ON LATTICE POINTS IN A CYLINDER

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DENOTE by x_1, x_2, x_3 rectangular coordinates in three-dimensional space, and by K a convex body with the origin $O = (0, 0, 0)$ as its centre. A lattice Λ,

$$x_h = \sum_{k=1}^{3} \alpha_{hk} u_k \quad (h = 1, 2, 3;\ u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$

say of determinant

$$d(\Lambda) = \left| |\alpha_{hk}|_{h,k=1,2,3} \right|,$$

is called *K-admissible* if O is its only point which is an *inner* point of K. Let $\Delta(K)$ be the lower bound of $d(\Lambda)$ extended over all K-admissible lattices. Then $\Delta(K) > 0$, and there is at least one *critical lattice*, i.e., a K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$.*

Minkowski's theorem on convex bodies may be expressed as

$$\Delta(K) \geqslant \tfrac{1}{8} V,$$

where V is the volume of K. In general, the sign '$>$' holds in this inequality, and so the problem arises of finding the exact value of $\Delta(K)$. Minkowski himself solved this problem for the cube, the octahedron, and the sphere. In this note I solve it for the cylinder

$$K: \qquad x_1^2 + x_2^2 \leqslant 1, \quad -1 \leqslant x_3 \leqslant +1,$$

by proving that†
$$\Delta(K) = \tfrac{1}{2}\sqrt{3}, \tag{1}$$

a result which surprisingly has escaped notice.

That $\Delta(K) \leqslant \frac{1}{2}\sqrt{3}$ is nearly trivial, because the following lattices of determinant $\frac{1}{2}\sqrt{3}$ are evidently K-admissible:

(i) The lattices Λ_1 derived from the particular lattice

$$x_1 = u_1 + \tfrac{1}{2} u_2 + \alpha u_3, \quad x_2 = \tfrac{1}{2}\sqrt{3}\, u_2 + \beta u_3, \quad x_3 = u_3 \quad (\alpha, \beta \text{ arbitrary})$$

by any rotation about the x_3-axis;

* For the two-dimensional case of these rather obvious statements see my note, *J. of London Math. Soc.* 17 (1942), 130–3.

† The same proof shows that $\Delta(K) = \frac{1}{2}\sqrt{3}$, where K is the n-dimensional convex body $x_1^2 + x_2^2 \leqslant 1$, $|x_3| \leqslant 1, \ldots, |x_n| \leqslant 1$ $(n \geqslant 2)$.

(ii) The lattices Λ_2 derived from the particular lattice

$$x_1 = u_1+\tfrac{1}{2}u_2, \quad x_2 = \tfrac{1}{2}\sqrt{3}\,u_2, \quad x_3 = \alpha u_1+\beta u_2+u_3 \quad (\alpha, \beta \text{ arbitrary})$$

by any rotation about the x_3-axis.*

It suffices therefore to show that

$$\Delta(K) \geqslant \tfrac{1}{2}\sqrt{3}, \tag{2}$$

in order to prove the assertion (1).

I use the following lemmas.

LEMMA 1. *Let π be a plane convex polygon of area A with angles not greater than* 120°, *and let $C_1, C_2,..., C_s$ be non-overlapping circles of radius r contained in π. Then*

$$s \leqslant \frac{A}{r^2\sqrt{12}}.\dagger$$

LEMMA 2. *Let n be a positive integer, and let W be the cube*

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad |x_3| \leqslant n.$$

Let further $Z_1, Z_2,..., Z_t$ be non-overlapping circular cylinders of radius $\frac{1}{2}$ *and height* 1, *all contained in W with their axes parallel to the x_3-axis. Then*

$$t \leqslant \frac{16}{\sqrt{3}}n^3.$$

Proof. Denote by x any number in the interval $-n \leqslant x \leqslant n$. The plane $x_3 = x$ intersects the cylinders $Z_1, Z_2,..., Z_t$ in a certain point set $J(x)$, say of area $Q(x)$. Then the integral $\int_{-n}^{+n} Q(x)\,dx$ equals the total volume of the cylinders $Z_1, Z_2,..., Z_t$, and so

$$\int_{-n}^{+n} Q(x)\,dx = \tfrac{1}{4}\pi t. \tag{3}$$

Now there are at most $2t$ different values of x for which the plane $x_3 = x$ contains either the base or the top of one of these cylinders; let x be different from these exceptional values. Then $J(x)$ consists of a finite number, say s, of circles of radius $\frac{1}{2}$; no two of these circles overlap, and all lie inside the square

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad x_3 = x$$

* The lattices Λ_1 and Λ_2 are the only critical lattices of K, as can be proved. One can further show that, if H is any convex body symmetrical in O which is contained in, but different from K, then $\Delta(H) < \Delta(K)$.

† For a proof see the note 'On the densest packing of circles' by B. Segre and myself, *American Math. Monthly*, 51 (1944), 261–70.

of area $A = 4n^2$. Hence, by Lemma 1,

$$s \leqslant \frac{4n^2}{(\frac{1}{2})^2\sqrt{12}} = \frac{8}{\sqrt{3}}n^2.$$

Then
$$Q(x) = \tfrac{1}{4}\pi s \leqslant \frac{2\pi}{\sqrt{3}}n^2,$$

and so, by (3),

$$\tfrac{1}{4}\pi t = \int_{-n}^{+n} Q(x)\,dx \leqslant \int_{-n}^{+n} \frac{2\pi}{\sqrt{3}}n^2\,dx = \frac{4\pi}{\sqrt{3}}n^3,$$

whence
$$t \leqslant \frac{16}{\sqrt{3}}n^3.$$

Proof of (2). Put

$$F(x_1, x_2, x_3) = \max(|\sqrt{(x_1^2+x_2^2)}|,\ |x_3|),$$

so that the cylinder K consists of all points satisfying $F(x_1, x_2, x_3) \leqslant 1$ Denote by Λ any K-admissible lattice. Then at every point $X = (x_1^0, x_2^0, x_3^0)$ of Λ place a cylinder

$Z(X)$:
$$F(x_1-x_1^0, x_2-x_2^0, x_3-x_3^0) \leqslant \tfrac{1}{2}$$

of half the linear dimensions of K, and with its centre at R and axis parallel to the x_3-axis. Since Λ is K-admissible and since K is convex, no two of these cylinders overlap.*

Let n now be a large positive integer. Since every lattice parallelepiped is of volume $d(\Lambda)$, the cube

$$|x_1| \leqslant n-\tfrac{1}{2}, \qquad |x_2| \leqslant n-\tfrac{1}{2}, \qquad |x_3| \leqslant n-\tfrac{1}{2}$$

contains
$$\frac{8n^3}{d(\Lambda)} + O(n^2)$$

points X of Λ; at least as many cylinders $Z(X)$ lie therefore in the cube

$$|x_1| \leqslant n, \qquad |x_2| \leqslant n, \qquad |x_3| \leqslant n.$$

Thus, by Lemma 2,

$$\frac{8n^3}{d(\Lambda)} + O(n^2) \leqslant \frac{16}{\sqrt{3}}n^3,$$

whence

$$d(\Lambda) \geqslant \tfrac{1}{2}\sqrt{3}-o(1), \quad \text{i.e.} \quad d(\Lambda) \geqslant \tfrac{1}{2}\sqrt{3}, \qquad \Delta(K) \geqslant \tfrac{1}{2}\sqrt{3},$$

as asserted.

* Minkowski, *Geometrie der Zahlen*, 74.