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On lattice points in n-dimensional star bodies

I. Existence theorems

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Let $F(X) = F(x_1, \ldots, x_n)$ be a continuous non-negative function of X satisfying $F(tX) = |t| F(X)$ for all real numbers t. The set K in n-dimensional Euclidean space R_n defined by $F(X) \leqslant 1$ is called a star body. The author studies the lattices Λ in R_n which are of minimum determinant and have no point except $(0, \ldots, 0)$ inside K. He investigates how many points of such lattices lie on, or near to, the boundary of K, and considers in detail the case when K admits an infinite group of linear transformations into itself.

Introduction

Let K be an arbitrary bounded or unbounded point set in the n-dimensional Euclidean space R_n of all points

$$X = (x_1, x_2, \ldots, x_n) \qquad (x_1, x_2, \ldots, x_n \text{ real numbers}).$$

A point lattice Λ,

$$x_h = \sum_{k=1}^{n} a_{hk} u_k \quad (h = 1, 2, \ldots, n,\ u_1, u_2, \ldots, u_n = 0, \pm 1, \pm 2, \ldots),$$

in R_n of determinant $\qquad d(\Lambda) = \left| \, |a_{hk}|_{h,k=1,2,\ldots,n} \right|$

is called K-admissible if no point P of Λ, except possibly the origin $O = (0, 0, \ldots, 0)$, is an inner point of K. (P is an inner point of K if there is an n-dimensional sphere with centre at P and contained in K.) The minimum determinant $\Delta(K)$ of K is defined as the lower bound of $d(\Lambda)$ extended over all K-admissible lattices. This function $\Delta(K)$ depends on K in a very complicated way and is, in general, not a continuous function of K. A K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$ is called a critical lattice of K; such critical lattices exist, for instance, if K contains O as an inner point and has at least one admissible lattice.

Minkowski proved in his classical theorem that if K is a convex body with centre at O, then

$$2^n \Delta(K) \geq V(K),$$

where $V(K)$ is the volume of K. He further gave a finite algorism for obtaining $\Delta(K)$ and the critical lattices of K if K is such a convex body and $n = 2$ or $n = 3$, or if K is of a certain type with $n = 4$ (Minkowski 1907, 1911).

Minkowski also considered another more general class of point sets, the star bodies (*Strahlenkörper*). These are point sets defined by an inequality

$$F(X) \leq 1,$$

where $F(X) = F(x_1, \ldots, x_n)$ is a continuous function of X such that

$$F(X) \geq 0 \quad \text{for all points } X,$$

$$F(tx_1, \ldots, tx_n) = |t| \, F(x_1, \ldots, x_n) \quad \text{for real } t.$$

The functional equation implies that K is symmetrical in O. This restriction is not made by Minkowski, but is in no way essential. He found (1911) for such point sets that

$$2\zeta(n)\, \Delta(K) \leq V(K),$$

but his proof was never published. Recently, Hlawka (1943) gave a very ingenious proof based on the theory of multiple integrals, and I found a geometrical proof (Mahler 1944) for a slightly less exact inequality.*

New progress was made in the years from 1938 onwards when important special examples of star bodies in two or three dimensions were investigated by Davenport (1938, 1939 and 1944) and Mordell (1942, 1943, 1944, and the general method 1945). In 1941 Mordell discovered a method for dealing with a certain important class of such problems. This work led me to ask myself whether Minkowski's method of evaluating $\Delta(K)$ when K is convex (Minkowski 1907, 1911) could be extended to arbitrary bounded star bodies. I succeeded in answering this question in the affirmative, and found an algorism for the evaluation of $\Delta(K)$ if K is two-dimensional and bounded; and I applied this method to a few special cases.

In the present paper, the aim is not to consider further special examples of star bodies, but rather to lay the foundations of a general theory of bounded or unbounded n-dimensional star bodies and their critical lattices.

In this first part, I begin by proving that if the star body K,

$$F(X) \leq 1,$$

* *Addition, May* 1946. A beautiful new proof of the Minkowski-Hlawka theorem was recently given by C. L. Siegel, *Ann. Math.* **46** (1945), 340–347.

has at least one admissible lattice, then K also admits at least one critical lattice. The points of such a critical lattice Λ on, or in the neighbourhood of, the boundary C of K are next studied. If K is bounded, then at least $2n$ points of Λ lie on C, as is almost obvious; an example is constructed in which this lower bound is attained. If K is not bounded, then Λ need not have a single point on C, as is also proved by means of an example. It is then easily proved that to every $\epsilon > 0$ there is at least one point P of Λ such that

$$1 \leqslant F(P) < 1 + \epsilon;$$

however, it remains an open question whether there are always n independent points of Λ with this property.

From § 14 onwards, unbounded star bodies are considered with an infinite group Γ of linear transformations into themselves; many of the most interesting lattice-point problems are of this type. Three different assumptions about Γ are made and applied to the study of the critical lattices. Then three general classes of star bodies are found with the following three properties respectively: (*a*) At least one critical lattice of K has a point on C (theorem 21). (*b*) For every $\epsilon > 0$, every critical lattice Λ of K contains an infinity of points P satisfying

$$1 \leqslant F(P) < 1 + \epsilon$$

(theorem 23). (*c*) For every $\epsilon > 0$, every critical lattice Λ of K contains n independent points $P_1, \ldots, P_n$ satisfying

$$1 \leqslant F(P_g) < 1 + \epsilon \quad (g = 1, 2, \ldots, n)$$

(theorem 25). The simplest example of an n-dimensional star body with all three properties (*a*), (*b*), (*c*) is that defined by the inequality

$$|x_1 x_2 \ldots x_n| \leqslant 1.$$

In the second part of this paper which is appearing in the *Proc. Royal Acad. Amsterdam*, I intend to study certain types of star bodies K according as they contain, or do not contain, smaller star bodies K' such that

$$\Delta(K') = \Delta(K).$$

1. Notation

The following notation is used in this paper:

If $x_1, x_2, \ldots, x_n$ $(n \geqslant 2)$ are real numbers, then

$$X = (x_1, x_2, \ldots, x_n) \tag{1·1}$$

is the point in n-dimensional Euclidean space R_n with rectangular co-ordinates $x_1, x_2, \ldots, x_n$. The non-negative number

$$|X| = +(x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \tag{1·2}$$

is called the distance of X from the origin $O = (0, 0, \ldots, 0)$. If

$$X_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}), \ldots, X_r = (x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}) \tag{1·3}$$

are any points in R_n, and $\lambda_1, \ldots, \lambda_r$ are any real numbers, then

$$\lambda_1 X_1 + \ldots + \lambda_r X_r$$

is written for the point

$$(\lambda_1 x_1^{(1)} + \ldots + \lambda_r x_1^{(r)}, \lambda_1 x_2^{(1)} + \ldots + \lambda_r x_2^{(r)}, \ldots, \lambda_1 x_n^{(1)} + \ldots + \lambda_r x_n^{(r)}).$$

The determinant of n points

$$X_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}), \ldots, X_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}) \tag{1·4}$$

is denoted by

$$\{X_1, X_2, \ldots, X_n\} = \begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \ldots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \ldots & x_n^{(2)} \\ \vdots & \vdots & & \vdots \\ x_1^{(n)} & x_2^{(n)} & \ldots & x_n^{(n)} \end{vmatrix}. \tag{1·5}$$

The points are called independent, if this determinant does not vanish.

The set Λ of all points

$$X = u_1 X_1 + \ldots + u_n X_n, \quad \text{where} \quad u_1, \ldots, u_n = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

is called a lattice if its determinant

$$d(\Lambda) = |\{X_1, X_2, \ldots, X_n\}| \tag{1·6}$$

is not zero; then $X_1, X_2, \ldots, X_n$ are said to form a basis of Λ. Any n points $Y_1, Y_2, \ldots, Y_n$ of Λ form a basis of this lattice if and only if

$$\{Y_1, Y_2, \ldots, Y_n\} = \pm d(\Lambda). \tag{1·7}$$

If $P, Q, R, \ldots$ are points of Λ, then $\Lambda - [P, Q, R, \ldots]$ denotes the set of all points of Λ different from $P, Q, R, \ldots$.

2. The reduced basis of a lattice

Theorem 1. *There exists a constant $\gamma_n > 0$ depending only on the dimension n of R_n, with the following property: every lattice Λ in R_n has a reduced basis, i.e. a basis $Y_1, Y_2, \ldots, Y_n$ for which*

$$|Y_1||Y_2| \ldots |Y_n| \leqslant \gamma_n d(\Lambda). \tag{2·1}$$

Proof. Let

$$X_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}), \ldots, X_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}) \tag{2·2}$$

be any basis of Λ. Then

$$\Phi(u_1, \ldots, u_n) = \sum_{g=1}^{n} (x_g^{(1)} u_1 + \ldots + x_g^{(n)} u_n)^2 = |u_1 X_1 + \ldots + u_n X_n|^2 \tag{2·3}$$

is a positive definite quadratic form of discriminant

$$d(\Lambda)^2 = \{X_1, X_2, \ldots, X_n\}^2. \tag{2·4}$$

There exists a linear unimodular substitution

$$u_g = \sum_{h=1}^{n} a_{gh} v_h, \quad \text{where} \quad g = 1, 2, \ldots, n, \tag{2·5}$$

with integral coefficients by which Φ is changed into a new form

$$\Phi(u_1, \ldots, u_n) = \Psi(v_1, \ldots, v_n) = \sum_{g=1}^{n} (y_g^{(1)} v_1 + \ldots + y_g^{(n)} v_n)^2, \tag{2·6}$$

which is reduced according to Minkowski (1911). Hence by his theorem

$$\Psi(1, 0, \ldots, 0)\, \Psi(0, 1, \ldots, 0) \ldots \Psi(0, 0, \ldots, 1) \leqslant \gamma_n^2 d(\Lambda)^2, \tag{2·7}$$

where $\gamma_n > 0$ depends only on n. The n points

$$Y_1 = (y_1^{(1)}, y_2^{(1)}, \ldots, y_n^{(1)}), \ldots, Y_n = (y_1^{(n)}, y_2^{(n)}, \ldots, y_n^{(n)}) \tag{2·8}$$

form a basis of Λ since

$$\{Y_1, \ldots, Y_n\} = \{X_1, \ldots, X_n\} = \pm d(\Lambda). \tag{2·9}$$

Moreover,

$$\Psi(1, 0, \ldots, 0) = |Y_1|^2,\ \Psi(0, 1, \ldots, 0) = |Y_2|^2,\ \ldots,\ \Psi(0, 0, \ldots, 1) = |Y_n|^2, \tag{2·10}$$

whence the assertion.

Theorem I may also be proved by the reduction method of Hermite (1905), which has the advantage that the proof of the product formula for the Ψ's is of an elementary character.

3. The convergence theorem

Definition 1. *An infinite sequence of lattices*

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is called bounded, if there exist two positive numbers c_1, c_2 such that

$$d(\Lambda_r) \leqslant c_1 \quad \textit{for} \quad r = 1, 2, 3, \ldots; \tag{3·1}$$

$$|X| \geqslant c_2 \quad \textit{for all points } X \neq O \textit{ of } \Lambda_r, \quad \textit{when} \quad r = 1, 2, 3, \ldots. \tag{3·2}$$

Definition 2. *An infinite sequence of lattices*

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is said to converge, and to have as its limit the lattice Λ, if there exist reduced bases

$$Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \textit{ of } \Lambda_r \quad \textit{for} \quad r = 1, 2, 3, \ldots \tag{3·3}$$

and a basis $Y_1, Y_2, \ldots, Y_n$ *of* Λ,

such that $\lim_{r\to\infty} |Y_g^{(r)} - Y_g| = 0, \quad \textit{where} \quad g = 1, 2, \ldots, n. \qquad (3·4)$

This definition implies that the points of Λ_r in any finite region independent of r tend to the points of Λ, as r tends to infinity.

From these two definitions is derived the following theorem which is fundamental for the study of star bodies:

THEOREM 2. *Every bounded infinite sequence of lattices contains a convergent infinite subsequence.*

Proof. Let $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ be any bounded sequence, and let $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$ be a reduced basis of Λ_r for $r = 1, 2, 3, \ldots$, then from definition 1,

$$d(\Lambda_r) \leqslant c_1, \quad |Y_g^{(r)}| \geqslant c_2, \quad \text{where} \quad g = 1, 2, \ldots, n \text{ and } r = 1, 2, 3, \ldots, \tag{3·5}$$

and from theorem 1,

$$|Y_1^{(r)}|\,|Y_2^{(r)}| \ldots |Y_n^{(r)}| \leqslant \gamma_n d(\Lambda_r), \quad \text{where} \quad r = 1, 2, 3, \ldots, \tag{3·6}$$

hence

$$|Y_g^{(r)}| \leqslant \gamma_n c_1 c_2^{-(n-1)}, \quad \text{where} \quad g = 1, 2, \ldots, n \text{ and } r = 1, 2, 3, \ldots. \tag{3·7}$$

All co-ordinates of the basis points $Y_g^{(r)}$ ($g = 1, 2, \ldots, n$; $r = 1, 2, 3, \ldots$) are therefore bounded, and so there exists an infinite sequence of indices

$$r_1, r_2, r_3, \ldots,$$

and a set of n points

$$Y_1, Y_2, \ldots, Y_n,$$

such that

$$\lim_{k\to\infty} |Y_g^{(r_k)} - Y_g| = 0, \quad \text{where} \quad g = 1, 2, \ldots, n, \tag{3·8}$$

whence

$$\lim_{k\to\infty} d(\Lambda_{r_k}) = \lim_{k\to\infty} |\{Y_1^{(r_k)}, Y_2^{(r_k)}, \ldots, Y_n^{(r_k)}\}| = |\{Y_1, Y_2, \ldots, Y_n\}|. \tag{3·9}$$

Further, from

$$\gamma_n d(\Lambda_{r_k}) \geqslant |Y_1^{(r_k)}|\,|Y_2^{(r_k)}| \ldots |Y_n^{(r_k)}| \geqslant c_2^n, \tag{3·10}$$

and

$$d(\Lambda_{r_k}) \geqslant \gamma_n^{-1} c_2^n, \tag{3·11}$$

it is deduced that

$$|\{Y_1, Y_2, \ldots, Y_n\}| \geqslant \gamma_n^{-1} c_2^n > 0, \tag{3·12}$$

and so the lattice Λ of basis $Y_1, Y_2, \ldots, Y_n$ satisfies the assertion.

4. DISTANCE FUNCTIONS AND STAR BODIES

DEFINITION 3. *A function*

$$F(X) = F(x_1, x_2, \ldots, x_n) \tag{4·1}$$

of the point $X = (x_1, x_2, \ldots, x_n)$ *in* R_n *is called a distance function if it satisfies the following conditions:*

(a) $F(X) \geqslant 0$ *for all points, and* $F(X) > 0$ *for at least one point;*

(b) $F(tX) = |t|\,F(X)$ *for all points* X *and all real numbers* t; *hence*

$$F(-X) = F(X) \quad \text{and} \quad F(O) = 0;$$

(c) $F(X)$ *is a continuous function of* X.

Definition 4. *The set K of all points X satisfying $F(X) \leqslant 1$ is called the star body of distance function $F(X)$; the subset C of all points of K with $F(X) = 1$ is called the boundary of K.*

It is evident that a star body K has the following properties:

(A) If X belongs to K, then tX, where $-1 \leqslant t \leqslant 1$, also belongs to K.

(B) The limit point of a convergent sequence of points of K also belongs to K.

(C) The origin O is an inner point of K; i.e. there exists a positive number ρ such that all points of the sphere $|X| \leqslant \rho$ belong to K.

For since $F(X)$ is continuous, it assumes on the sphere $|X| = 1$ a maximum value, say $1/\rho$. Then $F(X)|X|^{-1} \leqslant 1/\rho$ for all $X \neq O$, whence $F(X) \leqslant 1$, if X is a point of the sphere $|X| \leqslant \rho$.

Theorem 3. *The star body K is bounded if and only if*

$$F(X) > 0 \text{ for all points } X \neq O.$$

Proof. As a continuous function, $F(X)$ assumes on the sphere $|X| = 1$ a minimum, say μ. If $\mu = 0$, then $F(X)$ vanishes at a point $X \neq O$, and so it vanishes at all points of the line through O and X; hence K is not bounded. If, however, $\mu = 1/P > 0$, then $F(X)|X|^{-1} \geqslant 1/P$ for all $X \neq O$, hence $|X| \leqslant P$ if $F(X) \leqslant 1$, and so K is bounded.

5. The two types of star bodies

Definition 5. *The lattice Λ is called K-admissible if $\Lambda - [O]$ contains no inner points of K.*

Definition 6. *The star body K is called of the finite type if there exists at least one K-admissible lattice; it is called of the infinite type if no such lattice exists.*

Theorem 4. *Every bounded star body is of the finite type.*

Proof. Let $P > 0$ be a number such that $|X| \leqslant P$ for all points of K, and denote by Λ the lattice of basis

$$X_1 = (P, 0, \ldots, 0),\ X_2 = (0, P, \ldots, 0),\ \ldots,\ X_n = (0, 0, \ldots, P). \tag{5·1}$$

Then $|X| \geqslant P$ for all points $X \neq O$ of Λ; hence Λ is K-admissible.

Theorem 5. *Unbounded star bodies exist of the finite type, and also of the infinite type.*

Proof. (1) The star body K of distance function

$$F(X) = |x_1 x_2 \ldots x_n|^{1/n} \tag{5·2}$$

is not bounded. To show that K is of the finite type, denote by $\mathfrak{K}$ any totally real algebraic field of degree n, by

$$\omega_1^{(g)}, \omega_2^{(g)}, \ldots, \omega_n^{(g)}, \quad \text{where} \quad g = 1, 2, \ldots, n,$$

conjugate integral bases of the n fields $\mathfrak{K}^{(1)}, \mathfrak{K}^{(2)}, \ldots, \mathfrak{K}^{(n)}$ conjugate to $\mathfrak{K}$, and by Λ the lattice of basis

$$X_h = (\omega_h^{(1)}, \omega_h^{(2)}, \ldots, \omega_h^{(n)}), \quad \text{where} \quad h = 1, 2, \ldots, n. \tag{5·3}$$

Then, except for the sign, $F(X)$ is the norm of an integer $\alpha \neq 0$ in $\mathfrak{K}$ if X lies in $\Lambda - [O]$; hence $F(X) \geqslant 1$ for all lattice points $X \neq O$.

(2) The star body of distance function

$$F(X) = |\, x_1^2 x_2 \ldots x_n \,|^{1/(n+1)} \tag{5·4}$$

likewise is not bounded, but it is of the infinite type. For let Λ be any lattice, and denote by $t_1, t_2, \ldots, t_n$, n positive numbers of product $d(\Lambda)$. By Minkowski's theorem on linear forms, there exists a point $X = (x_1, x_2, \ldots, x_n) \neq O$ of Λ such that

$$|\, x_1 \,| \leqslant t_1, \; |\, x_2 \,| \leqslant t_2, \; \ldots, \; |\, x_n \,| \leqslant t_n, \tag{5·5}$$

hence

$$0 < F(X) \leqslant \{t_1 d(\Lambda)\}^{1/(n+1)}. \tag{5·6}$$

If it be assumed now that $t_1 < d(\Lambda)^{-1}$, then X is an inner point of K. Therefore Λ is not K-admissible.

Unless otherwise stated, all star bodies considered are from now on assumed to be of the finite type.

6. The determinant of a star body

Let K: $F(X) \leqslant 1$, be a star body of the finite type. By definition 6, the set $\Lambda(K)$ of all K-admissible lattices is not empty. Hence the lower bound

$$\Delta(K) = \text{l.b.}\, d(\Lambda) \tag{6·1}$$

extended over all elements of $\Lambda(K)$, exists; $\Delta(K)$ is called the determinant of K. For star bodies K of the infinite type, put $\Delta(K) = \infty$.

Theorem 6. *The determinant of a star body is positive.*

Proof. By the property (C) of a star body (§ 4), K contains the sphere $|\, X \,| \leqslant \rho$, hence also the cube

$$\max(|\, x_1 \,|, \; |\, x_2 \,|, \; \ldots, \; |\, x_n \,|) \leqslant \rho n^{-\frac{1}{2}}. \tag{6·2}$$

By Minkowski's theorem on linear forms, every lattice of determinant

$$d(\Lambda) < \rho^n n^{-\frac{1}{2}n}$$

contains an inner point $X \neq O$ of this cube, i.e. of K, and so such a lattice cannot be K-admissible. Hence, for every K-admissible lattice Λ,

$$d(\Lambda) \geqslant \rho^n n^{-\frac{1}{2}n}, \tag{6·3}$$

whence

$$\Delta(K) \geqslant \rho^n n^{-\frac{1}{2}n} > 0. \tag{6·4}$$

Theorem 7. *If the star body H is contained in the star body K, then*

$$\Delta(H) \leqslant \Delta(K). \tag{6·5}$$

Proof. Every K-admissible lattice is also H-admissible.

7. The existence of a critical lattice

Definition 7. *The lattice Λ is called a critical lattice of K if it is K-admissible and $d(\Lambda) = \Delta(K)$.*

The following theorem is fundamental for the theory:

Theorem 8. *Every star body of the finite type possesses at least one critical lattice.*

Proof. From the definition of $\Delta(K)$, there exists an infinite sequence of K-admissible lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots,$$

not necessarily all different, such that

$$\lim_{r\to\infty} d(\Lambda_r) = \Delta(K); \tag{7·1}$$

it may be assumed further, without loss of generality, that

$$d(\Lambda_r) \leqslant 2\Delta(K), \quad \text{where} \quad r = 1, 2, 3, \ldots. \tag{7·2}$$

Moreover, since the sphere $|X| \leqslant \rho$ is contained in K,

$$|X| \geqslant \rho \quad \text{for all points } X \neq O \text{ of } \Lambda_r, \quad \text{where} \quad r = 1, 2, 3, \ldots. \tag{7·3}$$

From (7·2) and (7·3) the sequence $\{\Lambda_r\}$ is bounded, and hence, from theorem 2, it contains a convergent infinite subsequence

$$\Lambda_{r_1}, \Lambda_{r_2}, \Lambda_{r_3}, \ldots,$$

say of limit Λ. Denote by $Y_1^{(r_k)}, Y_2^{(r_k)}, \ldots, Y_n^{(r_k)}$ a reduced basis of Λ_{r_k}, by $Y_1, Y_2, \ldots, Y_n$ a basis of Λ, taken such that

$$\lim_{k\to\infty} |Y_g^{(r_k)} - Y_g| = 0, \quad \text{where} \quad g = 1, 2, \ldots, n, \tag{7·4}$$

hence

$$d(\Lambda) = |\{Y_1, Y_2, \ldots, Y_n\}| = \lim_{k\to\infty} |\{Y_1^{(r_k)}, Y_2^{(r_k)}, \ldots, Y_n^{(r_k)}\}| = \lim_{k\to\infty} d(\Lambda_{r_k}) = \Delta(K). \tag{7·5}$$

Let further

$$Y = u_1 Y_1 + \ldots + u_n Y_n \neq O, \quad \text{where} \quad u_1, \ldots, u_n \text{ are integers} \tag{7·6}$$

be any point of Λ, and put

$$Y^{(r_k)} = u_1 Y_1^{(r_k)} + \ldots + u_n Y_n^{(r_k)}, \quad \text{where} \quad k = 1, 2, 3, \ldots; \tag{7·7}$$

then

$$\lim_{k\to\infty} |Y^{(r_k)} - Y| = 0. \tag{7·8}$$

Hence $Y^{(r_k)} \neq O$ for sufficiently large k, and so $F(Y^{(r_k)}) \geqslant 1$ since Λ_{r_k} is K-admissible, whence

$$F(Y) = \lim_{k\to\infty} F(Y^{(r_k)}) \geqslant 1. \tag{7·9}$$

From (7·5) and (7·9), Λ satisfies the assertion.

8. The continuity of $\Delta(K)$

If $K: F(X) \leqslant 1$, is any star body, and if t is a positive number, then we denote by tK the star body of distance function $t^{-1}F(X)$, i.e. the set of all points X for which $F(X) \leqslant t$. From homogeneity, it is evident that

$$\Delta(tK) = t^n \Delta(K). \tag{8·1}$$

The set of all points X in K for which $|X| \leqslant t$ is further denoted by K^t.

Theorem 9. *Let $K, K_1, K_2, \ldots$ be an infinity of star bodies of the finite type, satisfying the following conditions:*

(*a*) *To every $\epsilon > 0$, there is a positive integer $N(\epsilon)$ such that K_r is contained in $(1+\epsilon)K$ if $r \geqslant N(\epsilon)$.*

(*b*) *To every $t > 0$ and every $\epsilon > 0$, there is a positive integer $N(t, \epsilon)$ such that K^t is contained in $(1+\epsilon)K_r$ if $r \geqslant N(t, \epsilon)$.*

Then
$$\lim_{r\to\infty} \Delta(K_r) = \Delta(K). \tag{8·2}$$

Proof. From (*a*), by theorem 7,

$$\Delta(K_r) \leqslant \Delta((1+\epsilon)K) = (1+\epsilon)^n \Delta(K), \tag{8·3}$$

whence for $\epsilon \to 0$,
$$\limsup_{r\to\infty} \Delta(K_r) \leqslant \Delta(K). \tag{8·4}$$

It will now be shown that also

$$\liminf_{r\to\infty} \Delta(K_r) \geqslant \Delta(K). \tag{8·5}$$

Let this inequality be false. Then there exists an infinite sequence of indices $r_1, r_2, r_3, \ldots$ not smaller than $N(\rho, 1)$ such that

$$\Delta(K_{r_k}) \leqslant 2\Delta(K), \quad \text{and} \quad \lim_{k\to\infty} \Delta(K_{r_k}) < \Delta(K). \tag{8·6}$$

Denote by Λ_{r_k} a critical lattice of K_{r_k}; therefore

$$d(\Lambda_{r_k}) \leqslant 2\Delta(K). \tag{8·7}$$

Then from (*b*) above, on taking $t = \rho$, $\epsilon = 1$, the star body $2K_{r_k}$ contains K^ρ, i.e. the sphere $|X| \leqslant \rho$; hence K_{r_k} contains the sphere $|X| \leqslant \frac{1}{2}\rho$. Since Λ_{r_k} is K_{r_k}-admissible, this implies that

$$|X| \geqslant \tfrac{1}{2}\rho \text{ for all points } X \neq O \text{ of } \Lambda_{r_k}.$$

It is clear from this and (8·7) that the sequence of lattices $\{\Lambda_{r_k}\}$ is bounded. Therefore, from theorem 2, this sequence contains a convergent infinite subsequence

$$\Lambda^{(1)} = \Lambda_{r_{k_1}},\ \Lambda^{(2)} = \Lambda_{r_{k_2}},\ \Lambda^{(3)} = \Lambda_{r_{k_3}},\ \ldots, \tag{8·8}$$

of limit Λ, say. For shortness write

$$K^{(1)} = K_{r_{k_1}},\ K^{(2)} = K_{r_{k_2}},\ K^{(3)} = K_{r_{k_3}},\ \ldots, \tag{8·9}$$

then, as in the proof of the last theorem,

$$d(\Lambda) = \lim_{l\to\infty} d(\Lambda^{(l)}) = \lim_{l\to\infty} \Delta(K^{(l)}), \tag{8·10}$$

and so
$$d(\Lambda) = \lim_{k\to\infty} \Delta(K_{r_k}) < \Delta(K). \tag{8·11}$$

This means that Λ is not K-admissible; hence Λ contains at least one point $Y \neq O$ which is an inner point of K.

Denote now by $Y_1^{(l)}, Y_2^{(l)}, \ldots, Y_n^{(l)}$ a reduced basis of $\Lambda^{(l)}$, and by $Y_1, Y_2, \ldots, Y_n$ a basis of Λ taken such that

$$\lim_{l\to\infty} | Y_g^{(l)} - Y_g | = 0, \quad \text{where} \quad g = 1, 2, \ldots, n; \tag{8·12}$$

then Y can be written as
$$Y = u_1 Y_1 + \ldots + u_n Y_n \tag{8·13}$$

with integral coefficients $u_1, \ldots, u_n$ not all zero. Now put

$$Y^{(l)} = u_1 Y_1^{(l)} + \ldots + u_n Y_n^{(l)}, \tag{8·14}$$

then $Y^{(l)}$ belongs to $\Lambda^{(l)}$,

$$Y^{(l)} \neq O, \quad \text{and} \quad \lim_{l\to\infty} | Y^{(l)} - Y | = 0, \tag{8·15}$$

whence, for sufficiently large indices l,

$$| Y^{(l)} | \leqslant 2 | Y |. \tag{8·16}$$

Since Y is an inner point of K and different from O, there is an $\epsilon > 0$ such that

$$F(Y) \leqslant \frac{1}{1+3\epsilon}, \tag{8·17}$$

hence, if l is sufficiently large, from (8·15) it follows that

$$F(Y^{(l)}) \leqslant \frac{1}{1+2\epsilon}, \tag{8·18}$$

and so $(1+2\epsilon)\, Y^{(l)} \neq O$ belongs to K. This implies, from (8·16), that $(1+2\epsilon)\, Y^{(l)}$ is a point of K^t, where $t = 2(1+2\epsilon) | Y |$. Hence, from (*b*) above, the point $(1+2\epsilon)\, Y^{(l)}$ belongs to $(1+\epsilon) K^{(l)}$ if l is sufficiently large. This implies that $Y^{(l)}$ is a point of $\dfrac{1+\epsilon}{1+2\epsilon} K^{(l)}$ and so is an inner point of $K^{(l)}$. However, this is impossible since $Y^{(l)} \neq O$ and since $\Lambda^{(l)}$ is a critical lattice of $K^{(l)}$.

THEOREM 10. *Let $K: F(X) \leqslant 1$ be a star body of the finite type, $G(X)$ an arbitrary distance function, and t a positive parameter. Then the star body*

$$K_t: F_t(X) \leqslant 1, \quad \textit{where} \quad F_t(X) = \max(F(X), t^{-1}G(X)),$$

is also of the finite type, and further

$$\lim_{t\to\infty} \Delta(K_t) = \Delta(K). \tag{8·19}$$

Proof. It is evident from definition 3 that $F_t(X)$ is a distance function. Since $F_t(X) \geqslant F(X)$ for all X and t, K_t is contained in K and so is a star body of the finite type. Further, since the set H: $G(X) \leqslant 1$ is a star body, there exists a number $\tau > 0$ such that H contains the whole sphere $|X| \leqslant \tau$. The sphere $|X| \leqslant \tau t$ is then contained in tH, and so $K^{\tau t}$, which is a subset of this sphere, is contained in K_t. The hypothesis of theorem 9 is therefore satisfied, and so

$$\Delta(K) = \lim_{r\to\infty} \Delta(K_{t_r}) \tag{8·20}$$

for every sequence of positive numbers $t_1, t_2, t_3, \ldots$ of limit infinity. This proves the assertion.

The last theorem, for $G(X) = |X|$, shows that

$$\Delta(K) = \lim_{t\to\infty} \Delta(K^t). \tag{8·21}$$

Originally (Mahler 1943), I used this formula as the definition of $\Delta(K)$ for unbounded star bodies, so reducing the problem to one for the bounded case.

Remark. The results of this paragraph remain true when $\Delta(K) = \infty$.

9. Lattice points on the boundary of a bounded star body

THEOREM 11. *If K is a bounded star body, then every critical lattice of K has n independent points on the boundary C of K.*

Proof. Let Λ be a K-admissible lattice which does not contain n independent points on C. Then denote by Π the set of all points of Λ on C, and by L the linear space of lowest dimension f $(0 \leqslant f \leqslant n-1)$ containing Π. By Minkowski's method of adaptation of lattices, a basis $Y_1, \ldots, Y_n$ of Λ can be found such that $Y_1, \ldots, Y_f$ lie in and generate L, while $Y_{f+1}, \ldots, Y_n$ lie outside L. Let $\epsilon > 0$ be sufficiently small and denote by Λ^* the lattice of basis $Y_1, \ldots, Y_f$, $(1-\epsilon)\,Y_{f+1}, \ldots, (1-\epsilon)\,Y_n$. This lattice is K-admissible since O and the elements of Π are its only points belonging to K. Since $d(\Lambda^*) = (1-\epsilon)^{n-f}\,d(\Lambda) < d(\Lambda)$, Λ^* is of smaller determinant than Λ, and so Λ is not critical.

This theorem shows that in the case of a bounded star body K, every critical lattice Λ has at least $2n$ points on its boundary C, namely, n independent points $P_1, \ldots, P_n$ and their images $-P_1, \ldots, -P_n$ in O. If

$$\pm P_1,\ \pm P_2,\ \ldots,\ \pm P_n$$

are the only points on C of the lattice Λ, then Λ is called a singular lattice of K; otherwise it is called a regular lattice. The example in the next paragraph shows that star bodies with singular lattices do exist.

10. An example of a star body with a singular lattice

THEOREM 12. *There exists a bounded star body with just one critical lattice. Moreover, this lattice is singular.*

Proof. Let ϵ be so small a positive constant that

$$(1-\epsilon)^n > \tfrac{99}{100}, \quad (1-\epsilon)^{n-1}\sqrt[n]{\tfrac{3}{2}} > 1, \quad \epsilon < n^{-1}\left(\sqrt[n]{\left(\tfrac{11}{10}\right)}-1\right), \tag{10·1}$$

and let $Q \neq O$ be a point in R_n. The set $S_\epsilon(Q)$ of all points

$$P = tQ + (t-1)\,\epsilon R, \quad \text{where} \quad t \geqslant 1 \ \text{and} \ |R| \leqslant 1, \tag{10·2}$$

is a cone with vertex at Q and its open side away from O. For when t is fixed and R describes all points of the unit sphere $|R| \leqslant 1$, then P lies on or in a sphere centre at tQ and radius $(t-1)\epsilon$; on varying t, we obtain $S_\epsilon(Q)$ as the sum set of these spheres.

Denote further by Λ_0 the lattice of all points with integral co-ordinates, i.e. of basis

$$P_1 = (1, 0, \ldots, 0),\ P_2 = (0, 1, \ldots, 0),\ \ldots,\ P_n = (0, 0, \ldots, 1), \tag{10·3}$$

and of determinant $d(\Lambda_0) = 1$.

The cube $$W: \quad |x_1| \leqslant \sqrt[n]{\tfrac{3}{2}},\ |x_2| \leqslant \sqrt[n]{\tfrac{3}{2}},\ \ldots,\ |x_n| \leqslant \sqrt[n]{\tfrac{3}{2}} \tag{10·4}$$

contains 3^n points of Λ_0, namely, the origin O, the $2n$ points $\pm P_1, \pm P_2, \ldots, \pm P_n$, and the m points $P'_1, P'_2, \ldots, P'_m$, where

$$P'_h = (x_1^{(h)}, x_2^{(h)}, \ldots, x_n^{(h)}), \quad x_g^{(h)} = 0, 1, \text{or} -1, \quad \sum_{g=1}^{n} |x_g^{(h)}| \geqslant 2. \tag{10·5}$$

Denote by K the set of all those points of W which are not inner points of one of the cones

$$S_\epsilon(\pm P_g), \quad \text{where} \quad g = 1, 2, \ldots, n, \qquad \text{or} \qquad S_\epsilon[(1-\epsilon)\,P'_h], \quad \text{where} \quad h = 1, 2, \ldots, m.$$

Then K is a bounded star body, and the cube

$$V: \quad |x_1| \leqslant 1-\epsilon,\ |x_2| \leqslant 1-\epsilon,\ \ldots,\ |x_n| \leqslant 1-\epsilon, \tag{10·6}$$

obviously is a subset of K. Therefore from theorem 7, Minkowski's theorem on linear forms, and from (10·1)

$$\Delta(K) \geqslant \Delta(V) = (1-\epsilon)^n > \tfrac{5}{6}. \tag{10·7}$$

On the other hand $$\Delta(K) \leqslant d(\Lambda_0) = 1, \tag{10·8}$$

since, by the construction, Λ_0 is K-admissible. Hence, if Λ is any critical lattice of K, then

$$\tfrac{5}{6} < d(\Lambda) \leqslant 1. \tag{10·9}$$

Each one of the n parallelepipeds

$$U_g: \quad |x_g| \leqslant \sqrt[n]{\tfrac{3}{2}}, \quad |x_l| \leqslant 1-\epsilon \quad \text{for} \quad l = 1, 2, \ldots, g-1, g+1, \ldots, n \tag{10·10}$$

from (10·1) is of volume $$2^n(1-\epsilon)^{n-1}\sqrt[n]{\tfrac{3}{2}} > 2^n. \tag{10·11}$$

Hence, from Minkowski's theorem on linear forms, at least one point of $\Lambda - [O]$ is an inner point of U_g, say the point $P^*_g = (\xi_1^{(g)}, \xi_2^{(g)}, \ldots, \xi_n^{(g)})$. This point lies in one of the two cones $S_\epsilon(\pm P_g)$, since the other inner points of U_g are also inner points of K. There is no loss of generality in assuming that P^*_g belongs to $S_\epsilon(P_g)$ and so may be written as

$$P^*_g = t_g P_g + (t_g - 1)\,\epsilon R_g, \quad \text{where} \quad t_g \geqslant 1 \ \text{and} \ |R_g| \leqslant 1. \tag{10·12}$$

Therefore if, say, $R_g = (\eta_1^{(g)}, \eta_2^{(g)}, \ldots, \eta_n^{(g)})$, then

$$\xi_g^{(g)} = t_g + (t_g - 1)\,\epsilon\eta_g^{(g)} \tag{10·13}$$

and

$$\eta_g^{(g)} \geq -1; \tag{10·14}$$

and since P_g lies in U_g,

$$\sqrt[n]{\tfrac{3}{2}} \geq \xi_g^{(g)} = t_g + (t_g - 1)\,\epsilon\eta_g^{(g)} \geq t_g - (t_g - 1)\,\epsilon > (1-\epsilon)\,t_g, \tag{10·15}$$

whence

$$1 \leq t_g < \frac{\sqrt[n]{\frac{3}{2}}}{1-\epsilon}, \quad \text{where} \quad g = 1, 2, \ldots, n. \tag{10·16}$$

Denote now by D the determinant

$$D = \{P_1^*, P_2^*, \ldots, P_n^*\}, \tag{10·17}$$

by E the unit determinant

$$E = \{P_1, P_2, \ldots, P_n\} = d(\Lambda_0) = +1, \tag{10·18}$$

and by $\quad E(g_1, g_2, \ldots, g_r), \quad$ where $\quad 1 \leq r \leq n,\ 1 \leq g_1 < g_2 < \ldots < g_r < n,$

the determinant which is obtained from E if the points $P_{g_1}, P_{g_2}, \ldots, P_{g_r}$ in it are replaced by the points $R_{g_1}, R_{g_2}, \ldots, R_{g_r}$ of the same indices. Obviously $E(g_1, g_2, \ldots, g_r)$ is equal to its cofactor of order r belonging to the rows and columns of indices $g_1, g_2, \ldots, g_r$. Hence

$$|\, E(g_1, g_2, \ldots, g_r)\,| \leq r! \tag{10·19}$$

since the moduli of the co-ordinates of $R_{g_1}, R_{g_2}, \ldots, R_{g_r}$ are not larger than 1, and since a determinant of order r consists of $r!$ terms.

From (10·12), D can be split into a sum of 2^n determinants, namely,

$$D = t_1 t_2 \ldots t_n \left(E + \sum_{r=1}^{n}{}^{*}\, E(g_1, g_2, \ldots, g_r)\,\epsilon^r \frac{t_{g_1}-1}{t_{g_1}} \frac{t_{g_2}-1}{t_{g_2}} \ldots \frac{t_{g_r}-1}{t_{g_r}} \right), \tag{10·20}$$

with the abbreviation

$$\sum_{r=1}^{n}{}^{*} = \sum_{r=1}^{n} \sum_{\substack{g_1, g_2, \ldots, g_r = 1 \\ g_1 < g_2 < \ldots < g_r}}^{n}. \tag{10·21}$$

Now from (10·1) and (10·19)

$$|\, E(g_1, g_2, \ldots, g_r)\,|\, \epsilon^r \leq r!\,\epsilon^r \leq (r\epsilon)^r \leq (n\epsilon)^r \leq \{\sqrt[n]{(\tfrac{11}{10})} - 1\}^r, \tag{10·22}$$

hence

$$\left| \sum_{r=1}^{n}{}^{*}\, E(g_1, g_2, \ldots, g_r)\,\epsilon^r \frac{t_{g_1}-1}{t_{g_1}} \ldots \frac{t_{g_r}-1}{t_{g_r}} \right| \leq \sum_{r=1}^{n}{}^{*} \{\sqrt[n]{(\tfrac{11}{10})} - 1\}^r \frac{t_{g_1}-1}{t_{g_1}} \ldots \frac{t_{g_r}-1}{t_{g_r}}$$

$$= \prod_{g=1}^{n} \left(1 + \{\sqrt[n]{(\tfrac{11}{10})} - 1\} \frac{t_g - 1}{t_g} \right) - 1 = (t_1 t_2 \ldots t_n)^{-1} \prod_{g=1}^{n} \{1 + \sqrt[n]{(\tfrac{11}{10})}\,(t_g - 1)\} - 1, \tag{10·23}$$

whence

$$D \leq \prod_{g=1}^{n} \{1 + \sqrt[n]{(\tfrac{11}{10})}\,(t_g - 1)\}, \tag{10·24}$$

and

$$D \geq 2 \prod_{g=1}^{n} t_g - \prod_{g=1}^{n} \{1 + \sqrt[n]{(\tfrac{11}{10})}\,(t_g - 1)\}. \tag{10·25}$$

From (10·1), (10·16) and (10·24) then

$$D < \prod_{g=1}^{n} \{\sqrt[n]{(\tfrac{11}{10})} + \sqrt[n]{(\tfrac{11}{10})}\,(t_g - 1)\} = \prod_{g=1}^{n} \{\sqrt[n]{(\tfrac{11}{10})}\, t_g\} < \prod_{g=1}^{n} \left(\frac{\sqrt[n]{(\frac{11}{10})}\,\sqrt[n]{(\frac{3}{2})}}{1-\epsilon}\right) = \tfrac{33}{20}(1-\epsilon)^{-n} < \tfrac{5}{3}. \tag{10·26}$$

Further, since $2 - (\frac{11}{10})^{r/n} > 0$ for $r = 1, 2, \ldots, n$, then from (10·16) and (10·25),

$$D \geqslant 2 \prod_{g=1}^{n} \{1 + (t_g - 1)\} - \prod_{g=1}^{n} \{1 + \sqrt[n]{(\tfrac{11}{10})}\,(t_g - 1)\}$$

$$= 1 + \sum_{r=1}^{n}{}^{*} \{2 - (\tfrac{11}{10})^{r/n}\}\,(t_{g_1} - 1) \ldots (t_{g_r} - 1) \geqslant 1, \tag{10·27}$$

with $D = 1$ if and only if $t_1 = t_2 = \ldots = t_n = 1$.

This proves that
$$1 \leqslant D < \tfrac{5}{3}, \tag{10·28}$$
the lower bound being assumed if and only if $t_1 = t_2 = \ldots = t_n = 1$, i.e. if

$$P_1^* = P_1,\; P_2^* = P_2,\; \ldots,\; P_n^* = P_n.$$

Since $D > 0$, the n points $P_1^*, P_2^*, \ldots, P_n^*$ are independent; therefore

$$D = jd(\Lambda), \tag{10·29}$$

where j is a positive integer. From (10·9) and (10·28) it follows that

$$\tfrac{5}{3} > j \cdot \tfrac{5}{6}, \quad j < 2, \tag{10·30}$$

and so $j = 1$, $d(\Lambda) = D \geqslant 1$, with equality if and only if $\Lambda = \Lambda_0$. Since Λ_0 is K-admissible and since $d(\Lambda_0) = 1$, this completes the proof that Λ_0 is the only critical lattice of K, and also that Λ_0 is singular.

Corollary. *For any given integer $m \geqslant n$, there exists a bounded star body K with a critical lattice having just $2m$ points on the boundary of K.*

Proof. Nearly obvious, because any star body K' has the required property if it satisfies the following three conditions: (*a*) K, as defined in the last proof, is a subset of K'. (*b*) Λ_0, as defined in the last proof, is K'-admissible. (*c*) Just $2m$ points of Λ_0 lie on the boundary of K'.

Remark. In an earlier paper on star domains,* I discussed a method by which to obtain $\Delta(K)$ and the critical lattices for every bounded two-dimensional star body, provided the boundary consists of a finite number of analytical arcs. This method may be extended to the n-dimensional case, but, naturally, the calculations now become very complicated.

11. The lattice function $F(\Lambda)$

If Λ is a lattice, t a positive number, and $t\Lambda$ denotes the lattice of all points tP where P runs over Λ, it is obvious that

$$d(t\Lambda) = t^n d(\Lambda). \tag{11·1}$$

* Mahler—On lattice points in two-dimensional star domains, to appear in the *Proceedings of the London Mathematical Society.*

Further, if K denotes the star body (not necessarily bounded) of distance function $F(X)$, write

$$F(\Lambda) = \text{l.b.}\, F(P), \tag{11·2}$$

for the lower bound of $F(P)$ extended over all points $P \neq O$ of Λ. Then the symbol $F(\Lambda)$ has the following evident properties:

Λ is K-admissible if and only if $\quad F(\Lambda) \geqslant 1.$

Λ is a critical lattice of K if and only if $\quad F(\Lambda) = 1,\ d(\Lambda) = \Delta(K);$

further

$$F(t\Lambda) = tF(\Lambda) \quad \text{if} \quad t > 0. \tag{11·3}$$

A star body is therefore of the finite type if $F(\Lambda) > 0$ for at least one lattice, and is of the infinite type if $F(\Lambda) = 0$ for all lattices.

In the special case when K is a bounded star body, it is easily seen that $F(\Lambda)$ is a continuous function of Λ; i.e. if $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ is a convergent sequence of lattices of limit Λ, then

$$\lim_{r\to\infty} F(\Lambda_r) = F(\Lambda). \tag{11·4}$$

If, however, K is an unbounded star body, then $F(\Lambda)$ need not be continuous, as the following example shows. We choose

$$F(X) = |x_1 x_2 \ldots x_n|^{1/n}, \tag{11·5}$$

and take for Λ the lattice of basis

$$X_h = (\omega_h^{(1)}, \omega_h^{(2)}, \ldots, \omega_h^{(n)}), \quad \text{where} \quad h = 1, 2, \ldots, n, \tag{11·6}$$

as defined in the proof of part (1) of theorem 5; there is no restriction in assuming that this basis is reduced. Further, denote by

$$X_1^{(r)}, X_2^{(r)}, \ldots, X_n^{(r)}, \quad \text{where} \quad r = 1, 2, 3, \ldots,$$

an infinity of sets of n independent points with rational co-ordinates such that

$$\lim_{r\to\infty} |X_h^{(r)} - X_h| = 0, \quad \text{where} \quad h = 1, 2, \ldots, n, \tag{11·7}$$

and such that further $X_1^{(r)}, X_2^{(r)}, \ldots, X_n^{(r)}$ form a reduced basis of the lattice Λ_r generated by these n points. Then by the proof of theorem 5,

$$F(\Lambda) \geqslant 1, \tag{11·8}$$

while, on the other hand,

$$F(\Lambda_r) = 0 \tag{11·9}$$

and

$$\lim_{r\to\infty} F(\Lambda_r) = 0, \tag{11·10}$$

since a linear form with rational coefficients represents zero.

12. Lattice points near the boundary of an unbounded star body

It was seen in § 9 that a critical lattice of any bounded star body has at least $2n$ points on its boundary. For unbounded star bodies, this is no longer so; as will be seen in the next paragraph, there exists an unbounded star body of the finite type such that at least one of its critical lattices has no point on its boundary.

It may then be asked, however, whether lattice points lie arbitrarily near to the boundary of K. The answer is given by the nearly obvious

THEOREM 13. *If K: $F(X) \leqslant 1$ is a star body of the finite type and Λ is a critical lattice of K, then to every $\epsilon > 0$ there exists a point P of Λ such that*

$$1 \leqslant F(P) < 1+\epsilon. \tag{12·1}$$

Proof. If $F(P) \geqslant 1+\epsilon$ for every point $P \neq O$ of Λ, then

$$F(\Lambda) \geqslant 1+\epsilon, \tag{12·2}$$

whence

$$F\left(\frac{\Lambda}{1+\epsilon}\right) \geqslant 1. \tag{12·3}$$

Therefore $\dfrac{\Lambda}{1+\epsilon}$ is also K-admissible, but is of smaller determinant than Λ, and so Λ is not critical.

This theorem leads to:

PROBLEM A. *Let K: $F(X) \leqslant 1$ be a star body of the finite type, Λ a critical lattice of K, and $\epsilon > 0$ any arbitrarily small number. Do there exist n independent points $P_1, P_2, \ldots, P_n$ of Λ such that*

$$1 \leqslant F(P_g) < 1+\epsilon, \quad \textit{where} \quad g = 1, 2, \ldots, n? \tag{12·4}$$

I have not been able to decide this question. The difficulty lies in the fact that $F(\Lambda)$ may be discontinuous, and so the method of the proof of theorem 11 cannot be applied.

Remark. From theorems 8 and 13, for any given $\epsilon > 0$, every lattice of determinant $d(\Lambda) = \Delta(K)$ contains a point $P \neq O$ satisfying $F(P) < 1+\epsilon$.

13. AN EXAMPLE OF AN UNBOUNDED STAR BODY WITH NO CRITICAL LATTICE POINTS ON ITS BOUNDARY

THEOREM 14. *Let $F_0(X)$ be the distance function*

$$F_0(X) = |x_1 x_2 \ldots x_n|^{1/n}, \tag{13·1}$$

and let further $F(X)$ be any distance function satisfying the conditions

$$F(X) \geqslant F_0(X) \quad \textit{if} \quad F_0(X) > 0, \tag{13·2}$$

$$\frac{F(X)}{F_0(X)} \to 1 \quad \textit{if} \quad F_0(X) > 0,\ |X|^{-1} F_0(X) \to 0. \tag{13·3}$$

Denote by K_0 and K the star bodies of distance functions $F_0(X)$ and $F(X)$, respectively. Then

$$\Delta(K) = \Delta(K_0). \tag{13·4}$$

Proof. K is a subset of K_0, and so from theorem 7, it follows that

$$\Delta(K) \leqslant \Delta(K_0). \tag{13·5}$$

Now assume that
$$\Delta(K) < \Delta(K_0); \tag{13·6}$$
this assumption leads to a contradiction, as will be proved.

The function $f(X)$ defined by
$$f(X) = \begin{cases} \dfrac{F(X)}{F_0(X)} & \text{if } F_0(X) \neq 0, \\ 1 & \text{if } F_0(X) = 0, \ X \neq O, \end{cases} \tag{13·7}$$
and not defined if $X = O$, is continuous and therefore bounded for all points of the unit sphere $|X| = 1$. Let $c \geqslant 1$ be its upper bound on this sphere:
$$f(X) \leqslant c \quad \text{if} \quad |X| = 1. \tag{13·8}$$
Then, since
$$f(tX) = f(X) \quad \text{for} \quad t \neq 0, \tag{13·9}$$
c is the upper bound of $f(X)$ for all $X \neq O$, therefore
$$F(X) \leqslant cF_0(X) \tag{13·10}$$
for all X, since this inequality remains true if $X = O$.

Let now Λ be any critical lattice of K; then, from (13·6),
$$d(\Lambda) < \Delta(K_0), \tag{13·11}$$
or, say,
$$d(\Lambda) = (1+\alpha)^{-(n+1)} \Delta(K_0), \tag{13·12}$$
where α is some positive number. Put
$$(1+\alpha)\Lambda = \Lambda', \tag{13·13}$$
so that Λ' is $(1+\alpha)$ K-admissible, and
$$d(\Lambda') = (1+\alpha)^{-1} \Delta(K_0) < \Delta(K_0). \tag{13·14}$$

Denote further by Σ the set of all points of Λ' which are inner points of K_0. If P is any point of Σ, then
$$F(P) \geqslant 1+\alpha, \quad F_0(P) < 1, \tag{13·15}$$
whence
$$\frac{F(P)}{F_0(P)} > 1+\alpha, \tag{13·16}$$
and further, from (13·10),
$$F_0(P) \geqslant \frac{1}{c} F(P) \geqslant \frac{1+\alpha}{c} > 0. \tag{13·17}$$

But from (13·3) there exists a positive number β such that
$$\left| \frac{F(X)}{F_0(X)} - 1 \right| \leqslant \alpha \quad \text{if} \quad F_0(X) \neq 0, \ |X|^{-1} F_0(X) < \beta. \tag{13·18}$$

Hence, by the inequalities just proved,
$$|P|^{-1} F_0(P) \geqslant \beta, \tag{13·19}$$
and so
$$|P| \leqslant \frac{F_0(P)}{\beta} < \frac{1}{\beta}. \tag{13·20}$$

Next, if $P = (p_1, p_2, \ldots, p_n)$, then

$$|\, p_1 p_2 \cdots p_n \,| = F_0(P)^n \geqslant \left(\frac{1+\alpha}{c}\right)^n, \tag{13·21}$$

and
$$\max(|\, p_1 \,|, |\, p_2 \,|, \ldots, |\, p_n \,|) \leqslant |\, P \,| < \frac{1}{\beta}; \tag{13·22}$$

and so, finally,
$$|\, p_1 \,| \geqslant |\, p_2 \cdots p_n \,|^{-1} \left(\frac{1+\alpha}{c}\right)^n > \left(\frac{1+\alpha}{c}\right)^n \beta^{n-1}. \tag{13·23}$$

Denote by r any positive integer, and by $Y = \Omega_r X$ the unimodular linear transformation

$$y_1 = r^{n-1} x_1, \quad y_2 = r^{-1} x_2, \ldots, y_n = r^{-1} x_n. \tag{13·24}$$

Further denote by $\Lambda_r = \Omega_r \Lambda'$ the lattice of all points $Q = \Omega_r P$ where P runs over Λ', and by $\Sigma_r = \Omega_r \Sigma$ the set of all points $Q = \Omega_r P$ where P lies in Σ. Then obviously

$$d(\Lambda_r) = d(\Lambda'), \tag{13·25}$$

and Σ_r consists of all and only all those points of Λ_r which are inner points of K_0.

If $P = (p_1, p_2, \ldots, p_n)$ is a point of Σ and $Q = \Omega_r P = (q_1, q_2, \ldots, q_n)$ is the corresponding point of Σ_r, then, from (13·23)

$$|\, q_1 \,| = r^{n-1} |\, p_1 \,| > \left(\frac{1+\alpha}{c}\right)^n (\beta r)^{n-1}, \tag{13·26}$$

and so
$$|\, Q \,| \geqslant |\, q_1 \,| > \left(\frac{1+\alpha}{c}\right)^n (\beta r)^{n-1}. \tag{13·27}$$

As in § 8, denote by K_0^t, where $t > 0$, the set of all points X of K_0 for which $|\, X \,| \leqslant t$. Then the last inequality for Q shows that there exists a monotone increasing function $R(t)$ of t such that

$$\Lambda_r \quad \text{is} \quad K_0^t\text{-admissible} \quad \text{if} \quad r \geqslant R(t). \tag{13·28}$$

Now the sphere $|\, X \,| \leqslant 1$ is obviously a subset of K_0, hence also of K_0^t if $t \geqslant 1$. Therefore, from (13·28),

$$|\, Q \,| \geqslant 1 \text{ for all points } Q \neq O \text{ of } \Lambda_r \quad \text{if} \quad r \geqslant R(t) \text{ and } t \geqslant 1.$$

Also since
$$d(\Lambda_r) = d(\Lambda') \quad \text{for} \quad r = 1, 2, 3, \ldots, \tag{13·29}$$

the sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ is bounded.

But then, by theorem 2, this sequence contains a convergent infinite subsequence of lattices

$$\Lambda_{r_1}, \Lambda_{r_2}, \Lambda_{r_3}, \ldots,$$

say of limit Λ^*. Since, from (13·14),

$$d(\Lambda^*) = \lim_{k \to \infty} d(\Lambda_{r_k}) = d(\Lambda') < \Delta(K_0), \tag{13·30}$$

Λ^* cannot be K_0-admissible; there is then a point P^* of Λ^* which is an inner point of K_0 and so also an inner point of K_0^t if t is sufficiently large. Further, as in earlier proofs, it may be shown that there are points

$$P_{r_1}, P_{r_2}, P_{r_3}, \ldots \text{ of } \Lambda_{r_1}, \Lambda_{r_2}, \Lambda_{r_3}, \ldots \quad \text{respectively,}$$

such that

$$\lim_{k\to\infty} | P_{r_k} - P^* | = 0. \tag{13·31}$$

But then P_{r_k} is also an inner point of K_0^t if k is sufficiently large, contrary to (13·28). This completes the proof.

Theorem 15. *There exists an unbounded star body of the finite type with a critical lattice which has no points on the boundary of this body.*

Proof. The same notation is used as in theorem 14, but it is assumed that $F(X)$ satisfies, instead of (13·2), the stronger condition

$$F(X) > F_0(X) \quad \text{if} \quad F_0(X) > 0; \tag{13·32}$$

e.g. take

$$F(X) = F_0(X)\left\{1 + \frac{F_0(X)}{|X|}\right\}. \tag{13·33}$$

Let Λ be a critical lattice of K_0. Since K is a subset of K_0, Λ is K-admissible; further, since from theorem 14,

$$d(\Lambda) = \Delta(K_0) = \Delta(K), \tag{13·34}$$

Λ is a critical lattice of K. But the boundary of K consists only of inner points of K_0, and so no point of Λ may lie on the boundary of K, as asserted.†

It is easily proved from § 15 that K_0 and so also K have an infinity of critical lattices. The question also arises:

Problem B. *Do there exist critical lattices of K_0 which are not critical lattices of K, and do these lattices have points on the boundary of K?*

14. Star bodies with automorphisms

Let $X = (x_1, x_2, \ldots, x_n)$ and $X' = (x_1', x_2', \ldots, x_n')$ be two points in R_n. The linear substitution

$$\Omega: \qquad x_g' = \sum_{k=1}^{n} a_{gh} x_h, \quad \text{where} \quad g = 1, 2, \ldots, n, \tag{14·1}$$

of determinant

$$\omega = | a_{gh} |_{g,h=1,2,\ldots,n} \neq 0, \tag{14·2}$$

or shorter

$$X' = \Omega X \tag{14·3}$$

has an inverse

$$X = \Omega^{-1} X'. \tag{14·4}$$

The substitution defines a one-to-one mapping of R_n on itself.

† A much simpler proof of theorem 15 will be given in Part II of this paper.

If Λ is an arbitrary lattice, then $\Omega\Lambda$ denotes the lattice of all points $P' = \Omega P$ where P belongs to Λ; obviously

$$d(\Omega\Lambda) = |\omega| \, d(\Lambda). \tag{14·5}$$

THEOREM 16. *Let K: $F(X) \leqslant 1$ be a star body of the finite type, Ω a substitution of determinant $\omega \neq 0$, $F'(X)$ the distance function*

$$F'(X) = F(\Omega X), \tag{14·6}$$

and K' the star body $F'(X) \leqslant 1$. Then K' is also of the finite type, and

$$\Delta(K') = |\omega|^{-1} \Delta(K). \tag{14·7}$$

Proof. If Λ is any K-admissible lattice, then $\Lambda' = \Omega^{-1}\Lambda$ is evidently K'-admissible, and so K' is also of the finite type. Further $\Delta(K')$ is not greater than the lower bound of $d(\Omega^{-1}\Lambda) = |\omega|^{-1} d(\Lambda)$ extended over all K-admissible lattices, i.e.

$$\Delta(K') \leqslant |\omega|^{-1} \Delta(K). \tag{14·8}$$

Since $F(X) = F'(\Omega^{-1}X)$, conversely

$$\Delta(K) \leqslant |\omega| \, \Delta(K'). \tag{14·9}$$

From these two inequalities, the assertion follows at once.

DEFINITION 8. *The linear substitution $X' = \Omega X$ is called an automorphism of the star body K: $F(X) \leqslant 1$, if identically in X,*

$$F(X') = F(X). \tag{14·10}$$

It is obvious that such an automorphism leaves both K and its boundary C invariant.

THEOREM 17. *If the star body K is of the finite type and admits the automorphism $X' = \Omega X$ of determinant ω, then $\omega = \pm 1$.*

Proof. By theorem 16, $\Delta(K) = |\omega|^{-1} \Delta(K)$, whence $|\omega| = 1$ since $\Delta(K) \neq 0$.

This theorem shows that star bodies having automorphisms of determinant $\omega \neq \pm 1$, are necessarily of the infinite type, e.g. the star body of distance function $F(X) = |x_1^2 x_2 \ldots x_n|^{1/(n+1)}$ with the automorphism

$$x_1' = t^{-\frac{1}{2}(n-1)} x_1, \; x_2' = t x_2, \; \ldots, \; x_n' = t x_n \quad (t > 0). \tag{14·11}$$

It is obvious that if K is of the finite type, then the set of all automorphisms of K forms a group. Whether this group is finite or infinite, discrete or continuous, depends on K itself.

DEFINITION 9. *An unbounded star body K of the finite type is called automorphic if it admits a group Γ of automorphisms Ω with the following property: 'There exists a positive constant c depending only on K and Γ such that to every point X of K there is an element Ω of Γ satisfying*

$$|\Omega X| \leqslant c.\text{'} \tag{14·12}$$

A few examples of automorphic star bodies are given in the next section.

15. Examples of automorphic star bodies

(1) Let $r \geqslant 0$ and $s \geqslant 0$ be integers such that $r+2s = n$, and let $F(X)$ be the distance function

$$F(X) = \left| \prod_{\rho=1}^{r} x_\rho \prod_{\sigma=1}^{s} (x_{r+\sigma}^2 + x_{r+s+\sigma}^2) \right|^{1/n}. \tag{15·1}$$

It was shown in the first part of the proof of theorem 5 that the star body $K: F(X) \leqslant 1$ is of the finite type if $r = n, s = 0$. Just the same proof applies when $s > 0$, except that the field $\mathfrak{K}$ there must now be algebraic with r real and $2s$ complex conjugate fields. If the trivial cases $r = 1, s = 0$ and $r = 0, s = 1$ be excluded, then K is not bounded and admits a continuous group of automorphisms depending on $n-1$ parameters, namely, the group of substitutions

$$x'_\rho = t_\rho x_\rho, \quad \text{where} \quad \rho = 1, 2, \ldots, r, \tag{15·2}$$

$$x'_{r+\sigma} = t_{r+\sigma} x_{r+\sigma} - t_{r+s+\sigma} x_{r+s+\sigma}, \quad x'_{r+s+\sigma} = t_{r+s+\sigma} x_{r+\sigma} + t_{r+\sigma} x_{r+s+\sigma},$$

where $$\sigma = 1, 2, \ldots, s, \tag{15·3}$$

while $t_1, t_2, \ldots, t_n$ are n real numbers such that

$$\prod_{\rho=1}^{r} t_\rho \prod_{\sigma=1}^{s} (t_{r+\sigma}^2 + t_{r+s+\sigma}^2) = \pm 1. \tag{15·4}$$

The star body K is automorphic since obviously every point X of K can be transformed into a point X' of bounded distance from O by one of these automorphisms.

(2) Let r be an integer such that $1 \leqslant r \leqslant n-1$, and let K be the star body of distance function

$$F(X) = \left| \sum_{\rho=1}^{r} x_\rho^2 - \sum_{\sigma=r+1}^{n} x_\sigma^2 \right|^{\frac{1}{2}}. \tag{15·5}$$

By the theory of quadratic forms, K admits a group of automorphisms depending on $\frac{1}{2}n(n-1)$ real parameters. It is again possible to show that every point in K can be transformed by one of these automorphisms into a point of bounded distance from O. Hence K is automorphic provided it is of the finite type, and so the following problem arises:

Problem C. *Is the star body of distance function*

$$F(X) = \left| \sum_{\rho=1}^{r} x_\rho^2 - \sum_{\sigma=r+1}^{n} x_\sigma^2 \right|^{\frac{1}{2}} \tag{15·6}$$

of the finite or of the infinite type?†

For $2 \leqslant n \leqslant 4$, K is of the finite type, because there exist indefinite quadratic forms in n variables with integral coefficients and of given signature which do not represent zero non-trivially. If, however, $n \geqslant 5$, then, by Meyer's theorem (Bachmann 1898), every indefinite quadratic form with integral coefficients does represent zero; so the solution of problem C may be difficult.

† *Addition, May* 1946. In a joint paper, H. Davenport and H. Heilbron have just shown that K is of the infinite type if $n \geqslant 5$.

(3) Let $n = 2$, and denote by θ any number with $0 < \theta < 1$. The line segments joining the pairs of points

$$(\theta^k, \theta^{-k}) \quad \text{and} \quad (\theta^{k+1}, \theta^{-k-1}), \quad \text{where} \quad k = 0, \pm 1, \pm 2, \ldots,$$

form an infinite polygon Π; let C be the curve consisting of Π and the images of Π in O and the two axes. Then C forms the complete boundary of a two-dimensional star body K. There is no difficulty in proving that K is of the finite type and that it admits the infinite group of automorphisms

$$x_1' = \pm \theta^k x_g, \quad x_2' = \pm \theta^{-k} x_h, \quad \text{where} \quad k = 0, \pm 1, \pm 2, \ldots$$

and
$$g = 1, h = 2 \quad \text{or} \quad g = 2, h = 1. \tag{15·7}$$

It can be shown that every point of K can be transformed by one of these automorphisms into a point of bounded distance from O; hence K is an automorphic star body.

16. Properties of the lattice function $F(\Lambda)$

It was seen in § 11 that $F(\Lambda)$ need not be a continuous function of Λ. The next two theorems on sequences of lattices have therefore some interest:

Theorem 18. *Let $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ be a convergent sequence of lattices, say of limit Λ. Then*

$$F(\Lambda) \geqslant \liminf_{r \to \infty} F(\Lambda_r). \tag{16·1}$$

Proof. Choose reduced bases $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$ of Λ_r, and a basis $Y_1, Y_2, \ldots, Y_n$ of Λ such that

$$\lim_{r \to \infty} | Y_g^{(r)} - Y_g | = 0, \quad \text{where} \quad g = 1, 2, \ldots, n. \tag{16·2}$$

Every point $P \neq O$ of Λ can be written as

$$P = u_1 Y_1 + \ldots + u_n Y_n \tag{16·3}$$

with integral coefficients $u_1, \ldots, u_n$ not all zero. Put

$$P_r = u_1 Y_1^{(r)} + \ldots + u_n Y_n^{(r)}, \tag{16·4}$$

then
$$P_r \neq O, \text{ and } P_r \text{ lies in } \Lambda_r. \tag{16·5}$$

Hence
$$F(P_r) \geqslant F(\Lambda_r). \tag{16·6}$$

Therefore by the continuity of $F(X)$,

$$F(P) = \lim_{r \to \infty} F(P_r) \geqslant \liminf_{r \to \infty} F(\Lambda_r), \tag{16·7}$$

as asserted.

Theorem 19. *Let $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ be a convergent sequence of lattices, say of limit Λ, and assume that $\phi = \lim_{r \to \infty} F(\Lambda_r)$ exists and is positive. Let there also be a constant $c > 0$ and an infinite sequence of points $P_1, P_2, P_3, \ldots$ such that*

$$P_r \neq O; \quad | P_r | \leqslant c; \quad P_r \textit{ lies in } \Lambda_r, \quad \textit{where} \quad r = 1, 2, 3, \ldots, \tag{16·8}$$

$$\lim_{r \to \infty} F(P_r) \textit{ exists and is equal to } \phi.$$

Then
$$\lim_{r\to\infty} F(\Lambda_r) = F(\Lambda), \tag{16·9}$$

and there exists a point $P \neq O$ of Λ such that
$$F(P) = F(\Lambda). \tag{16·10}$$

Proof. There is a positive number ρ such that the sphere $|X| \leqslant \rho$ is contained in the star body F: $F(X) \leqslant 1$. Put
$$\sigma = \tfrac{1}{2}\rho\phi. \tag{16·11}$$

Then the sphere $|X| \leqslant \sigma$ is contained in the star body $F(X) \leqslant \sigma/\rho$, i.e. in $F(X) \leqslant F(\Lambda_r)$, for all sufficiently large r, say for $r \geqslant r_0$. Therefore for every point $Q \neq O$ of Λ_r, since $F(Q) \geqslant F(\Lambda_r)$,
$$|Q| \geqslant \sigma \quad \text{if} \quad r \geqslant r_0. \tag{16·12}$$

Let, in particular, $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$ be a reduced basis of Λ_r and $Y_1, Y_2, \ldots, Y_n$ a basis of Λ taken such that
$$\lim_{r\to\infty} |Y_g^{(r)} - Y_g| = 0, \quad \text{where} \quad g = 1, 2, \ldots, n. \tag{16·13}$$

Then
$$|Y_g^{(r)}| \geqslant \sigma \quad \text{for} \quad r \geqslant r_0, \quad g = 1, 2, \ldots, n. \tag{16·14}$$

On the other hand, from theorem 1,
$$|Y_1^{(r)}|\,|Y_2^{(r)}| \ldots |Y_n^{(r)}| \leqslant \gamma_n d(\Lambda_r). \tag{16·15}$$

Also, from the hypothesis,
$$\lim_{r\to\infty} d(\Lambda_r) = d(\Lambda), \tag{16·16}$$

hence
$$\tfrac{1}{2}d(\Lambda) \leqslant d(\Lambda_r) \leqslant 2d(\Lambda) \quad \text{for} \quad r \geqslant r_1, \text{ say}, \tag{16·17}$$

and so
$$|Y_g^{(r)}| \leqslant 2\sigma^{-(n-1)}\gamma_n d(\Lambda) \quad \text{for} \quad r \geqslant \max(r_0, r_1), \quad \text{where} \quad g = 1, 2, \ldots, n. \tag{16·18}$$

Since P_r is a point of Λ_r different from O,
$$P_r = u_1^{(r)} Y_1^{(r)} + \ldots + u_n^{(r)} Y_n^{(r)} \tag{16·19}$$

with integral coefficients $u_1^{(r)}, \ldots, u_n^{(r)}$ not all zero. On solving this vector equation for $u_1^{(r)}, \ldots, u_n^{(r)}$,
$$d(\Lambda_r)\,|u_g^{(r)}| = |\{Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}\}|\,|u_g^{(r)}| = |\{Y_1^{(r)}, \ldots, Y_{g-1}^{(r)}, P_r, Y_{g+1}^{(r)}, \ldots, Y_n^{(r)}\}|. \tag{16·20}$$

Hence the lower bound for $d(\Lambda_r)$ and the upper bounds for $Y_g^{(r)}$ and P_r imply that
$$|u_g^{(r)}| \leqslant c', \tag{16·21}$$

where c' is a positive number independent of r and g.

There exists then an infinite sequence of indices
$$r = r_1, r_2, r_3, \ldots, \quad \text{where} \quad \lim_{k\to\infty} r_k = \infty, \tag{16·22}$$

such that the coefficients
$$u_g^{(r_k)} = u_g \text{ say}, \quad \text{where} \quad k = 1, 2, 3, \ldots; \; g = 1, 2, \ldots, n, \tag{16·23}$$

assume integral values independent of k, and such that at least one of these integers $u_1, \ldots, u_n$ is different from zero. Further

$$P_{r_k} = u_1 Y_1^{(r_k)} + \ldots + u_n Y_n^{(r_k)}, \quad \text{where} \quad k = 1, 2, 3, \ldots, \tag{16·24}$$

and so the points P_{r_k} tend to the limit point

$$P = u_1 Y_1 + \ldots + u_n Y_n \neq O \tag{16·25}$$

which is a point of Λ. From the hypothesis

$$F(P) = \lim_{k\to\infty} F(P_{r_k}) = \lim_{r\to\infty} F(P_r) = \lim_{r\to\infty} F(\Lambda_r), \tag{16·26}$$

whence
$$F(\Lambda) \leqslant \lim_{r\to\infty} F(\Lambda_r). \tag{16·27}$$

Moreover, from the last theorem,

$$F(\Lambda) \geqslant \lim_{r\to\infty} F(\Lambda_r), \tag{16·28}$$

and so the assertion follows at once.

17. Lattice points on the boundary of an automorphic star body

Theorem 20. *Let K: $F(X) \leqslant 1$ be an automorphic star body, and let Λ be any lattice such that $F(\Lambda) > 0$. Then there exists a lattice Λ^* and a point P^* of Λ^* such that*

$$F(P^*) = F(\Lambda^*) = F(\Lambda), \quad d(\Lambda^*) = d(\Lambda). \tag{17·1}$$

(*Remark.* Λ^* need not be different from Λ. The theorem remains valid if $F(\Lambda) = 0$, but then is nearly trivial.)

Proof. Assume that Λ contains no point P such that

$$F(P) = F(\Lambda); \tag{17·2}$$

otherwise the assertion is certainly true. There exists then an infinite sequence of points $P_1, P_2, P_3 \ldots$ of Λ such that

$$\lim_{r\to\infty} F(P_r) = F(\Lambda) > 0; \tag{17·3}$$

assume that all these points are different from O.

For each point P_r select an automorphism Ω_r of K such that

$$|\Omega_r P_r| \leqslant c. \tag{17·4}$$

Put
$$\Omega_r P_r = Q_r, \quad \Omega_r \Lambda = \Lambda_r, \tag{17·5}$$

so that Q_r belongs to Λ_r, is different from O, and satisfies the inequality

$$|Q_r| \leqslant c. \tag{17·6}$$

By the invariance of K,
$$F(Q_r) = F(\Omega_r P_r) = F(P_r), \tag{17·7}$$

hence from the hypothesis
$$\lim_{r\to\infty} F(Q_r) = F(\Lambda) > 0. \tag{17·8}$$

Further, from theorem 17, Ω_r is of determinant ± 1, and so

$$d(\Lambda_r) = d(\Lambda). \tag{17·9}$$

Next, it is shown that

$$F(\Lambda_r) = F(\Lambda). \tag{17·10}$$

For if P runs over all points of $\Lambda - [O]$, then $Q = \Omega_r P$ runs over all points of $\Lambda_r - [O]$, and vice versa. But by the invariance assumption,

$$F(Q) = F(P), \tag{17·11}$$

and by definition,

$$F(\Lambda) = \underset{P \text{ in } \Lambda - [O]}{\text{l.b.}} F(P), \quad F(\Lambda_r) = \underset{Q \text{ in } \Lambda_r - [O]}{\text{l.b.}} F(Q), \tag{17·12}$$

whence (17·10) follows at once.

Finally, the sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is bounded. For from (17·9), the determinants $d(\Lambda_r)$ are bounded, and from (17·10),

$$F(Q) \geqslant F(\Lambda) \text{ for all points } Q \neq O \text{ of } \Lambda_r. \tag{17·13}$$

Hence, if ρ is any number such that K contains the sphere $|X| \leqslant \rho$, i.e. $F(\Lambda)K$ contains the sphere $|X| \leqslant \rho F(\Lambda)$, then

$$|Q| \geqslant F(\Lambda)\rho \text{ for all points } Q \neq O \text{ of } \Lambda_r. \tag{17·14}$$

From theorem 2, there exists then an infinite subsequence of lattices

$$\Lambda_{r_1}, \Lambda_{r_2}, \Lambda_{r_3}, \ldots$$

which tends to a limit, say the lattice Λ^*; from (17·9)

$$d(\Lambda^*) = \lim_{k\to\infty} d(\Lambda_{r_k}) = d(\Lambda). \tag{17·15}$$

Hence the supposition of theorem 19 is satisfied if one substitutes therein for the sequence of lattices $\{\Lambda_r\}$, the lattice Λ, and the sequence of points $\{P_r\}$ respectively, the sequence of lattices $\{\Lambda_{r_k}\}$, the lattice Λ^*, and the sequence of points $\{Q_{r_k}\}$ of the present proof. The assertion follows therefore at once from theorem 19.

Remark. Theorem 20 does not assert that every lattice Λ^* satisfying

$$F(\Lambda^*) = F(\Lambda), \quad d(\Lambda^*) = d(\Lambda) \tag{17·16}$$

contains a point P^* such that $F(P^*) = F(\Lambda^*)$. Thus take $n = 2$ and $F(X) = |x_1 x_2|^{\frac{1}{2}}$. Then, as follows from results in the theory of indefinite binary quadratic forms (Koksma 1936), there exists an infinity of lattices Λ^* such that

$$F(\Lambda^*) = 1, \quad d(\Lambda^*) = 3, \tag{17·17}$$

and some, but not all, of these lattices contain points P^* such that

$$F(P^*) = 1. \tag{17·18}$$

The following particular case of the last theorem is of special interest.

THEOREM 21. *Every automorphic star body K has a critical lattice with at least one point on the boundary of K.*

Proof. A lattice Λ is a critical lattice of K if and only if

$$F(\Lambda) = 1, \quad d(\Lambda) = \Delta(K). \tag{17·19}$$

Now, from theorem 8, critical lattices of K do exist; the assertion follows therefore at once from theorem 20.

PROBLEM D. *Does* every *critical lattice of an automorphic star body K have at least one point on the boundary of K?*

The example in theorem 20 does not answer this question, but makes it probable that the answer is in the negative.

Theorem 20 further suggests the following:

PROBLEM E. *To study the set d_F of the values of $d(\Lambda)$ where Λ runs over all lattices Λ satisfying $F(\Lambda) = 1$.*

The set d_F has a smallest element which is, of course, $\Delta(K)$; this number and the other elements of the set may be considered as the successive minima of the lattice point problem for the body K: $F(X) \leqslant 1$. Even in the case $F(X) = |x_1 x_2|^{\frac{1}{2}}$, d_F is a very complicated set (Koksma 1936), and the same is to be expected for other unbounded star bodies. It is then rather surprising that in the case of automorphic star bodies, all these minima are actually attained in the sense that to every element δ of d_F there exists a lattice Λ^* and a point P^* of Λ^* such that

$$F(P^*) = F(\Lambda^*) = 1, \quad d(\Lambda^*) = \delta. \tag{17·20}$$

18. THE INVARIANT SUBSET OF AN AUTOMORPHIC STAR BODY

Let K: $F(X) \leqslant 1$ be an automorphic star body, and let Γ be a group of automorphisms Ω of K. We denote by Σ_Γ the set of the points X in R_n which have the following property:

'There exists a positive number $a(X)$ depending only on X such that

$$|\Omega X| \leqslant a(X) \text{ for all } \Omega \text{ in } \Gamma.\text{'} \tag{18·1}$$

This set Σ_Γ is called the invariant manifold of K. It may contain only the origin, and it has the following properties:

(*a*) If X lies in Σ_Γ, and Ω is an element of Γ, then $Y = \Omega X$ also lies in Σ_Γ, and we may take

$$a(Y) = a(X). \tag{18·2}$$

For let Ω_1 be an arbitrary element of Γ. Then $\Omega_2 = \Omega_1 \Omega$ also belongs to Γ, and so by the definition of $a(X)$,

$$|\Omega_1 Y| = |\Omega_2 X| \leqslant a(X). \tag{18·3}$$

(*b*) If $X_1, X_2, \ldots, X_m$ is any number of points of Σ_Γ, and if $t_1, t_2, \ldots, t_m$ are real numbers, then $t_1X_1+t_2X_2+\ldots+t_mX_m$ also lies in Σ_Γ, and we may take

$$a(t_1X_1+t_2X_2+\ldots+t_mX_m) = |t_1|\,a(X_1)+|t_2|\,a(X)+\ldots+|t_m|\,a(X_m). \quad (18\cdot4)$$

For if Ω is any element of Γ, then

$$|\Omega(t_1X_1+\ldots+t_mX_m)| = |t_1\Omega X_1+\ldots+t_m\Omega X_m|$$
$$\leqslant |t_1|\,|\Omega X_1|+\ldots+|t_m|\,|\Omega X_m| \leqslant |t_1|\,a(X_1)+\ldots+|t_m|\,a(X_m). \quad (18\cdot5)$$

From (*b*), Σ_Γ is a linear manifold. Let it be of dimension δ where $0 \leqslant \delta \leqslant n$, and let $P_1, \ldots, P_\delta$ be a set of δ independent points of Σ_Γ. Then the points X of Σ_Γ may be written as

$$X = \xi_1P_1+\ldots+\xi_\delta P_\delta \quad (18\cdot6)$$

with real coefficients $\xi_1, \ldots, \xi_\delta$; conversely, every such point X belongs to Σ_δ. On considering this vector equation as a system of n equations for the n co-ordinates, we find on solving for $\xi_1, \ldots, \xi_\delta$ that

$$\max(|\xi_1|, \ldots, |\xi_\delta|) \leqslant \gamma|X|, \quad (18\cdot7)$$

where γ is a positive number depending only on the choice of $P_1, \ldots, P_\delta$.

(*c*) There exists a positive constant b such that if X is any point of Σ_Γ, Ω any element of Γ, and $Y = \Omega X$, then

$$b^{-1}|X| \leqslant |Y| \leqslant b|X|. \quad (18\cdot8)$$

For let $X = \xi_1P_1+\ldots+\xi_\delta P_\delta$. Then

$$|Y| = |\xi_1\Omega P_1+\ldots+\xi_\delta\Omega P_\delta| \leqslant \max(|\xi_1|, \ldots, |\xi_\delta|)(|\Omega P_1|+\ldots+|\Omega P_\delta|)$$
$$\leqslant \gamma|X|\{a(P_1)+\ldots+a(P_\delta)\} = b|X|, \quad (18\cdot9)$$

where
$$b = \gamma\{a(P_1)+\ldots+a(P_\delta)\}.$$

Further if X is in Σ_Γ and $Y = \Omega X$, then Y is also in Σ_Γ and $X = \Omega^{-1}Y$. Hence by the same proof $|X| \leqslant b|Y|$, whence the assertion.

Let now $J_\Gamma = K \times \Sigma_\Gamma$ be the set of all points of Σ_Γ which belong to K; we call J_Γ the invariant subset of K.

(*d*) The invariant subset J_Γ is a bounded set. For let X be any point of J_Γ. By definition 9, there exists a positive constant c and an element Ω of Γ such that

$$|\Omega X| \leqslant c. \quad (18\cdot10)$$

Hence from (*c*),
$$|X| \leqslant b|\Omega X| \leqslant bc, \quad (18\cdot11)$$

as asserted.

This result shows that the dimension δ of Σ_Γ and J_Γ is at most $n-1$. For let this assertion be false so that $\delta = n$. Then Σ_Γ coincides with the whole space R_n, and therefore J_Γ is identical with K. Hence, from (*d*), K is a bounded set, contrary to the definition of an automorphic star body.

Probably δ satisfies the stronger inequality $\delta \leqslant n-2$. The following example shows, however, that δ can be any integer in the interval

$$0 \leqslant \delta \leqslant n-2.$$

Take for K the star body of distance function

$$F(X) = \max\left(\{x_1^2 + \dots + x_\delta^2\}^{\frac{1}{2}}, |x_{\delta+1} \dots x_n|^{1/(n-\delta)}\right), \tag{18·12}$$

and for Γ the group of automorphisms

$$x_1' = x_1, \dots, x_\delta' = x_\delta, \quad x_{\delta+1}' = t_{\delta+1}x_{\delta+1}, \dots, x_n' = t_n x_n, \tag{18·13}$$

where $t_{\delta+1}, \dots, t_n$ are real numbers of product $t_{\delta+1} \dots t_n = 1$; then Σ_Γ is the δ-dimensional linear manifold

$$x_{\delta+1} = \dots = x_n = 0. \tag{18·14}$$

The automorphic star bodies with $\delta = 0$ are of particular interest; then both Σ_Γ and J_Γ reduce to the single point O. To this type belong, for instance, all the star bodies considered in § 15. In § 20, a general property of star bodies with $\delta = 0$ will be proved.

19. An improvement on theorem 13

Theorem 22. *Let K: $F(X) \leqslant 1$ be any star body of the finite type. Then there exists to every number $\epsilon > 0$ a positive number $t = t(\epsilon)$ such that every critical lattice Λ of K contains at least one point P satisfying the inequalities*

$$1 \leqslant F(P) < 1 + \epsilon, \quad |P| \leqslant t. \tag{19·1}$$

Proof. By the remark to theorem 10, there is a positive number $t^* = t^*(\epsilon)$ such that the star body

$$K^* = K^{(t^*)}: \qquad F(X) \leqslant 1, \quad |X| \leqslant t^*$$

is of determinant

$$\Delta(K^*) \geqslant \left(1 + \frac{\epsilon}{2}\right)^{-n} \Delta(K). \tag{19·2}$$

Put

$$t = \left(1 + \frac{\epsilon}{2}\right) t^*, \quad K^{**} = \left(1 + \frac{\epsilon}{2}\right) K^*, \tag{19·3}$$

so that K^{**} consists of the points satisfying

$$F(X) \leqslant 1 + \frac{\epsilon}{2}, \quad |X| \leqslant \left(1 + \frac{\epsilon}{2}\right) t^* = t, \tag{19·4}$$

then

$$\Delta(K^{**}) = \left(1 + \frac{\epsilon}{2}\right)^n \Delta(K^*) \geqslant \Delta(K). \tag{19·5}$$

Hence every lattice of determinant $\Delta(K)$ contains a point $P \neq O$ for which

$$F(P) \leqslant 1 + \frac{\epsilon}{2} < 1 + \epsilon, \quad |P| \leqslant t; \tag{19·6}$$

if the lattice is critical with respect to K, then moreover

$$F(P) \geqslant 1, \tag{19·7}$$

whence the assertion.

20. Automorphic star bodies with $\Sigma_\Gamma = J_\Gamma = \{O\}$

Theorem 23. *Let K: $F(X) \leqslant 1$ be an automorphic star body for which Σ_Γ and so also J_Γ consist of the single point O. Further let Λ be any critical lattice of K, and ϵ any positive number. Then there exists an infinite sequence of different points $P_1, P_2, P_3, \ldots$ of Λ such that*

$$1 \leqslant F(P_\mu) < 1+\epsilon, \quad \textit{where} \quad \mu = 1, 2, 3, \ldots. \tag{20·1}$$

Proof. Assume the assertion is false. There is then a positive number ϵ and a critical lattice Λ of K such that the inequality

$$1 \leqslant F(P_\mu) < 1+\epsilon \tag{20·2}$$

is satisfied by only a finite number of points of Λ, say by only the m points

$$P_1,\ P_2,\ \ldots,\ P_m;$$

by the last theorem, m is not zero. It may be assumed, without loss of generality, that ϵ and Λ have been chosen so as to make m a minimum, that is,

> There does not exist any positive number ϵ^* and any critical lattice Λ^* of K such that the inequality
>
> $$1 \leqslant F(P_\mu^*) < 1+\epsilon^* \tag{20·3}$$
>
> is satisfied by less than m points P_μ^* of Λ^*.

This minimum assumption implies, in particular, that

$$F(P_\mu) = 1, \quad \text{where} \quad \mu = 1, 2, \ldots, m; \tag{20·4}$$

for if, for instance, $F(P_m) = 1+\delta > 1$, then, on putting $\epsilon^* = \delta$, $\Lambda^* = \Lambda$, there are less than m points P^* of Λ^* such that

$$1 \leqslant F(P^*) < 1+\epsilon^*. \tag{20·5}$$

Let now Ω be any automorphism in Γ. Then from (20·2), (20·4) and theorem 22, the lattice $\Omega\Lambda$ has the following properties:

> There are just m points P^* of $\Omega\Lambda$ for which
>
> $$1 \leqslant F(P^*) < 1+\epsilon, \tag{20·6}$$
>
> viz. the points $$P^* = \Omega P_1,\ \Omega P_2,\ \ldots,\ \Omega P_m; \tag{20·7}$$
>
> and, in fact, $$F(\Omega P_\mu) = 1, \quad \text{where} \quad \mu = 1, 2, \ldots, m. \tag{20·8}$$
>
> There is, moreover, a positive number t independent of Ω and μ such that
>
> $$|\Omega P_\mu| \leqslant t \text{ for at least one index } \mu \text{ with } 1 \leqslant \mu \leqslant m. \tag{20·8$\tfrac{1}{2}$}$$

From (20·4), P_m is different from O, and so does not belong to Σ_Γ. Hence there exists an infinite sequence

$$\{\Omega_r^{(m)}\} = \{\Omega_1^{(m)}, \Omega_2^{(m)}, \Omega_3^{(m)}, \ldots\} \tag{20·9}$$

of automorphisms $\Omega_r^{(m)}$ of K such that

$$\lim_{r\to\infty} |\Omega_r^{(m)} P_m| = \infty. \tag{20·10}$$

Now construct $m-1$ infinite subsequences

$$\{\Omega_r^{(\mu)}\} = \{\Omega_1^{(\mu)}, \Omega_2^{(\mu)}, \Omega_3^{(\mu)}, \ldots\} \tag{20·11}$$

of $\{\Omega_r^{(m)}\}$ according to the following rule:

Suppose the sequence $\{\Omega_r^{(\mu+1)}\}$ has been defined. If now

$$\lim_{r\to\infty} |\Omega_r^{(\mu+1)} P_\mu| = \infty, \tag{20·12}$$

then let $\{\Omega_r^{(\mu)}\}$ be identical with $\{\Omega_r^{(\mu+1)}\}$:

$$\Omega_r^{(\mu)} = \Omega_r^{(\mu+1)}, \quad \text{where} \quad r = 1, 2, 3, \ldots. \tag{20·13}$$

If, however,

$$\liminf_{r\to\infty} |\Omega_r^{(\mu+1)} P_\mu| \tag{20·14}$$

is finite, then choose for $\{\Omega_r^{(\mu)}\}$ an infinite subsequence

$$\Omega_1^{(\mu)} = \Omega_{r_1}^{(\mu+1)},\ \Omega_2^{(\mu)} = \Omega_{r_2}^{(\mu+1)},\ \Omega_3^{(\mu)} = \Omega_{r_3}^{(\mu+1)},\ \ldots \tag{20·15}$$

of $\{\Omega_r^{(\mu+1)}\}$ such that the point sequence

$$\{\Omega_1^{(\mu)} P_\mu,\ \Omega_2^{(\mu)} P_\mu,\ \Omega_3^{(\mu)} P_\mu,\ \ldots\} \tag{20·16}$$

tends to a limit, say the point P_μ^*.

This means that the last sequence $\{\Omega_r^{(1)}\}$ has the following properties:

$$\lim_{r\to\infty} |\Omega_r^{(1)} P_m| = \infty. \tag{20·17}$$

If μ is one of the indices $1, 2, \ldots, m-1$, then either

$$\lim_{r\to\infty} |\Omega_r^{(1)} P_\mu| = \infty, \tag{20·18}$$

or there exists a finite point P_μ^* such that

$$\lim_{r\to\infty} |\Omega_r^{(1)} P_\mu - P_\mu^*| = 0. \tag{20·19}$$

Denote then by $\mu_1, \mu_2, \ldots, \mu_g$ those different indices μ with $1 \leqslant \mu \leqslant m$ for which

$$\lim_{r\to\infty} |\Omega_r^{(1)} P_\mu| = \infty, \tag{20·20}$$

by $\mu_1^*, \mu_2^*, \ldots, \mu_h^*$ those for which

$$\lim_{r\to\infty} |\Omega_r^{(1)} P_\mu - P_\mu^*| = 0; \tag{20·21}$$

hence $g+h = m$. From (20·11) and (20·17), then

$$g \geqslant 1, \quad h \geqslant 1. \tag{20·22}$$

Since $\Omega_r^{(1)}$ is an automorphism of K, it is evident that

$$d(\Omega_r^{(1)}\Lambda) = d(\Lambda) = \Delta(K), \quad F(\Omega_r^{(1)}\Lambda) = F(\Lambda) = 1, \quad \text{where} \quad r = 1, 2, 3, \ldots \tag{20·23}$$

(for the second equation, compare the proof of theorem 20), and so the lattices

$$\{\Omega_1^{(1)}\Lambda, \Omega_2^{(1)}\Lambda, \Omega_3^{(1)}\Lambda, \ldots\} \tag{20·24}$$

form a bounded sequence. From theorem 2, one can therefore choose an infinite subsequence $\{\Omega_r\}$ of automorphisms

$$\Omega_1 = \Omega_{r'}^{(1)}, \ \Omega_2 = \Omega_{r''}^{(1)}, \ \Omega_3 = \Omega_{r'''}^{(1)} \ldots \tag{20·25}$$

in $\{\Omega_r^{(1)}\}$ such that the corresponding sequence of lattices

$$\Lambda_1 = \Omega_1\Lambda, \ \Lambda_2 = \Omega_2\Lambda, \ \Lambda_3 = \Omega_3\Lambda, \ \ldots \tag{20·26}$$

tends to a limit, the lattice Λ^*, say.

Then from (20·23) it follows that

$$\lim_{r\to\infty} d(\Lambda_r) = \Delta(K), \quad \lim_{r\to\infty} F(\Lambda_r) = 1. \tag{20·27}$$

Further, from (20·8½), and the construction of $\{\Omega_r^{(\mu)}\}$ and $\{\Lambda_r\}$, each lattice

$$\Lambda_r = \Omega_r\Lambda, \quad \text{where} \quad r = 1, 2, 3, \ldots, \tag{20·28}$$

contains a point $$P^{(r)} = \Omega_r P_{\mu(r)} \quad \text{with} \quad 1 \leqslant \mu(r) \leqslant m-1, \tag{20·29}$$

such that $$P^{(r)} \neq O, \quad |P^{(r)}| \leqslant t, \quad F(P^{(r)}) = 1. \tag{20·30}$$

An application of theorem 19 therefore gives

$$d(\Lambda^*) = \lim_{r\to\infty} d(\Lambda_r) = \Delta(K), \quad F(\Lambda^*) = \lim_{r\to\infty} F(\Lambda_r) = 1, \tag{20·31}$$

which means that Λ^* is a critical lattice. Now a consideration analogous to that in earlier proofs makes it evident that the points

$$P^*_{\mu_1^*}, \ P^*_{\mu_2^*}, \ \ldots, \ P^*_{\mu_h^*}$$

as defined in (20·21), are the only points P^* of Λ^* such that

$$1 \leqslant F(P^*) < 1 + \frac{\epsilon}{2}; \tag{20·32}$$

moreover $$F(P^*_{\mu_1^*}) = F(P^*_{\mu_2^*}) = \ldots = F(P^*_{\mu_h^*}) = 1. \tag{20·33}$$

Hence Λ^* is a lattice of the same type as Λ, except that m is replaced by the smaller number h. This contradicts the minimum assumption (20·3); the hypothesis is therefore false and the assertion is true.

Problem F. *Does the assertion of theorem 23 remain true if Σ_Γ is of positive dimension δ?*

Closely related to problem F is the following question which I also have not been able to solve:

PROBLEM G. *To decide whether there exists an automorphic star body K: $F(X) \leqslant 1$ with the following two properties: (a) The invariant manifold Σ_Γ is of positive dimension. (b) There exists a critical lattice Λ of K and a positive number α such that*

$$F(P) \geqslant 1+\alpha \tag{20·34}$$

for all points P of Λ which do not belong to Σ_Γ.

21. STAR BODIES OF RANK δ

The considerations in § 18 can be generalized and lead to the following definition:

DEFINITION 10. *Let K: $F(X) \leqslant 1$ be a star body of the finite type with a group Γ of automorphisms Ω, and let δ be an integer such that $1 \leqslant \delta \leqslant n-1$. Then K is said to be of rank δ with respect to Γ if δ is the largest integer such that to every positive number t^* and to every δ-dimensional linear manifold M containing O there is an element $\Omega = \Omega(t^*, M)$ of Γ satisfying*

$$|\Omega X| \geqslant t^* F(X) \quad \textit{for all points } X \textit{ of } M. \tag{21·1}$$

An example on this definition is given by

THEOREM 24. *Let K be the star body of distance function*

$$F(X) = \left| \prod_{\rho=1}^{r} x_\rho \prod_{\sigma=1}^{s} (x_{r+\sigma}^2 + x_{r+s+\sigma}^2) \right|^{1/n}, \quad \textit{where} \quad r+2s = n, \tag{21·2}$$

and let Γ be the group of all automorphisms Ω of K defined by

$$\Omega: \begin{cases} x'_\rho = t_\rho x_\rho, & \textit{where} \quad \rho = 1, 2, \ldots, r, \quad (21·3) \\ \left.\begin{aligned} x'_{r+\sigma} &= t_{r+\sigma} x_{r+\sigma} - t_{r+s+\sigma} x_{r+s+\sigma} \\ x'_{r+s+\sigma} &= t_{r+s+\sigma} x_{r+\sigma} + t_{r+\sigma} x_{r+s+\sigma} \end{aligned}\right\} & \textit{where} \quad \sigma = 1, 2, \ldots, s, \quad (21·4),\ (21·5) \end{cases}$$

where $t_1, t_2, \ldots, t_n$ are real numbers satisfying

$$\prod_{\rho=1}^{r} t_\rho \prod_{\sigma=1}^{s} (t_{r+\sigma}^2 + t_{r+s+\sigma}^2) = 1. \tag{21·6}$$

Further let $\qquad r \geqslant 0, \quad s \geqslant 0, \quad r+s > 1.$

Then K is of rank $r+s-1$ with respect to Γ.

Proof. An arbitrary linear manifold M through O of dimension $r+s-1$ can be defined by $n-(r+s-1) = s+1$ independent homogeneous linear equations

$$a_{h1} x_1 + a_{h2} x_2 + \ldots + a_{hn} x_n = 0, \quad \text{where} \quad h = 1, 2, \ldots, s+1, \tag{21·7}$$

and where the a's are real numbers. Two cases may now be distinguished:

(a) First assume that $r > 0$, and that at least one coefficient

$$a_{hk} \quad \text{with} \quad 1 \leqslant h \leqslant s+1,\ 1 \leqslant k \leqslant r$$

is different from zero, say the coefficient a_{11}.

Then, on solving the equation,

$$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=0 \tag{21·8}$$

for x_1,
$$x_1=b_2x_2+\ldots+b_nx_n, \tag{21·9}$$

where $b_2, \ldots, b_n$ are real numbers; hence there is a positive constant γ such that

$$|x_1| \leqslant \gamma\{x_2^2+\ldots+x_n^2\}^{\frac{1}{2}} \tag{21·10}$$

for all points X of M. Put now $t=\gamma^{1/n}t^*$, and apply the automorphism $X'=\Omega X$ defined by

$$x_1'=t^{-(n-1)}x_1,\ x_2'=tx_2,\ \ldots,\ x_n'=tx_n, \tag{21·11}$$

that is
$$x_1=t^{n-1}x_1',\ x_2=t^{-1}x_2',\ \ldots,\ x_n=t^{-1}x_n'. \tag{21·12}$$

Then
$$F(X)=F(X'), \tag{21·13}$$

and from (21·10) it follows that

$$|t^{n-1}x_1'| \leqslant \gamma\left\{\left(\frac{x_2'}{t}\right)^2+\ldots+\left(\frac{x_n'}{t}\right)^2\right\}^{\frac{1}{2}}, \tag{21·14}$$

whence

$$t^nF(X')^n \leqslant \gamma\{x_2'^2+\ldots+x_n'^2\}^{\frac{1}{2}}\left|\prod_{\rho=2}^{r}x_\rho'\prod_{\sigma=1}^{s}(x_{r+\sigma}'^2+x_{r+s+\sigma}'^2)\right| \leqslant \gamma\{x_2'^2+\ldots+x_n'^2\}^{\frac{1}{2}\{1+(r-1)+2s\}}. \tag{21·15}$$

Hence

$$t^nF(X)^n=t^nF(X')^n \leqslant \gamma|X'|^n, \quad |\Omega X|=|X'| \geqslant \gamma^{-1/n}tF(X)=t^*F(X), \tag{21·16}$$

as asserted.

(*b*) Secondly, let either $r=0$, or assume that $r>0$, but that all coefficients

$$a_{hk} \quad \text{with} \quad 1 \leqslant h \leqslant s+1,\ 1 \leqslant k \leqslant r$$

vanish.

Then the equations defining M are of the form

$$a_{h,r+1}x_{r+1}+a_{h,r+2}x_{r+2}+\ldots+a_{h,n}x_n=0, \quad \text{where} \quad h=1,2,\ldots,s+1. \tag{21·17}$$

Arrange the $2s$ co-ordinates $x_{r+1}, x_{r+2}, \ldots, x_n$ as s pairs

$$(x_{r+\sigma}, x_{r+s+\sigma}), \quad \text{where} \quad \sigma=1,2,\ldots,s. \tag{21·18}$$

Since the $s+1$ equations defining M are independent, and since there are only s such pairs of co-ordinates, it must be possible to express at least one such pair of these co-ordinates in terms of the others. Now assume this is the pair (x_{r+1}, x_{r+s+1}), and that on solving for x_{r+1}, x_{r+s+1}, the following equations are obtained:

$$x_{r+1}=\sum_{\sigma=2}^{s}(b_\sigma x_{r+\sigma}+b_\sigma' x_{r+s+\sigma}), \quad x_{r+s+1}=\sum_{\sigma=2}^{s}(c_\sigma x_{r+\sigma}+c_\sigma' x_{r+s+\sigma}), \tag{21·19}$$

where the coefficients b_σ, b_σ', c_σ, c_σ' are real numbers. Hence there is a positive constant γ such that

$$x_{r+1}^2+x_{r+s+1}^2 \leqslant \gamma\sum_{\sigma=2}^{s}(x_{r+\sigma}^2+x_{r+s+\sigma}^2), \tag{21·20}$$

for all points X of M. Put now $t = \gamma^{1/2s} t^{*n/2s}$, and apply the automorphism $X' = \Omega X$ defined by

$$x'_\rho = x_\rho, \quad \text{where} \quad \rho = 1, 2, \ldots, r, \tag{21·21}$$

$$x'_{r+1} = t^{-(s-1)} x_{r+1}, \quad x'_{r+s+1} = t^{-(s-1)} x_{r+s+1}, \tag{21·22}$$

$$x'_{r+\sigma} = t x_{r+\sigma}, \quad x'_{r+s+\sigma} = t x_{r+s+\sigma}, \quad \text{where} \quad \sigma = 2, 3, \ldots, s, \tag{21·23}$$

or conversely,

$$x_\rho = x'_\rho, \quad \text{where} \quad \rho = 1, 2, \ldots, r \tag{21·24}$$

$$x_{r+1} = t^{s-1} x'_{r+1}, \quad x_{r+s+1} = t^{s-1} x'_{r+s+1}, \tag{21·25}$$

$$x_{r+\sigma} = t^{-1} x'_{r+\sigma}, \quad x_{r+s+\sigma} = t^{-1} x'_{r+s+\sigma}, \quad \text{where} \quad \sigma = 2, 3, \ldots, s. \tag{21·26}$$

Then again

$$F(X) = F(X'), \tag{21·27}$$

and from (21·20)

$$t^{2(s-1)}(x'^2_{r+1} + x'^2_{r+s+1}) \leqslant \gamma t^{-2} \sum_{\sigma=2}^{s} (x'^2_{r+\sigma} + x'^2_{r+s+\sigma}) \leqslant \gamma t^{-2} \mid X' \mid^2, \tag{21·28}$$

whence

$$t^{2s} F(X')^n \leqslant \gamma \mid X' \mid^2 \left| \prod_{\rho=1}^{r} x'_\rho \prod_{\sigma=2}^{s} (x'^2_{r+\sigma} + x'^2_{r+s+\sigma}) \right| \leqslant \gamma \mid X' \mid^{2+r+2(s-1)} = \gamma \mid X' \mid^n. \tag{21·29}$$

Hence

$$\mid \Omega X \mid = \mid X' \mid \geqslant \gamma^{-1/n} t^{2s/n} F(X) = t^* F(X), \tag{21·30}$$

as asserted.

Up to now it has only been proved that the rank δ of K with respect to Γ is at least $r+s-1$; one now proves that $\delta < r+s$. This is trivial from definition 10 if $s = 0$. Let therefore $s > 0$. Consider the special $(r+s)$-dimensional linear manifold M_0 defined by the equations

$$x_{r+s+\sigma} = 0, \quad \text{where} \quad \sigma = 1, 2, \ldots, s. \tag{21·31}$$

It suffices to prove that, however Ω is chosen in Γ, there is at least one point X of M_0 such that

$$\mid \Omega X \mid < \sqrt{(n+1)}\, F(X). \tag{21·32}$$

There is no loss of generality in assuming that the point X is such that

$$F(X) = 1; \tag{21·33}$$

hence the point $X = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}, 0, \ldots, 0)$ of M_0 satisfies the equation

$$\left| \prod_{\rho=1}^{r} x_\rho \prod_{\sigma=1}^{s} x^2_{r+\sigma} \right| = 1, \tag{21·34}$$

but is otherwise arbitrary.

Let now Ω be any element of Γ, and X the point above of M_0. Then the co-ordinates of $X' = \Omega X$ take the form

$$x'_\rho = t_\rho x_\rho, \quad \text{where} \quad \rho = 1, 2, \ldots, r, \tag{21·35}$$

$$x'_{r+\sigma} = t_{r+\sigma} x_{r+\sigma}, \quad x'_{r+s+\sigma} = t_{r+s+\sigma} x_{r+\sigma}, \quad \text{where} \quad \sigma = 1, 2, \ldots, s, \tag{21·36}$$

and where

$$\prod_{\rho=1}^{r} t_\rho \prod_{\sigma=1}^{s} (t^2_{r+\sigma} + t^2_{r+s+\sigma}) = 1. \tag{21·37}$$

Choose now X in M_0 such that

$$x_\rho = t_\rho^{-1}, \quad \text{where} \quad \rho = 1, 2, \ldots, r, \tag{21·38}$$

$$\left.\begin{aligned} x_{r+\sigma} &= (t_{r+\sigma}^2 + t_{r+s+\sigma}^2)^{-\frac{1}{2}}, \\ x_{r+s+\sigma} &= 0, \end{aligned}\right\} \quad \text{where} \quad \sigma = 1, 2, \ldots, s; \qquad (21·39),\ (21·40)$$

then evidently $F(X) = 1$, as assumed. This choice of X implies that

$$x'_\rho = 1, \quad \text{where} \quad \rho = 1, 2, \ldots, r, \tag{21·41}$$

$$x'^2_{r+\sigma} + x'^2_{r+s+\sigma} = 1, \quad \text{where} \quad \sigma = 1, 2, \ldots, s, \tag{21·42}$$

and so
$$|X'|^2 = r+s < n+1, \tag{21·43}$$

whence
$$|\Omega X| = |X'| < \sqrt{(n+1)} = \sqrt{(n+1)}\, F(X), \tag{21·44}$$

as asserted. This completes the proof.

THEOREM 25. *Let K: $F(X) \leqslant 1$ be a star body of rank δ with respect to Γ, Λ a critical lattice of K, and ϵ an arbitrary positive number. Then there exist $\delta+1$ independent points $P_1, P_2, \ldots, P_{\delta+1}$ of Λ such that*

$$1 \leqslant F(P_\mu) < 1+\epsilon, \quad \textit{where} \quad \mu = 1, 2, \ldots, \delta+1. \tag{21·45}$$

Proof. Let the assertion be false, i.e. assume that there is a critical lattice Λ_0 of K and a positive number ϵ such that all lattice points P_0 of Λ_0 satisfying

$$1 \leqslant F(P_0) < 1+\epsilon \tag{21·46}$$

lie in a certain δ-dimensional linear manifold M containing O.

From theorem 22, there is a positive number t such that every critical lattice Λ of K contains at least one point P such that

$$1 \leqslant F(P) < 1+\epsilon, \quad |P| \leqslant t. \tag{21·47}$$

Further, by the last definition applied with $t^* = t+1$, there exists an automorphism Ω in Γ such that

$$|\Omega X| \geqslant (t+1)\, F(X) \quad \text{for all points } X \text{ in } M. \tag{21·48}$$

Denote now by
$$P_1, P_2, P_3, \ldots$$
the points of Λ_0 for which

$$1 \leqslant F(P_r) < 1+\epsilon, \quad \text{where} \quad r = 1, 2, 3, \ldots; \tag{21·49}$$

by hypothesis, these points belong to M. Then the only points Q_r of the lattice $\Lambda = \Omega\Lambda_0$ satisfying

$$1 \leqslant F(Q_r) < 1+\epsilon \tag{21·50}$$

are those given by
$$Q_r = \Omega P_r, \quad \text{where} \quad r = 1, 2, 3, \ldots, \tag{21·51}$$

and for these points
$$|Q_r| = |\Omega P_r| \geqslant (t+1)\, F(P_r) \geqslant t+1, \tag{21·52}$$

contrary to the existence result (21·47). Hence the assertion is true.

From theorems 24 and 25, it is deduced that if K is the star body of distance function

$$F(X) = |x_1 x_2 \dots x_n|^{1/n}, \tag{21·53}$$

and Λ is any critical lattice of K, then there exist n independent points $P_1, P_2, \dots, P_n$ of Λ such that

$$1 \leqslant F(P_g) < 1 + \epsilon, \quad \text{where} \quad g = 1, 2, \dots, n, \tag{21·54}$$

however small ϵ is chosen. Hence problem A can be solved in this special case, and the answer is in the affirmative.

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