LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (I)

By KURT MAHLER.

[Received 14 May, 1942.—Read 21 May, 1942.]

Last year, after earlier special results by H. Davenport†, L. J. Mordell extended Minkowski's classical work on lattice points in convex bodies‡ to certain two-dimensional non-convex domains. Then, very recently, he has found very general results for finite or infinite non-convex domains§. His method depends on an application of Minkowski's theorem on linear forms. He thus obtains point sets abutting on the given domain and containing at least one lattice point. By forming other lattice points from these so obtained, he succeeds in finding the lattices of smallest determinant that contain no inner points of the given domain, except the origin of the coordinate system.

In the present paper I consider the same problem for a general class of two-dimensional domains which I call *simple star domains*. A simple star domain is a bounded and closed set that contains the origin of the coordinate system as an inner point and is symmetrical in the origin||. Further, its boundary is a Jordan curve that is composed of a finite number of analytical arcs, and which is intersected in just *one* point by every radius vector from the origin. For these domains I derive a finite algorithm to find the smallest determinant for which lattices exist containing no inner points of the set except the origin. This method is a generalization of one due to Minkowski¶. In the second part of this paper I shall apply the algorithm

† *Proc. London Math. Soc.* (2), 44 (1938), 412–431, and 45 (1939), 98–125.

‡ *Geometrie der Zahlen* (Leipzig und Berlin, 1910).

§ *Proc. London Math. Soc.* (2), 48 (1943), 198–228 and 339–390; *Journal London Math. Soc.*, 17 (1942), 107–115.

|| The restriction to *symmetrical* domains can easily be removed (§ 16).

¶ *Ges. Abh.*, II, 3–42 (Leipzig und Berlin, 1911).

to some special domains. The method can probably be extended to more than two dimensions; but the work required will be excessive. In special cases the method can be applied to infinite domains.

I should like to express my gratitude to Prof. Mordell for his criticisms and his help with the manuscript.

CHAPTER I. THE GENERAL THEORY.

1. *Lattices in the* (x, y)*-plane.*

Let an arbitrary rectangular coordinate system in the plane be given. The unit square with vertices at the points

$$(0, 0),\ (1, 0),\ (1, 1),\ (0, 1)$$

is therefore of area unity.

We identify the point of coordinates (x, y) with the vector of components (x, y), and apply the usual vector notation. If

$$R : (\alpha, \beta) \quad \text{and} \quad S : (\gamma, \delta)$$

are two vectors, u, v two real numbers, then $P = uR+vS$ denotes the vector of components

$$x = u\alpha+v\gamma, \quad y = u\beta+v\delta.$$

Further

$$(R, S) = \alpha\delta-\beta\gamma$$

stands for the determinant of the two vectors. Hence

$$A = |(R, S)|,$$

where A is the area of the parallelogram Π with vertices at O, R, $R+S$, S; here $O : (0, 0)$ denotes the origin of the coordinate system.

The points or vectors R and S are called

$$\textit{dependent}, \quad \text{if} \quad (R, S) = 0,$$

$$\textit{independent}, \quad \text{if} \quad (R, S) \neq 0.$$

In the first case the parallelogram Π degenerates into a line segment, and R, S are collinear with O.

Assume now that $(R, S) \neq 0$, and let Λ be the lattice of all points

$$P = uR+vS \qquad (u, v = 0, \pm 1, \pm 2, \ldots).$$

The positive number

$$d = d(\Lambda) = |(R, S)|$$

is called the *determinant* of the lattice, and R, S are called a *basis* of Λ. If

$$R^* = u_1 R + v_1 S \quad \text{and} \quad S^* = u_2 R + v_2 S$$

are any two points of Λ, then the non-negative integer

$$n = \text{ind}\,(R^*, S^*) = |u_1 v_2 - u_2 v_1| = \frac{|(R^*, S^*)|}{|(R, S)|}$$

is called the *index* of R^*, S^*. This index vanishes if, and only if, R^* and S^* are dependent. If $n \neq 0$, then the area A^* of the parallelogram Π^* with vertices at O, R^*, R^*+S^*, S^* is given by

$$A^* = |(R^*, S^*)| = nA.$$

In the special case $n = 1$, the two points R^*, S^* form a new basis of Λ, and the equality $A^* = A$ holds.

From the definition of n,

$$(1) \qquad |(R^*, S^*)| = d(\Lambda)\,\text{ind}\,(R^*, S^*).$$

We shall apply this simple relation repeatedly.

We require the following lemma† ‡:

The lattice Λ has at least one reduced basis. *This is characterized by the property that no diagonal of Π^* is smaller than the sides of this parallelogram. Hence all angles of Π^* lie between $60°$ and $120°$, and the area $A^* = d(\Lambda)$ of Π^* satisfies the inequalities*

$$(2) \qquad \sqrt{\tfrac{3}{4}}\,.\,\overline{OR^*} \times \overline{OS^*} \leqslant d(\Lambda) \leqslant \overline{OR^*} \times \overline{OS^*}.$$

A point $$P = uR + vS$$

of Λ is called *primitive* when the greatest common divisor $Gcd(u, v) = 1$. The point P is primitive if, and only if, there is no lattice point on the line segment connecting O with P except the endpoints O and P. It is well known† that corresponding to every primitive lattice point $R^* = P$ there exist lattice points S^* such that R^*, S^* form a basis of Λ.

† Bachmann, *Quadratische Formen*, 2 (Leipzig und Berlin, 1923), Kap. 5.

‡ $\overline{PQ}$ denotes the distance between the two points P and Q.

2. *Star domains.*

A *star domain* K in the (x, y)-plane is a point set with the following properties:

(a) *K is a bounded and closed set.*

(b) *The origin O is an inner point of K.*

(c) *If K contains a point P, K also contains the point $-P$, which is symmetrical to P in the origin.*

(d) *If C denotes the boundary† of K, then every radius vector from the origin intersects C in just one point.*

From this definition some simple properties of K follow at once. For an arbitrary $r > 0$, let Γ_r be the circle

$$x^2+y^2 \leqslant r^2.$$

By (a) and (b) there exist two numbers P and ρ with

$$0 < \rho \leqslant \mathrm{P}$$

such that Γ_{P} contains K, and that K contains Γ_ρ. We call Γ_{P} an *outer circle* and Γ_ρ an *inner circle* of K‡.

Further, by (d), corresponding to every angle ϕ there is exactly one point

$$P(\phi) : (x, y)$$

of coordinates

$$x = r(\phi) \cos\phi, \quad y = r(\phi) \sin\phi,$$

such that

$$r(\phi) > 0,$$

and that $P(\phi)$ lies on C. From (c) and (d), the point symmetrical to $P(\phi)$ also lies on C and is given by

$$P(\phi+\pi) = -P(\phi).$$

† *I.e.* the set of all those points of K which are not inner points.

‡ Instead of using Γ_{P} and Γ_ρ, we may compare K with other circumscribed and inscribed convex domains. This would improve some of the later inequalities.

Obviously for all angles ϕ,

$$\rho \leqslant r(\phi) \leqslant \mathrm{P}. \tag{3}$$

THEOREM 1. *$r(\phi)$ is a continuous function of ϕ.*

Proof. Suppose the assertion is false. Then there is an angle ϕ_0 and an infinite sequence of angles

$$\phi_1, \phi_2, \phi_3, \ldots, \tag{4}$$

with limit ϕ_0, such that

$$\lim_{\nu \to \infty} r(\phi_\nu)$$

either does not exist, or exists and is different from $r(\phi_0)$. By (3), we can find an infinite subsequence

$$\phi_{\nu_1}, \phi_{\nu_2}, \phi_{\nu_3}, \ldots$$

of (4) for which

$$\lim_{k \to \infty} \phi_{\nu_k} = \phi_0, \quad \text{but} \quad \lim_{k \to \infty} r(\phi_{\nu_k}) = r^* \neq r(\phi_0).$$

By property (*a*), the point P^* of coordinates

$$x = r^* \cos \phi_0, \quad y = r^* \sin \phi_0$$

belongs to K. It also belongs to C, since it is the limit of the boundary points

$$P(\phi_{\nu_k}) \qquad (k = 1, 2, 3, \ldots).$$

Hence there are two different points $P(\phi_0)$ and P^* of C on the same radius vector from O, in contradiction to (*d*).

By Theorem 1, C is a *Jordan curve*. It therefore divides the plane into two separate parts: its interior which contains only points of K, and its exterior which contains no points of K.

Hence with a point P also all points tP, where $0 \leqslant t \leqslant 1$, belong to K. Further, if P_1 and P_2 are two points on C such that the whole line segment from P_1 to P_2 belongs to K, then the triangle with vertices at O, P_1, P_2 consists only of points of K.

3. *The lattice problem.*

A lattice Λ is called *admissible* with respect to the star domain K, say, for shortness, K-admissible, if none of its points except O is an inner point of K. The lattice may, however, contain points on the boundary C of K.

There are admissible lattices. For the lattice generated by

$$(\mathrm{P}, 0) \quad \text{and} \quad (0, \mathrm{P})$$

is Γ_{P}-admissible, and so is also K-admissible.

Hence the lower bound

$$\Delta(K) = \text{l. b.}\, d(\Lambda)$$

extended over all K-admissible lattices, exists and is a finite non-negative number.

THEOREM 2. *If the star domain K_1 is contained in a second star domain K_2, then*

$$\Delta(K_1) \leqslant \Delta(K_2).$$

Proof. Every K_2-admissible lattice is also K_1-admissible.

THEOREM 3. *For a circle Γ_r,*

$$\Delta(\Gamma_r) = \sqrt{\tfrac{3}{4}} \cdot r^2.$$

Proof. The assertion is equivalent to the classical result on the minimum of a positive definite binary quadratic form†.

THEOREM 4. *Let Γ_{P} be an outer circle and Γ_ρ an inner circle of the star domain K. Then*

$$0 < \sqrt{\tfrac{3}{4}} \cdot \rho^2 \leqslant \Delta(K) \leqslant \sqrt{\tfrac{3}{4}} \cdot \mathrm{P}^2.$$

Proof. Evident from Theorems 2 and 3.

The problem now is to determine $\Delta(K)$ for a given star domain K. I generalize a method of Minkowski for convex domains, and show how the problem can be reduced to a finite number of elementary extremal problems.

4. *Critical lattices.*

A K-admissible lattice Λ is called a *critical lattice* if

$$d(\Lambda) = \Delta(K),$$

i.e. if the lower bound is actually attained by its determinant. I now prove a theorem fundamental for what follows. It should be remarked that it need not hold for *infinite* domains.

† Bachmann, *Quadratische Formen*, 2 (Leipzig und Berlin, 1923), Kap. 5.

THEOREM 5. *Corresponding to every star domain K, there is at least one critical lattice.*

Proof. By the definition of $\Delta(K)$, there exists an infinite sequence of admissible lattices†.

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

such that

$$\lim_{\nu\to\infty} d(\Lambda_\nu) = \Delta(K).$$

By Theorem 4, we may assume that, for all indices ν,

$$(5) \qquad d(\Lambda_\nu) \leqslant \mathrm{P}^2.$$

Let R_ν, S_ν, for $\nu = 1, 2, 3, \ldots$, be a *reduced basis* of Λ_ν (§ 1). Then by formula (2) of § 1,

$$d(\Lambda_\nu) \geqslant \sqrt{\tfrac{3}{4}} \,.\, \overline{OR}_\nu \times \overline{OS}_\nu.$$

Further, since Λ_ν is Γ_ρ-admissible,

$$\overline{OR}_\nu \geqslant \rho, \quad \overline{OS}_\nu \geqslant \rho.$$

Therefore, by (5),

$$\overline{OR}_\nu \leqslant \frac{2}{\sqrt{3}} \frac{\mathrm{P}^2}{\rho} \quad \text{and} \quad \overline{OS}_\nu \leqslant \frac{2}{\sqrt{3}} \frac{\mathrm{P}^2}{\rho}.$$

Hence reduced basis points R_ν and S_ν of all lattices Λ_ν lie at a *bounded* distance from the origin.

It is therefore possible to choose an infinite subsequence

$$\Lambda_{\nu_1}, \Lambda_{\nu_2}, \Lambda_{\nu_3}, \ldots$$

of the Λ's such that R_{ν_k} tends to a limit point R^*, and S_{ν_k} to a limit point S^*. Denote by Λ^* the lattice of basis R^*, S^*. This lattice is K-admissible. It is also critical, since

$$d(\Lambda^*) = \lim_{k\to\infty} d(\Lambda_{\nu_k}),$$

and since, by hypothesis, this limit is equal to $\Delta(K)$.

It may be mentioned that there are star domains with any finite or infinite number of critical lattices.

† These lattices need not be all different.

5. *Regular and singular lattices.*

By property (*c*) and (*d*) of a star domain, two different points on its boundary are collinear with O if, and only if, they are symmetrical in this point. Hence three different points on the boundary C cannot all be collinear with O.

THEOREM 6. *Every critical lattice contains at least two independent points P_1 and P_2 on C.*

This theorem is an immediate consequence of the

THEOREM 7. *Let Λ be a K-admissible lattice which has either no points or only one pair of symmetrical points P, $-P$ on C. Then there exists a K-admissible lattice Λ^* such that $d(\Lambda^*) < d(\Lambda)$.*

Proof. First assume that no points of Λ lie on C. Denote by R, S an arbitrary basis of Λ and by ϑ a sufficiently small positive number. The lattice Λ^* of basis R and $(1-\vartheta)S$ is evidently K-admissible; but its determinant $d(\Lambda^*) = (1-\vartheta)d(\Lambda)$ is smaller than $d(\Lambda)$.

Secondly, assume that Λ has only the two points P and $-P$ on C. Then P is a primitive lattice point (§ 1); hence there exists a second lattice point S which with $R = P$ forms a basis of Λ. Let again ϑ be a sufficiently small positive number. The lattice Λ^* of basis R and $(1-\vartheta)S$ still contains only the two points P and $-P$ on C, and is therefore admissible; again $d(\Lambda^*) < d(\Lambda)$.

It is useful to make the following distinction.

A critical lattice of K is called *singular* or *regular*, according as it has *four points* or *more than four points* on the boundary C of K.

6. *The index of two lattice points on C.*

Let Λ be a K-admissible lattice which contains two independent points P_1 and P_2 on C. Upper bounds for the index of these two points (§ 1) are given by the following theorems.

THEOREM 8. $$\text{ind}(P_1, P_2) \leq \frac{2}{\sqrt{3}}\left(\frac{\mathrm{P}}{\rho}\right)^2.$$

Proof. Let R, S be a basis of Λ. By § 1,

$$\text{ind}(P_1, P_2) = \frac{|(P_1, P_2)|}{|(R, S)|}.$$

Since K lies entirely in Γ_P, obviously

$$|(P_1, P_2)| \leqslant P^2.$$

Further, since K contains Γ_ρ, Λ is Γ_ρ-admissible. Hence, by Theorem 3,

$$d(\Lambda) = |(R, S)| \geqslant \Delta(\Gamma_\rho) = \sqrt{\tfrac{3}{4}} \cdot \rho^2.$$

The assertion follows at once from these two inequalities.

THEOREM 9. *Let ϑ with $0 < \vartheta \leqslant 1$ be a number such that all points of the parallelogram Π_0 with vertices at*

$$\vartheta(P_1+P_2), \quad \vartheta(P_1-P_2), \quad \vartheta(-P_1-P_2), \quad \vartheta(-P_1+P_2)$$

belong to K. Then

$$\operatorname{ind}(P_1, P_2) \leqslant \vartheta^{-2}.$$

Proof. Let R, S be a basis of Λ. Every lattice point P can be written as $P = uR+vS$ with integral u, v; in particular

$$P_1 = u_1 R + v_1 S, \quad P_2 = u_2 R + v_2 S.$$

When we eliminate R and S, P takes the form

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{u_1 v_2 - u_2 v_1}, \quad \text{where} \quad \omega_1 = v_2 u - u_2 v, \ \omega_2 = v_1 u + u_1 v.$$

By § 1, the denominator has absolute value

$$n = |u_1 v_2 - u_2 v_1| = \operatorname{ind}(P_1, P_2).$$

Except for the sign, n is the determinant of the two linear forms ω_1 and ω_2 in u and v. By Minkowski's theorem on linear forms, integral values of u and v not both zero exist such that

$$|\omega_1| \leqslant n^{\frac{1}{2}}, \quad |\omega_2| \leqslant n^{\frac{1}{2}}.$$

Hence there is a lattice point

$$P = \lambda_1 P_1 + \lambda_2 P_2 \neq O$$

with coefficients

$$\lambda_1 = \frac{\omega_1}{u_1 v_2 - u_2 v_1}, \quad \lambda_2 = \frac{\omega_2}{u_1 v_2 - u_2 v_1}$$

which satisfy the inequalities

$$|\lambda_1| \leqslant n^{\frac{1}{2}-1} = n^{-\frac{1}{2}}, \quad |\lambda_2| \leqslant n^{\frac{1}{2}-1} = n^{-\frac{1}{2}}.$$

If $n^{-\frac{1}{2}} < \vartheta$, then P would be an inner point of Π_0, hence also an inner point of K; this is impossible since Λ is K-admissible. Therefore

$$n^{-\frac{1}{2}} \geqslant \vartheta \quad i.e., \quad n \leqslant \vartheta^{-2},$$

as was to be proved.

THEOREM 10. *Let ϑ with $0 < \vartheta \leqslant 1$ be a number such that all points of the parallelogram with vertices at*

$$\vartheta P_1,\ \vartheta P_2,\ -\vartheta P_1,\ -\vartheta P_2$$

belong to K. Then

$$\text{ind}\,(P_1, P_2) \leqslant 2\vartheta^{-2}.$$

I omit the proof, which is similar to that of Theorem 9.

THEOREM 11. *If the two points P_1 and P_2 of Λ on C have the property that all points of the line segment joining P_1 to P_2 are inner points of K, then*

$$\text{ind}\,(P_1, P_2) = 1,$$

i.e., P_1 and P_2 form a basis of Λ.

Proof. Assume that the assertion is false; then the lattice Λ^* generated by P_1 and P_2 is only a sub-lattice of Λ. Hence there is at least one point Q of Λ which does not belong to Λ^*. We can find a point P of Λ^* such that $P + \epsilon Q$, where $\epsilon = 1$ or $\epsilon = -1$, belongs to the triangle T with vertices at O, P_1, P_2. The point $P + \epsilon Q$ so found is an element of Λ, but not of Λ^*, and is therefore different from the vertices of T. It cannot be an inner point of K since Λ is admissible. This forms a contradiction to the hypothesis, since, by §2, all points of T different from P_1 and P_2 are inner points of K.

7. *Lattices with at least four points on C.*

Let Λ be an arbitrary lattice which contains two *primitive* and *independent* points P_1 and P_2 on the boundary C of K. Then there is a lattice point S^* which together with P_1 generates Λ. Further, P_1 and $S = gP_1 + \epsilon S^*$ also form a basis of Λ, where g is an arbitrary integer and $\epsilon = \pm 1$. The second point P_2 can be written as

$$P_2 = u^* P_1 + v^* S^* = (u^* - \epsilon g v^*)\, P_1 + \epsilon v^* S$$

with two integers u^* and v^*. Here

$$|v^*| = \text{ind}\,(P_1, P_2) > 0,$$

since P_1 and P_2 are independent. Determine ϵ and g such that

$$v_2 = \epsilon v^* > 0 \quad \text{and} \quad 0 \leqslant u_2 = u^* - \epsilon g v^* < |v^*| = v_2.$$

Then
$$P_2 = u_2 P_1 + v_2 S,$$

where

$$v_2 = \text{ind}\,(P_1, P_2), \quad 0 \leqslant u_2 < v_2, \quad Gcd(u_2, v_2) = 1;$$

the third formula holds, since P_2 is primitive. For given v_2, u_2 is one of $\phi(v_2)$ different integers, where $\phi(n)$ is Euler's function.

An arbitrary point P of Λ can be written as

$$P = uP_1 + vS$$

with integral coefficients u and v. On eliminating S,

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \omega_1 = uv_2 - u_2 v, \quad \omega_2 = v.$$

The two coefficients ω_1 and ω_2 are integers satisfying the congruence

$$\omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

but are otherwise arbitrary. Obviously

$$|\omega_1| = \text{ind}\,(P, P_2), \quad |\omega_2| = \text{ind}\,(P, P_1).$$

Now, if Λ is an *admissible* lattice and P lies on C, then by Theorem 8,

$$(6) \qquad 0 < v_2 \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_1| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_2| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2.$$

Hence there are only a *finite* number of possibilities for the coefficients v_2, ω_1, ω_2. Better approximations for these coefficients are obtained if Theorems 9–11 can be applied.

The important question now arises: *When is the lattice Λ of all points*

$$(7) \qquad P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

admissible, where u_2 and v_2 are two arbitrary integers such that

$$0 \leqslant u_2 < v_2, \quad Gcd\,(u_2, v_2) = 1\,?$$

An answer is given by

THEOREM 12. *The lattice Λ is admissible if no point $P \neq O$ of Λ, with*

$$|\omega_1| < \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}, \quad |\omega_2| < \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}, \tag{8}$$

is an inner point of K.

Proof. It suffices to show that a point P for which at least one of the inequalities (8) is false does not lie in Γ_{P}, for then the point cannot belong to K.

Suppose then that, say,

$$|\omega_1| > \frac{v_2 \mathrm{P}^2}{|(P_1, P_2)|}. \tag{9}$$

Since P_2 lies on C and therefore in Γ_{P},

$$\overline{OP}_2 \leqslant \mathrm{P}.$$

By § 1, $|(P_1, P_2)|$ is the area of the parallelogram with vertices at O, P_1, P_1+P_2, P_2. Hence the distance of P_1 from the line L through O and P_2 is

$$\frac{|(P_1, P_2)|}{\overline{OP}_2} \geqslant \frac{|(P_1, P_2)|}{\mathrm{P}}.$$

The distance of

$$P = \lambda_1 P_1 + \lambda_2 P_2, \qquad \lambda_1 = \frac{\omega_1}{v_2}, \quad \lambda_2 = \frac{\omega_2}{v_2},$$

from L is the same as that of the point $\lambda_1 P_1$ from this line, and so by the last inequality and by (9) is not less than

$$|\lambda_1| \cdot \frac{|(P_1, P_2)|}{\mathrm{P}} > \mathrm{P}.$$

Hence P lies at a distance greater than P from the origin and does not belong to Γ_{P}.

It is useful to remark that the inequalities (8) in Theorem 12 can be replaced by

$$|\omega_1| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2, \quad |\omega_2| \leqslant \frac{2}{\sqrt{3}} \left(\frac{\mathrm{P}}{\rho}\right)^2,$$

provided that

$$v_2 \leqslant \frac{2}{\sqrt{3}} \frac{|(P_1, P_2)|}{\rho^2};$$

and if this last condition is not satisfied, then the lattice is not admissible, by the proof of Theorem 8.

8. *Simple star domains.*

Henceforth we consider the special class of star domains defined below.

A bounded rectifiable curve A in the (x, y)-plane is called an *analytical arc*, if it has the following properties:

(*a*) *To every point P on A there exists an infinity of vectors*

$$P', P'', P''', \ldots$$

not all zero such that every point P^* of A sufficiently near to P can be written as

$$P^* = P + P' \frac{s}{1!} + P'' \frac{s^2}{2!} + P''' \frac{s^3}{3!} + \ldots,$$

where s is a real parameter, and where this power series converges if $|s|$ is sufficiently small†.

(*b*) *The coefficient P' is different from 0 except at most in the two end points of A.*

It follows that in every point of A there is a unique tangent and an osculating circle.

THEOREM 13. *The common points of two analytical arcs are either finite in number, or they form a third analytical arc.*

Proof. The statement follows from the classical properties of regular analytical functions.

A star domain K is called *simple* if its boundary C consists of a *finite* number $A_1, A_2, \ldots, A_l$ of analytical arcs. Let

$$\Sigma = \{Q_1, Q_2, \ldots, Q_l\}$$

be the set of end points of these arcs, where the notation is such that

$$A_\lambda \text{ and } A_{\lambda+1} \text{ meet in } Q_\lambda \quad (\lambda = 1, 2, \ldots, l;\ A_{l+1} = A_1).$$

† By the Heine-Borel theorem, a finite number of these series suffices to represent the whole arc.

A straight line T is called an *inner* or an *outer tac-line* of K at a point P of C, according as *all* or *no* points of T sufficiently near to, but different from, P belong to K. There is evidently no tac-line at a point of inflexion, and there may not be one at the points of Σ. Excluding these points, there is just one tac-line at every other point of C, namely the tangent. In a point of Σ, the tac-lines fill up a complete angle if the two arcs of C meeting there have different tangents. When, however, these tangents coincide, there are three possibilities. Either there is *no* tac-line, or just *one* tac-line, or *all lines through this point except the tangent* are tac-lines.

Excluding the points of C which have no tac-line, the remaining points will be called *convex* or *concave*, according as they have an *outer* or *inner* tac-line.

Then C can be divided into a finite number of open arcs, consisting of only convex, or concave, points. The corresponding arcs are also called *convex* or *concave*.

It is clear that there are at most a finite number of tac-lines of K which have a given direction.

9. *Properties of the singular lattices.*

Let Λ be a singular lattice of K, having the four points $\pm P_1$ and $\pm P_2$ on C. Then for the two integers u_2 and v_2 associated with these points, all points of Λ are of the form

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Denote by P_1^* and P_2^* any two points of C sufficiently near to P_1 and P_2, respectively. Form the new lattice Λ^* of all points

$$P^* = \frac{\omega_1 P_1^* + \omega_2 P_2^*}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 = 0 \pmod{v_2}.$$

This lattice evidently is also admissible. Since Λ is critical, Λ^* must satisfy the inequality

$$d(\Lambda^*) \geqslant d(\Lambda). \tag{10}$$

We may assume without loss of generality that

$$(P_1, P_2) > 0, \text{ and therefore also } (P_1^*, P_2^*) > 0.$$

By formula (1) in § 1,

$$(P_1, P_2) = v_2 d(\Lambda), \quad (P_1{}^*, P_2{}^*) = v_2 d(\Lambda^*),$$

and so (10) takes the simpler form

$$(P_1{}^*, P_2{}^*) \geqslant (P_1, P_2). \tag{11}$$

THEOREM 14. *Let $\pm P_1$ and $\pm P_2$ be the four points of the singular lattice Λ on C. Then the straight lines through $\pm P_1$ parallel to the vector P_2, and the straight lines through $\pm P_2$ parallel to the vector P_1, are inner tac-lines of K.*

Proof. If the assertion is false, we may assume, without loss of generality, that a point P_2^{**} arbitrarily near to P_2 lies on the line through P_2 parallel to the vector P_1 and does not belong to K. The lattice Λ^{**} through P_1 and P_2^{**} and defined by the integers u_2 and v_2 is still admissible, but has on C only the lattice points O, P_1 and $-P_1$. It is therefore not critical. It has the same determinant as Λ, and so, by Theorem 7, there exists an admissible lattice of smaller determinant.

Theorem 14 enables us to find the singular lattices containing points of Σ; these lattices are *finite* in number.

We may assume from now on that neither P_1 nor P_2 belongs to Σ. Hence the points $P_1{}^*$ and $P_2{}^*$ above can be written as

$$P_1{}^* = P_1 + P_1{}' \frac{s}{1!} + P_1{}'' \frac{s^2}{2!} + \dots,$$

$$P_2{}^* = P_2 + P_2{}' \frac{t}{1!} + P_2{}'' \frac{t^2}{2!} + \dots$$

where $|s|$, $|t|$ are sufficiently small; also by (*b*) of § 8,

$$P_1{}' \neq 0, \quad P_2{}' \neq 0.$$

When we substitute these series for $P_1{}^*$ and $P_2{}^*$, (11) takes the form

$$\{(P_1{}', P_2)s + (P_1, P_2{}')t\} + \tfrac{1}{2}\{(P_1{}'', P_2)s^2 + 2(P_1{}', P_2{}')st + (P_1, P_2{}'')t^2\} + \dots$$

$$\geqslant 0.$$

This inequality is to hold if $|s|$ and $|t|$ are sufficiently small. Hence, first,

$$(P_1{}', P_2) = (P_1, P_2{}') = 0, \tag{12}$$

i.e. the tangents to C at P_1, P_2 are parallel to the vectors P_2, P_1 respectively. This result is contained in Theorem 14.

Secondly,

$$(13)\quad (P_1'', P_2)s^2+2(P_1', P_2')st+(P_1, P_2'')t^2 \geqslant 0 \text{ for all real } s, t.$$

If the quadratic form on the left-hand side is positive definite, then (11) holds in the stronger form

$$(P_1^*, P_2^*) > (P_1, P_2)$$

for all pairs of points P_1^*, P_2^* sufficiently near to P_1, P_2, except for the pair $P_1^* = P_1$, $P_2^* = P_2$.

From (12) and (13) we can find all pairs of points P_1, P_2 not in Σ through which a singular lattice may be drawn. In general, the equations (12) have only a *finite* number of solutions, since they reduce to a finite number of pairs of equations whose left-hand sides are analytical functions.

An exceptional case may arise when the two equations (12) lead to an *infinity* of points P_1, P_2. Then both equations are satisfied if P_1 lies anywhere on an arc B_1 of C, and P_2 determined by P_1 lies on an arc B_2 of C†. This is the case if B_2 is the envelope of the straight lines

$$uy-vx = \kappa,$$

where $\kappa \neq 0$ is a constant and $P_1:(u, v)$ runs over all points of B_1. Geometrically, B_2 is obtained by rotating through a right angle about the origin the polar reciprocal of B_1 with respect to the circle

$$x^2+y^2 = \kappa.$$

Evidently for all points P_1 and P_2 of these arcs associated by (12);

$$(P_1, P_2) = \kappa.$$

Hence, even if there is an infinity of admissible lattices, their determinants can only have one of the *finite* number of values

$$d(\Lambda) = \frac{\kappa}{v_2}.$$

Hence Theorem 14 and the formulae (12), (13) enable us to determine a finite or infinite set

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$$

† The two arcs may have common points or even be identical.

of lattices amongst which the singular lattices will be found. Even if the set is infinite, their determinants

$$d(\Lambda_1),\ d(\Lambda_2),\ d(\Lambda_3),\ \ldots$$

have only a *finite* number of different values. It is therefore possible to decide which is the *smallest* determinant.

10. *An example of a star domain with a singular lattice.*

The results of the last paragraph do not assert that there exist domains with singular lattices, and so I give an example of a simple star domain with this property.

Let θ be a number in the interval

$$\tfrac{3}{4} < \theta < 1.$$

Then K, the set of all points (x, y) which lie in at least one of the two rectangles

$$|x| \leqslant \theta,\ |y| \leqslant \tfrac{1}{2} \quad \text{and} \quad |x| \leqslant \tfrac{1}{2},\ |y| \leqslant \theta,$$

is a simple star domain; it has the form of a cross.

Let Λ be a critical lattice of K. If Λ is singular, then by Theorem 14 it must be the lattice Λ_0 which passes through

$$P_1\colon (\tfrac{1}{2}, -\tfrac{1}{2}) \quad \text{and} \quad P_2\colon (\tfrac{1}{2}, \tfrac{1}{2}).$$

Hence, by Theorem 11, P_1 and P_2 form a basis of Λ_0 which then consists of all points $P = \omega_1 P_1 + \omega_2 P_2$ with integral ω_1 and ω_2. The determinant of Λ_0 is

$$d(\Lambda_0) = \tfrac{1}{2}.$$

We show now that no regular lattice exists; therefore any critical lattice must be singular and so identical with Λ_0.

A regular lattice would contain at least three pairs of symmetrical points $\pm P_1$, $\pm P_2$, $\pm P_3$ on the boundary C of K. Hence, from the symmetry of the figure, we may assume without loss of generality that there are at least two independent lattice points P_1 and P_2 on that part of C formed by the three line segments

$$L_1\colon x = \theta,\ -\tfrac{1}{2} \leqslant y \leqslant \tfrac{1}{2};\quad L_2\colon \tfrac{1}{2} \leqslant x \leqslant \theta,\ y = \tfrac{1}{2};\quad L_3\colon \tfrac{1}{2} \leqslant x \leqslant \theta,\ y = -\tfrac{1}{2}.$$

By Theorem 11,

$$\operatorname{ind}(P_1, P_2) = 1,$$

and we may also assume that $(P_1, P_2) > 0$; therefore

$$d(\Lambda) = (P_1, P_2).$$

We now show that

$$(P_1, P_2) > \tfrac{1}{2}$$

by distinguishing the following four cases:

(*a*) *Both* P_1 *and* P_2 *lie on* L_1. Since $P_1 - P_2$ is not an inner point of K, P_1 and P_2 must lie at least a distance θ apart. Then, for admissible lattices, (P_1, P_2) is a minimum for an obvious infinity of lattices; for example,

$$P_1: (\theta, -\tfrac{1}{2}\theta), \quad P_2: (\theta, \tfrac{1}{2}\theta), \quad \text{and then} \quad (P_1, P_2) = \theta^2 > \tfrac{1}{2}.$$

(*b*) P_1 *lies on* L_1, P_2 *on* L_2, say

$$P_1: (\theta, \eta), \quad P_2: (\xi, \tfrac{1}{2}).$$

Since $P_1 - P_2$ is not an inner point of K, necessarily

$$-\tfrac{1}{2} \leqslant \eta \leqslant \tfrac{1}{2} - \theta.$$

This condition is easily proved to be also sufficient for the lattice to be admissible. If Λ is critical, then $P_1 - P_2$ must lie on C, and we have $\eta = \tfrac{1}{2} - \theta$. Hence

$$(P_1, P_2) = \tfrac{1}{2}\theta - \xi(\tfrac{1}{2} - \theta),$$

and is a minimum for $\xi = \tfrac{1}{2}$, when

$$(P_1, P_2) = \theta - \tfrac{1}{4} > \tfrac{1}{2}.$$

A similar proof holds when P_1 lies on L_3 and P_2 on L_1.

(*c*) P_1 *lies on* L_3, P_2 *on* L_2. It is easily seen that no admissible lattice of this kind has more than *two* pairs of symmetrical points on C.

(*d*) *Both* P_1 *and* P_2 *lie on* L_2, *or both lie on* L_3. It is obvious that no such lattice is admissible.

Hence Λ_0 is critical and so is singular. There are no other critical lattices. Hence*

$$\Delta(K) = d(\Lambda_0) = \tfrac{1}{2}.$$

---

* For another example of a star domain with a singular lattice, see the last part of this paper, p. 182.

11. *Determination of the regular lattices* (1).

We obtained in § 9 a method of finding all existing singular lattices. In this and the following paragraphs we give a similar method for the determination of the regular lattices. We construct a finite or infinite set of lattices

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$$

which must contain every existing regular lattice. This set has the important property that the set

$$\{d(\Lambda_1), d(\Lambda_2), d(\Lambda_3), \ldots\}$$

of determinants contains only a *finite* number of different values.

Let Λ be an arbitrary regular lattice of K and let

$$\pm P_1, \pm P_2, \ldots, \pm P_q \qquad (q \geqslant 3)$$

be the different pairs of symmetrical points of Λ which lie on C; then any two of these points with different indices are linearly independent. Every point P of Λ can be written as

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Here u_2, v_2 are the two integers associated with P_1 and P_2, and ω_1 and ω_2 are integers. In particular

$$P_\kappa = \frac{\Omega_{\kappa 1} P_1 + \Omega_{\kappa 2} P_2}{v_2}, \quad \Omega_{\kappa 1} + u_2 \Omega_{\kappa 2} \equiv 0 \pmod{v_2} \quad (\kappa = 3, 4, \ldots, q).$$

Since $$|\Omega_{\kappa 1}| = \text{ind}\,(P_2, P_\kappa), \quad |\Omega_{\kappa 2}| = \text{ind}\,(P_1, P_\kappa),$$

these integers are bounded, by Theorem 8.

We choose the notation so that $(P_1, P_2) > 0$, and hence by (1),

$$(P_1, P_2) = v_2 d(\Lambda),$$

and remark that these formulae remain satisfied if P_1 and P_2 are replaced by P_2 and $-P_1$.

It is now convenient to distinguish several types of lattices.

FIRST TYPE. *At least two of the points*

$$(14) \qquad P_1, P_2, \ldots, P_q$$

belong to the set Σ (§ 8).

Since Σ has only a finite number of points and u_2, v_2 are bounded, there exist only a finite number of lattices of this kind.

SECOND TYPE. *One, and only one, of the points* (14) *belongs to* Σ.

Call this now the point P_2. Denote by C^* the curve in the (x, y)-plane described by a point

$$P_3^* = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2}{v_2},$$

where the point P_1^* runs over the whole boundary of K. Since both P_1 and P_3 lie on C, the point P_3 must be a point of intersection of C and C^*. The two curves C and C^* consist of a finite number of analytical arcs, and so, by Theorem 13, have in general only a finite number of points of intersection. Then there exist at most a finite number of admissible lattices which contain P_2 and two independent points P_1, P_3 on C.

It is possible, however, that the curves C and C^* have an infinity of points and so an arc B in common. Then, when P_3^* runs over B, P_1^* describes an arc A on C, and *vice versa*. From the lattice Λ defined by

$$P = \frac{\omega_1 P_1 + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2},$$

we derive the lattice Λ^* given by

$$P^* = \frac{\omega_1 P_1^* + \omega_2 P_2}{v_2}, \quad \text{where} \quad \omega_1 + u_2 \omega_2 \equiv 0 \pmod{v_2}.$$

Then if we vary Λ^* by making P_1^* run over the arc A, each point P^* of Λ^* describes a curve.

Consider a point P of Λ independent of P_2; *i.e.* $\omega_1 \neq 0$. The corresponding point P^* is then a point of K if P_1^* is a point of the set $S(\omega_1, \omega_2)$ of all points

$$P_1^* = \frac{v_2 P^* - \omega_2 P_2}{\omega_1},$$

where P^* runs over all points of K. Evidently, since $S(\omega_1, \omega_2)$ is a domain similar and similarly situated to K, its boundary consists of a finite number of analytical arcs, and so has in common with A a finite number of points† and a finite number of arcs†.

† This, of course, includes the case when there are none.

Suppose now that Λ^* is an admissible lattice. If the domain $S(\omega_1, \omega_2)$ has points in common with K, then by Theorem 12 both $|\omega_1|$ and $|\omega_2|$ are less than a number depending only on K, *i.e.* at most a finite number of sets $S(\omega_1, \omega_2)$ have points in common with A. Hence the set of those points of A which are not inner points of any $S(\omega_1, \omega_2)$ consists of a finite number of points† and a finite number of sub-arcs $A_1, A_2, \ldots, A_h$ of A†. Let P_1^* run over the whole extent, say of A_1. The determinant $d(\Lambda^*)$ of Λ^* is a minimum for critical lattices; hence the area of the parallelogram $O, P_1^*, P_1^*+P_2, P_2$ is a minimum and so P_1^* is as near as possible to the line through O and P_2. Then either there is only a finite number of points P_1^* of A_1, or A_1 contains segments s of straight lines parallel to the vector P_2, and then $d(\Lambda^*)$ is constant however P_1^* is chosen on s.

Hence, in the second type, there exists a finite or infinite number of possible regular lattices, and in either case their determinants have only a finite number of different values.

12. *Determination of the regular lattices* (2).

We suppose from now on that no point of Λ belongs to Σ. Hence, if P_κ is an arbitrary point of the set (14), then every point P_κ^* on C sufficiently near to P_κ can be defined by a convergent power series

$$(15) \qquad P_\kappa^* = P_\kappa + P_\kappa' \frac{s_\kappa}{1!} + P_\kappa'' \frac{s_\kappa^2}{2!} + \ldots \qquad (\kappa = 1, 2, \ldots, q),$$

where s_κ is a real parameter and $|s_\kappa|$ is small.

We further assume, without loss of generality, that

$$(16) \qquad (P_1', P_2) \neq 0.$$

For, by hypothesis (§ 8), $P_1' \neq 0$. Hence, if $(P_1', P_2) = 0$, then $(P_1', P_3) \neq 0$, since P_3 is independent of P_2. We then can interchange P_2 and P_3.

The inequality (16) implies that corresponding to every point P_1^* on C in the neighbourhood of P_1 there is just *one* point P_2^* on C in the neighbourhood of P_2 such that

$$(17) \qquad (P_1^*, P_2^*) = (P_1, P_2).$$

† This, of course, includes the case when there are none.

For on substituting the series (15) for P_1^* and P_2^*, this equation takes the form

$$\{(P_1', P_2)s_1+(P_1, P_2')s_2\} + \tfrac{1}{2}\{(P_1'', P_2)s_1^2+2(P_1', P_2')s_1s_2+(P_1, P_2'')s_2^2\}+\ldots = 0,$$

where the series converges for all sufficiently small $|s_1|$, $|s_2|$. By (16), this equation can be solved with respect to s_1 as a power series

$$(18) \quad s_1 = -\frac{(P_1, P_2')}{(P_1', P_2)}s_2 - \frac{(P_1'', P_2)(P_1, P_2')^2-2(P_1', P_2')(P_1, P_2')(P_1', P_2)+(P_1, P_2'')(P_1', P_2)^2}{(P_1', P_2)^3}\frac{s_2^2}{2} + \ldots,$$

convergent for sufficiently small $|s_2|$. We write s instead of s_2, and substitute the series (18) for s_2 in the expression (15) for P_1^*. Then we have the parametric representation

$$(19) \quad \begin{cases} P_1^* = P_1+P_1^{(1)}\dfrac{s}{1!}+P_1^{(2)}\dfrac{s^2}{2!}+\ldots, \\ P_2^* = P_2+P_2'\dfrac{s}{1!}+P_2''\dfrac{s^2}{2!}+\ldots, \end{cases}$$

for points P_1^* and P_2^* on C sufficiently near to P_1 and P_2 and satisfying (17). In the first series the coefficient $P_1^{(1)}$ is given by

$$(20) \quad P_1^{(1)} = -\frac{(P_1, P_2')}{(P_1', P_2)}P_1'.$$

$P_1^{(1)}$ has the same direction as P_1', unless $(P_1, P_2')=0$ when $P_1^{(1)}$ becomes the null vector. Then the series for P_1^* may reduce to its first term. This happens if, and only if, the part of C in the neighbourhood of P_2 is a line segment parallel to the vector P_1.

Let u_2, v_2 be the integers associated with P_1 and P_2. For every pair of points P_1^*, P_2^* on C sufficiently near to P_1, P_2 and connected by (17), we form the lattice Λ^*,

$$P^* = \frac{\omega_1 P_1^*+\omega_2 P_2^*}{v_2}, \quad \text{where} \quad \omega_1+u_2\omega_2 \equiv 0 \pmod{v_2}.$$

In particular, let

$$P_\kappa^{**} = \frac{\Omega_{\kappa 1}P_1^*+\Omega_{\kappa 2}P_2^*}{v_2} \quad (\kappa = 3, 4, \ldots, q)$$

be the points of this lattice in the neighbourhood of P_κ. By (19), these points P_κ^{**} can be written as power series in s. From (17),

$$d(\Lambda^*) = d(\Lambda).$$

THEOREM 15. *If $|s|$ is sufficiently small, then for each s at least one of the $q-2$ points $P_3^{**}, P_4^{**}, \ldots, P_q^{**}$ belongs to K.*

Proof. Suppose the assertion to be false; then, for some s, arbitrarily small, none of the points $P_3^{**}, P_4^{**}, \ldots, P_q^{**}$ belongs to K. Hence, for these values of s, the lattice Λ^* contains only the four points $\pm P_1^*$, $\pm P_2^*$ on C and is admissible. From (16), it is not a singular lattice, and so it is not a critical lattice. Hence there is a critical lattice of determinant less than $d(\Lambda^*) = d(\Lambda)$, which is impossible.

We now apply this theorem to the types when no points of (14) belong to Σ. We distinguish two cases, according as $q = 3$ or $q > 3$.

13. *Determination of the regular lattices* (3).

THIRD TYPE ($q = 3$). *The regular lattice Λ has exactly six points $\pm P_1, \pm P_2, \pm P_3$ on C.*

By theorem 15, the point

$$P_3^{**} = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2^*}{v_2} \tag{21}$$

belongs to K for sufficiently small $|s|$. When we substitute the power series (19) for P_1^* and P_2^*, (21) takes the form

$$P_3^{**} = P_3 + P_3^{(1)} \frac{s}{1!} + P_3^{(2)} \frac{s^2}{2!} + \ldots,$$

where, by (20),

$$P_3^{(1)} = \frac{\Omega_{31} P_1^{(1)} + \Omega_{32} P_2'}{v_2} = \frac{-\Omega_{31}(P_1, P_2') P_1' + \Omega_{32}(P_1', P_2) P_2'}{v_2(P_1', P_2)}.$$

Since P_1 and P_2 are independent, $(P_1, P_2) \neq 0$. Hence

$$\Omega_{31} = -\frac{v_2(P_2, P_3)}{(P_1, P_2)}, \quad \Omega_{32} = \frac{v_2(P_1, P_3)}{(P_1, P_2)},$$

since $$P_3 = \frac{\Omega_{31} P_1 + \Omega_{32} P_2}{v_2}.$$

Hence the coefficient $P_3^{(1)}$ can be written as

$$P_3^{(1)} = \frac{(P_2, P_3)(P_1, P_2')P_1' + (P_1, P_3)(P_1', P_2)P_2'}{(P_1, P_2)(P_1', P_2)}.$$

On noting the terms in $P_3^{(1)}$, we have

$$(P^{(1)}, P_3') = \frac{(P_2, P_3)(P_1, P_2')(P_1', P_3') + (P_1, P_3)(P_1', P_2)(P_2', P_3')}{(P_1, P_2)(P_1', P_2)}$$

i.e., with a more symmetrical numerator,

$$(22)\quad (P_3^{(1)}, P_3') = -\frac{(P_1, P_2')(P_2, P_3')(P_3, P_1') + (P_1', P_2)(P_2', P_3)(P_3', P_1)}{(P_1, P_2)(P_1', P_2)}.$$

This follows easily from the well-known identity

$$(Q_1, Q_2)(Q_3, Q_4) = (Q_1, Q_3)(Q_2, Q_4) - (Q_1, Q_4)(Q_2, Q_3).$$

Suppose first that $P_3^{(1)} \neq 0$. For sufficiently small values of $|s|$, P_3^{**} describes a curve passing through P_3; from the series for P_3^{**}, its tangent at P_3 is parallel to the vector $P_3^{(1)}$. Since all points of this curve sufficiently near to P_3 belong to K, its tangent at P_3 coincides with that of C which is parallel to the vector P_3'. Hence we have a vector equation

$$(23)\qquad P_3^{(1)} = \mu P_3',$$

with a scalar μ. Suppose, secondly, that $P_3^{(1)} = 0$; then this formula remains true with $\mu = 0$.

From (23), $$(P_3^{(1)}, P_3') = 0,$$

and so by (22),

$$(24)\qquad (P_1, P_2')(P_2, P_3')(P_3, P_1') + (P_1', P_2)(P_2', P_3)(P_3', P_1) = 0$$

for all regular lattices of the third type.

The argument leading to (23) shows further that the three points P_1, P_2, P_3 satisfy an *inequality* condition arising from terms of the second order and corresponding to the sign of the second derivative in minima problems. I omit this condition since I have not succeeded in expressing it in a simple form similar to (24).

We can now determine all regular lattices of the third type. For given integers v_2, Ω_{31}, Ω_{32}, we have to find all sets of three points P_1, P_2, P_3 on C which satisfy the equations (24) and

$$(25) \qquad \Omega_{31} P_1 + \Omega_{32} P_2 - v_2 P_3 = 0.$$

Since the boundary C consists of a finite number of analytical arcs, (24) and (25) are in general independent and so have only a finite number of solutions; to these correspond only a finite number of possible regular lattices.

It may happen that only two of the three equations given by (24) and (25) are independent. Then one of the three points, say P_1, may be taken arbitrarily on an arc A of C, and then P_2 and P_3, being determined by P_1, lie on other arcs of the boundary C. It is clear from the construction of P_3 that all the lattices Λ so obtained have the same determinant. By using Theorem 12, we can decide which of these lattices are admissible, and then which are critical.

Finally, we note that the three equations given by (24) and (25) never reduce to *one* independent equation. For then P_1, P_2 would have arbitrary positions on C, and so the area of the parallelogram O, P_1, P_1+P_2, P_2 would not be constant.

14. *Determination of the regular lattices* (4).

FOURTH TYPE. *The regular lattice* Λ *has* $2q \geqslant 8$ *points*

$$\pm P_1, \pm P_2, \ldots, \pm P_q$$

on C.

In the equations

$$(26) \qquad \Omega_{\kappa 1} P_1 + \Omega_{\kappa 2} P_2 - v_2 P_\kappa = 0 \quad (\kappa = 3, 4, \ldots, q),$$

the coefficients v_2, $\Omega_{\kappa 1}$, $\Omega_{\kappa 2}$ are bounded integers, by Theorem 8. There are, in general, only a finite number of sets of points $P_1, P_2, \ldots, P_q$ on C satisfying (26), since the number $2q-4$ of conditions in (26) is at least equal to the number q of parameters of the p points. For these points, Theorem 15 gives a necessary inequality condition.

In exceptional cases there may exist an infinity of lattices with at least eight points on C. Since the boundary C consists of a finite number of analytical arcs, this can occur only if P_1 runs over an arc B_1 of C, while $P_2, P_3, \ldots, P_q$, as functions of P_1, describe other arcs of C. The determinants of the lattices through these points depend on the positions of P_1; for a critical lattice, the determinant is a minimum. This leads to the following considerations which might have been applied exactly the same way to find the condition (24) for the third type.

Let Λ be one of these critical lattices, let Λ^* be a neighbouring lattice, and denote by $\pm P_1, \pm P_2, \ldots, \pm P_q$ and $\pm P_1^*, \pm P_2^*, \ldots, \pm P_q^*$ the points on C of these two lattices. Then

$$P_3 = \frac{\Omega_{31} P_1 + \Omega_{32} P_2}{v_2}, \quad P_3^* = \frac{\Omega_{31} P_1^* + \Omega_{32} P_2^*}{v_2}. \tag{27}$$

By the last paragraph,

$$\Omega_{31} = -\frac{v_2(P_2, P_3)}{(P_1, P_2)}, \quad \Omega_{32} = \frac{v_2(P_1, P_3)}{(P_1, P_2)}.$$

The equations (27) can then be written as

$$(P_2, P_3) P_1 + (P_3, P_1) P_2 + (P_1, P_2) P_3 = 0, \tag{28}$$

$$(P_2, P_3) P_1^* + (P_3, P_1) P_2^* + (P_1, P_2) P_3^* = 0. \tag{29}$$

On substituting the power series

$$P_\kappa^* = P_\kappa + P_\kappa' \frac{s_\kappa}{1!} + \ldots \qquad (\kappa = 1, 2, 3).$$

in (29), we get by (28) the vector equation

$$Q \equiv (P_2, P_3) P_1' s_1 + (P_3, P_1) P_2' s_2 + (P_1, P_2) P_3' s_3 + \ldots = 0,$$

where the dots denote terms in s_1, s_2, s_3 of the second and higher degrees. Since $Q \equiv 0$, $(Q, P_3') = 0$, *i.e.*

$$(P_2, P_3)(P_1', P_3') s_1 + (P_3, P_1)(P_2', P_3') s_2 + \ldots = 0. \tag{30}$$

Suppose first that (P_1', P_3') and (P_2', P_3') are not both zero. Let, *e.g.*, $(P_2', P_3') \neq 0$; then the equation (30) can be solved for s_2 as a power series in s_1,

$$s_2 = -\frac{(P_2, P_3)(P_1', P_3')}{(P_3, P_1)(P_2', P_3')} s_1 + \ldots.$$

The lattice Λ^* contains therefore points P_1^*, P_2^* given by

$$P_1^* = P_1 + P_1' s_1 + \ldots, \quad P_2^* = P_2 + \frac{(P_2, P_3)(P_3', P_1')}{(P_3, P_1)(P_2', P_3')} P_2' s_1 + \ldots.$$

By hypothesis, Λ is critical; hence

$$(P_1^*, P_2^*) \geqslant (P_1, P_2),$$

that is
$$c' s_1 + c'' \frac{s_1^2}{2!} + \ldots \geqslant 0,$$

where
$$c' = \frac{(P_2, P_3)(P_3', P_1')(P_1, P_2')}{(P_3, P_1)(P_2', P_3')} + (P_1', P_2).$$

This inequality must hold for sufficiently small $|s_1|$. Hence $c' = 0$ or

$$(P_2, P_3)(P_1, P_2')(P_1', P_3') + (P_1, P_3)(P_1', P_2)(P_2', P_3') = 0,$$

a condition which clearly also holds if $(P_1', P_3') = (P_2', P_3') = 0$.

Hence the points P_1, P_2, P_3 satisfy equation (24) of the last paragraph. Similar equations hold with P_3 replaced by $P_4, \ldots, P_q$.

Therefore, as in § 13, the infinity of lattices Λ^* have all the same determinant.

Such cases actually occur. For, if $R > 0$ is sufficiently large, then the simple star domain

$$|xy| \leqslant 1, \quad x^2 + y^2 \leqslant R^2$$

has an infinity of regular lattices, and these, by suitable choice of R, will contain an arbitrary large number of lattice points on C.

15. *Conclusion.*

Let us now collect our results for simple star domains. We have developed a method of constructing a finite or infinite set of lattices containing all critical lattices. This set has the remarkable property that the determinants of its elements assume only a *finite* number of different values, the smallest of which is the minimum $\Delta(K)$.

The members of the set are:

1. *The singular lattices with at least one point in* Σ. (§ 9.)

2. *The singular lattices with no points in* Σ. These are obtained by solving the equations (12). (§ 9.)

3. *The regular lattices with either one point in* Σ *or with two independent points in* Σ. (§ 11.)

4. *The regular lattices with no points in* Σ *and having exactly six points on* C. These are obtained by solving the equations (24) and (25). (§ 13.)

5. *The regular lattices with no points in Σ and having at least eight points on C.* (§ 14.)

6. The set may further contain non-critical lattices either satisfying the conditions for stationary solutions, or having a certain number of points in Σ or on C, or possessing tac-lines of a certain direction. (§§ 9, 11, 13, 14.)

The method of the paper reduces the problem of finding $\Delta(K)$ to a finite number of elementary questions no longer involving number theory, namely, the solving of sets of equations in a small number of unknowns. I show in further parts of this paper that the method is practicable.

16. *Unsymmetrical star domains.*

We have considered so far only star domains K which are symmetrical in the origin. This restriction can now be easily removed.

Let H be a closed set in the (x, y) plane containing the origin as an inner point and bounded by a Jordan curve J of the following kind:

(*a*) *J consists of a finite number of analytical arcs.*

(*b*) *Every radius vector from the origin intersects J in a single point.*

If H is not symmetrical in O, then denote by K the set of all points P for which at least one of the two points P, $-P$ belongs to H. Then K obviously is a simple star domain (§ 8).

As in §§ 3 and 4, we define H-admissible lattices, the lower bound $\Delta(H)$, and critical lattices with respect to H. Then a lattice which contains at least one point $P \neq 0$, and so both points P and $-P$ of K, contains also at least one point $\pm P$ of H, and *vice versa*. Therefore the H-admissible lattices are identical with the K-admissible lattices, the critical lattices of H are identical with those of K, and

$$\Delta(H) = \Delta(K). \tag{31}$$

Hence there is no difference between the lattice problem for H and the lattice problem for K.

17. *The use of sub-domains.*

By Theorem 2,

$$\Delta(K') \leqslant \Delta(K),$$

when the star domain K' is contained in the star domain K. Hence if there exists at least one critical lattice of K' which is admissible and therefore critical with respect to K, then

$$\Delta(K') = \Delta(K).$$

This remark enables us to derive from known results new ones for larger domains. For instance, since

$$\Delta(\Gamma_r) = \sqrt{\tfrac{3}{4}} \cdot r^2,$$

we find without difficulty for h, a regular hexagon or a regular star hexagon of side a,

$$\Delta(h) = \frac{3\sqrt{3}}{8} a^2.$$

This method of extension applies in particular to *infinite star domains.* These are defined in the following way. Let S be an arbitrary point set in the (x, y)-plane, and $S(r)$ the subset of all points of S which lie in the circle Γ_r. Then S is called an infinite star domain if it is not bounded, and if $S(r)$ for every $r > 0$ is a simple star domain†.

There need not be admissible lattices with respect to an infinite star domain. If there are such lattices, then we can again define $\Delta(S)$. It does not now follow that critical lattices exist. But for certain infinite star domains, critical lattices may exist, and these may be already critical for all subsets $S(r)$ where r is sufficiently large. Then, as above, we can consider $S(r)$ and find the value of $\Delta(S)$. As examples, I mention the domains

$$|F(x, y)| \leqslant 1,$$

where $F(x, y)$ is a cubic binary form; these were considered for the first time by L. J. Mordell‡.

Addition (*March* 1945). In the three years since this paper was written, I have extended the theory of star domains to bounded or unbounded star bodies in any number of dimensions, and have proved a number of existence theorems. I have also evaluated $\Delta(K)$ for some special domains, *e.g.* the domain

$$|x+y| \leqslant \sqrt{5}, \quad |xy| \leqslant 1.$$

† By the preceding paragraph, it suffices to consider infinite domains which are *symmetrical* in the origin.

‡ *Proc. London Math. Soc.* (2), 48 (1943), 198–228.

These results are contained in the following papers:

(1) *Journal London Math. Soc.* 18 (1943), 233–238.

(2) *Proc. Cambridge Phil. Soc.*, 40 (1944), 107–120.

(3) "On lattice points in a cylinder", *Quarterly Journal of Math.*, 17 (1946), 16–18.

(4) "On lattice points in n-dimensional star bodies, I, II".

My student, Mrs. K. Ollerenshaw, D.Phil., has evaluated $\Delta(K)$ for some very interesting classes of star domains; see *Journal London Math. Soc.*, 19 (1944), 178–184; *Proc. Cambridge Phil. Soc.*, 41 (1945), 77–96; and a paper to appear in the *Quarterly Journal of Math.* One of the domains considered by her possesses a continuous infinity of singular lattices.

The University,
Manchester, 13.