(K)$ in (2) is equivalent to that in the case of a finite domain.]

Proof. The origin is an inner point of K; there is therefore a positive number ρ such that the circle K of centre O and radius ρ lies entirely in K_1. Hence K is also a subset of K_n for $n = 1, 2, 3, \ldots$.

Denote by Λ_n a critical lattice of K_n, and by R_n, S_n a basis of Λ_n. This basis can be chosen so as to be *reduced*; *i.e.* all angles of the parallelogram with vertices at O, R_n, $R_n + S_n$, S_n lie between 60° and 120°. Then by a well-known property of reduced lattices,

$$\sqrt{(\tfrac{3}{4})}\,\overline{OR_n}\times\overline{OS_n}\leqslant d(\Lambda_n)=\Delta(K_n)\leqslant\Delta(K).$$

Further, since no element of Λ_n can be an inner point of K,

$$\overline{OR_n}\geqslant\rho,\quad \overline{OS_n}\geqslant\rho.$$

Hence
$$\overline{OR_n}\leqslant\frac{2\Delta(K)}{\rho\sqrt{3}},\quad \overline{OS_n}\leqslant\frac{2\Delta(K)}{\rho\sqrt{3}},$$

and so the two basis points R_n, S_n of Λ_n lie at a bounded distance from O. Hence there exists an infinite sequence of indices

$$n_1,\ n_2,\ n_3,\ \ldots,$$

such that the two basis points

$$R_n,\quad S_n \qquad (n=n_1,\ n_2,\ n_3,\ \ldots)$$

tend to limit points R and S, respectively.

Denote by Λ the lattice of basis R, S. Then

$$d(\Lambda)=\lim_{\nu\to\infty} d(\Lambda_{n_\nu})=\Delta(K).$$

This lattice Λ is K-admissible. For if this be false, let

$$P=hR+kS \qquad (h,\ k \text{ integers})$$

be a point of Λ different from O which is an inner point of K. The sequence of points

$$P_n=hR_n+kS_n \qquad (n=n_1,\ n_2,\ n_3,\ \ldots)$$

tends to P, and so P_n is arbitrarily near to P for large n. Hence also P_{n_ν} is an inner point of K if ν is sufficiently large. Let r be the distance of P from O. Then, for $n_\nu>r$, P is also an inner point of K_{n_ν}. This, however, is contrary to the assumption that Λ_{n_ν} is a K_{n_ν}-admissible lattice.

There cannot be a K-admissible lattice Λ^* for which

$$d(\Lambda^*)<\Delta(K).$$

For, if such a lattice should exist, let n be an index such that

$$d(\Lambda^*)<\Delta(K_n).$$

Then at least one point $P\neq O$ of Λ^* is an *inner* point of K_n, and hence an inner point of K, contrary to hypothesis. This completes the proof.

It was proved in my paper that every critical lattice of a finite star domain has at least *four* points on its boundary. This is not so for infinite domains.

THEOREM 2. *There exists an infinite star domain K of boundary C such that no critical lattice of K has a point on C.*

Proof. Denote by K a domain with the properties:

(1) K is an infinite star domain.

(2) All points of K are *inner* points of the infinite star domain K^* defined by

$$|xy| \leqslant 1.$$

(3) If the point $P = (x, y)$ on C is at the distance r from O, then

$$\lim_{r\to\infty} |xy| = 1.$$

By a theorem of Hurwitz,

$$\Delta(K^*) = \sqrt{5};$$

hence, since K is contained in K^*,

$$\text{(I)} \qquad \Delta(K) \leqslant \Delta(K^*) = \sqrt{5}.$$

Let further ϵ and t be two positive numbers, of which ϵ is sufficiently small, and denote by $K(\epsilon, t)$ the finite star domain

$$|xy| \leqslant (1-\epsilon)^2, \quad \left|tx+\frac{1}{t}\,y\right| \leqslant \sqrt{5}(1-\epsilon).$$

Then, by a theorem of mine (in my paper: "On lattice points in the star domain $|xy| \leqslant 1$, $|x+y| \leqslant \sqrt{5}$", which is to appear in the *Proceedings* of the Cambridge Philosophical Society),

$$\Delta\big(K(\epsilon, t)\big) = \sqrt{5}(1-\epsilon)^2$$

is independent of the value of t. I assert that, for all sufficiently large t, $K(\epsilon, t)$ is contained in K, so that

$$\text{(II)} \qquad \Delta(K) \geqslant \Delta\big(K(\epsilon, t)\big) = \sqrt{5}(1-\epsilon)^2.$$

For choose a positive number $r(\epsilon)$ such that

$$|xy| > (1-\epsilon)^2$$

for all points P of C for which $r > r(\epsilon)$; such a constant exists by the property (3) of K. It is clear from this definition of $r(\epsilon)$ that no point P on C with $r > r(\epsilon)$ belongs to $K(\epsilon, t)$; hence it suffices to show that no point on C with $r \leqslant r(\epsilon)$ belongs to $K(\epsilon, t)$. Now the two coordinate axes are asymptotes of C, but do not intersect C. Hence there exists a positive number $\delta(\epsilon)$ such that

$$|x| \geqslant \delta(\epsilon), \quad |y| \leqslant r$$

for all points $P=(x, y)$ on C with $r \leqslant r(\epsilon)$. Choose t so large that

$$t > \frac{1+\sqrt{5}}{\delta(\epsilon)}, \quad t > r;$$

then

$$\left| tx + \frac{1}{t}\, y \right| > \frac{1+\sqrt{5}}{\delta(\epsilon)}\,\delta(\epsilon) - \frac{1}{r}\, r = \sqrt{5} > \sqrt{5}(1-\epsilon),$$

as asserted.

Since ϵ may be arbitrarily small, from (I) and (II),

$$\Delta(K) = \sqrt{5}.$$

Hence, if

$$(\Lambda) \qquad (x, y) = (\alpha h + \beta k,\ \gamma h + \delta k) \qquad (h, k = 0, \pm 1, \pm 2, \ldots)$$

is a critical lattice of K, then

$$d(\Lambda) = |\alpha\delta - \beta\gamma| = \sqrt{5}.$$

To Λ, we make correspond the indefinite quadratic form

$$\Phi(h, k) = (\alpha h + \beta k)(\gamma h + \delta k) = ah^2 + 2bhk + ck^2$$

of determinant

$$b^2 - ac = \left(\frac{\alpha\delta - \beta\gamma}{2}\right)^2 = \tfrac{5}{4}.$$

By the property (3) of K, this form satisfies the inequality

$$|\Phi(h, k)| \geqslant (1-\epsilon)^2 \qquad (\epsilon > 0)$$

for all integers h, k with sufficiently large h^2+k^2.

Now, by a theorem of Markoff, the forms equivalent to

$$\mp(h^2-hk-k^2)$$

are the only ones of determinant $\frac{5}{4}$ which do not assume values numerically less than 1 for integral h, k not both zero; every other form of determinant $\frac{5}{4}$ represents numbers numerically not greater than

$$\sqrt{\tfrac{5}{8}}$$

for an infinity of integral h, k.

Hence, since ϵ may be chosen so small that

$$(1-\epsilon)^2 > \sqrt{\tfrac{5}{8}},$$

$\Phi(h, k)$ must be the form

$$\Phi(h, k) = \mp(h^2-hk-k^2),$$

and so

$$|xy| = |\Phi(h, k)| \geqslant 1$$

for all points of Λ different from O. Therefore, as asserted, no point of Λ lies on the boundary C of K. We also see that K has actually an infinity of critical lattices, namely

$$(x, y) = \left(\lambda\left(h-\frac{1+\sqrt{5}}{2}k\right), \frac{1}{\lambda}\left(h-\frac{1-\sqrt{5}}{2}k\right)\right) \quad (h, k = 0, \pm 1, \pm 2, \ldots),$$

where λ is any positive or negative number.

A slight modification of this proof proves that suitable infinite star domains possess critical lattices with any even number of points on C.

There is no difficulty in extending Theorem 1 to more than two dimensions, if use is made of the theory of reduced quadratic forms to find n points forming a basis of the lattice.

Theorem 2 has an analogue in three dimensions, as can be deduced from results of Davenport on the product of three linear forms, but I do not know whether such an analogue holds in more than three dimensions†.

January, 1944. I have recently extended the result of this note to more dimensions, and proved some further existence theorems.

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