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Lattice points in n-dimensional star bodies II

BY

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Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems).* By K. MAHLER. (First communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

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The star body K is called *reducible* if there exists a star body H contained in, but different from, K such that $\triangle(H) = \triangle(K)$, and is otherwise called *irreducible*. We say further that K is *boundedly reducible* if a bounded star H contained in, but different from, K with $\triangle(H) = \triangle(K)$ exists. In this Part II, I prove conditions for K to be reducible (irreducible) or boundedly reducible. The irreducible star bodies, as well as the boundedly reducible star bodies, seem to have many interesting properties, and I believe that a further study might lead to results of importance in themselves and for applications to other problems.

As in Part I, I have stated a number of problems which seem to deserve further study *).

In order to make this paper intelligible to a reader who has not seen the first part, I repeat here the main definitions and theorems.

Let R_n be the n-dimensional space of all points $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, etc., with real coordinates; $O = (0, \ldots, 0)$ is the origin of R_n. We put $|X| = (x_1^2 + \ldots + x_n^2)^{1/2}$, and use the notation $X + Y$ and tX for the points $(x_1 + y_1, \ldots, x_n + y_n)$ and $(tx_1, \ldots tx_n)$. The determinant $\| x_{hk} \|$ of n points $X_h = (x_{h2}, \ldots, x_{hn})$ is denoted by $\{X_1, \ldots, X_n\}$. If this determinant does not vanish, then the set of all points

$$X = u_1X_1 + \ldots + u_nX_n \qquad (u_1, \ldots, u_n = 0, \mp 1, \mp 2, \ldots)$$

forms a lattice Λ of basis $X_1, \ldots, X_n$ and determinant

$$d(\Lambda) = |\{X_1, \ldots, X_n\}|.$$

There always exist reduced bases, i.e. such that the quadratic form

$$\Phi(u_1, \ldots, u_n) = \sum_{k=1}^{n} (x_{1k}\, u_1 + \ldots + x_{nk}\, u_n)^2$$

is reduced in the sense of MINKOWSKI; for such reduced bases,

$$|X_1||X_2| \ldots |X_n| \leqq \gamma_n\, d(\Lambda),$$

where $\gamma_n > 0$ depends only on n.

An infinite sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is called *bounded* if there are two constants $c > 0$ and $c' > 0$ such that

$$d(\Lambda_r) \leq c, \text{ and } |X| \geq c' \text{ for every point } X \neq O \text{ of } \Lambda_r \quad (r = 1, 2, \ldots).$$

The following theorem is fundamental for the whole paper:

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*) Part I of this paper is to appear in the Proceedings of the Royal Society.

Theorem 2: *Every bounded infinite sequence S of lattices contains a convergent infinite subsequence S′* (i.e., reduced bases of the elements of S' tend to a basis of the limiting lattice Λ, and so Λ consists just of the limit points of the lattices in S').

A function $F(X) = F(x_1, \ldots, x_n)$ of the variable point X in R_n is called a distance function if

(i) $F(X) \geqq 0$ for all $X \neq O$ in R_n, and $F(O) = 0$.
(ii) $F(tX) = |t| F(X)$ for all X and all real t.
(iii) $F(X)$ is a continuous function of X.

We then say that the point set K: $F(X) \leq 1$ is a *star body*. Such a star body contains, and is symmetrical in, the origin; and with every point X the whole line segment OX belongs to it. The boundary C of K may extend to infinity, but is continuous in the finite part.

A lattice Λ is *K-admissible* if no point of Λ except O is an *inner point* of K. According as to whether K-admissible lattices do, or do not, exist, we say that K is of the finite or the infinite type; in the first case, $\Delta(K)$ denotes the lower bound of $d(\Lambda)$ extended over all admissible lattices, and we put $\Delta(K) = \infty$ in the second case. It is nearly trivial that bounded star bodies are of the finite type, and that unbounded star bodies of both types exist. One proves easily:

Theorem 7: *If the star body H is contained in the star body K, then* $\Delta(H) \leq \Delta(K)$.

From Theorem 2, the following existence theorem is derived:

Theorem 8: *Every star body of the finite type possesses at least one critical lattice*, i.e. a lattice Λ such that (i) $d(\Lambda) = \Delta(K)$, and (ii) Λ is K-admissible.

Part 1 deals mainly with the properties of critical lattices, and discusses, in particular, their points on, or near to, the boundary of the star body. I quote the following theorems, since I refer to them in the present paper:

Theorem 9: *Let $K, K_1, K_2, \ldots$ be an infinity of star bodies of the finite type such that for every $\varepsilon > 0$ and every $t > 0$, (i) K_r is contained in $(1+\varepsilon)K$ if $r \geq r_0(\varepsilon)$, and (ii) the subset $|X| \leq t$, $F(X) \leq 1$ of K is contained in $(1+\varepsilon)K_r$ if $r \geq r_1(\varepsilon, t)$. Then $\lim_{r \to \infty} \Delta(K_r) = \Delta(K)$.*

Theorem 11: *Every critical lattice of a bounded star body has n independent points on its boundary.*

Theorem 15: *There exists an unbounded star body of the finite type with a critical lattice which has no points on its boundary.*

Theorem 16: *Let K: $F(X) \leq 1$ be of the finite type, Ω a linear transformation of R_n into itself of determinant $\omega \neq 0$, $F'(X) = F(\Omega X)$, and let K' be the star body $F'(X) \leq 1$. Then also K' is of the finite type, and $\Delta(K') = |\omega|^{-1} \Delta(K)$.*

Theorem 17: *If the star body K of the finite type admits the automorphism Ω* (i.e., $F(\Omega X) = F(X)$ identically in X), *then Ω is of determinant $\omega = \pm 1$.*

Theorem 19: *Let $\Lambda_1, \Lambda_2, \ldots$ be a convergent sequence of lattices of limit Λ, and let $\varphi = \lim_{r\to\infty} F(\Lambda_r) > 0$ exist* ($F(\Lambda_r)$ denotes the lower bound of $F(X)$ extended over all points $X \neq O$ of Λ_r). *Let there be a constant $c > 0$ such that in each Λ_r there is a point $P_r \neq O$ satisfying $|P_r| \leq c$ and $\lim_{r\to\infty} F(P_r) = \varphi$. Then $\lim_{r\to\infty} F(\Lambda_r) = F(\Lambda)$, and there is a point $P \neq O$ of Λ such that $F(P) = F(\Lambda)$.*

Theorem 23: *Let K: $F(X) \leq 1$ be an unbounded star body of the finite type, and let Γ be the group of its automorphisms. Let there be a constant $c > 0$ such that, if P is any point of K, then $|\Omega P| \leq c$ for at least one element Ω of Γ; and if $t > 0$ is arbitrary and $P \neq O$ is any point of K,* let $|\Omega P| > t$ *for at least one element Ω of Γ. Then, however small $\varepsilon > 0$, every critical lattice of K contains an infinity of different points P satisfying $1 \leq F(P) < 1 + \varepsilon$.*

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§ 1. *Preliminary remark.*

The theorems and definitions in Part I of this paper were numbered with Arabic numerals, and the problems with Latin capitals. In order to simplify references, theorems and definitions in the present Part II shall be numbered with Latin capitals, and problems with Arabic numerals.

§ 2. *The relations $H < K$ and $H \prec K$.*

Let H: $G(X) \leq 1$ and K: $F(X) \leq 1$ be two star bodies of the finite type in R_n. If H is a proper subset of K, then we write

$$H < K \quad \text{or} \quad K > H.$$

By Theorem 7, this relation implies that

$$\triangle(H) \leq \triangle(K),$$

where the equality sign may, or may not, hold. We therefore use the further symbol

$$H \prec K \text{ or } K \succ H$$

to denote that both

$$H < K \text{ and } \triangle(H) = \triangle(K).$$

From this definition,

$$\textit{if } H \prec K \textit{ and } K \prec L, \textit{ then } H \prec L,$$

and

$$\textit{if } H < K < L \textit{ and } H \prec L, \textit{ then } H \prec K \prec L.$$

It is further clear that

$H < K$ implies $H \prec K$, if and only if at least one critical lattice of H is K-admissible.

Theorem A: *To every star body H of the finite type, there exist star bodies K of the finite type such that $H \prec K$.*

Proof: Choose for Λ any critical lattice of H, and for K any star body such that $K > H$ and that Λ is K-admissible.

On the other hand, when K is given, then it is not always possible to find a star body H such that $H \prec K$. We therefore define:

Definition A: *The star body K is called reducible if there exists a star body H such that $H \prec K$, and it is called irreducible if no star body H with $H \prec K$ exists.*

Theorem A shows that there are reducible star bodies. We shall show later that also irreducible star bodies exist, and give some examples of such bodies. But we shall first obtain necessary, respectively sufficient, conditions of irreducibility. So far, I have not yet succeeded in finding conditions which are both necessary and sufficient.

§ 3. *A necessary condition for irreducibility.*

We need the following lemma which is closely related to Theorem 19 in Part 1:

Theorem B: *Let K: $F(X) \leq 1$ be a star body of the finite type, and $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ an infinite sequence of K-admissible lattices with the following properties:*

(a): $$\lim_{r \to \infty} d(\Lambda_r) = \triangle(K);$$

(b): *Every lattice Λ_r contains a point P_r on the boundary C: $F(X) = 1$ of K;*

(c): *The points $P_1, P_2, P_3, \ldots$ tend to a limit point P.*

Then P lies on C, and there exists a critical lattice Λ of K containing P.

Proof: The first part of the assertion follows at once from

$$F(P) = \lim_{r \to \infty} F(P_r) = \lim_{r \to \infty} 1 = 1.$$

For the second part, we first remark that the sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is bounded. Hence, by Theorem 2, there is an infinite subsequence

$$\Lambda^{(1)} = \Lambda_{k_1}; \Lambda^{(2)} = \Lambda_{k_2}; \Lambda^{(3)} = \Lambda_{k_3}; \ldots, \qquad (k_1, < k_2 < k_3 < \ldots)$$

which tends to a limit, the lattice Λ say. We write $P^{(r)} = P_{k_r}$. Choose in every lattice $\Lambda^{(r)}$ a reduced basis

$$Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$$

and in Λ a basis

$$Y_1, Y_2, \ldots, Y_n$$

such that

$$\lim_{r \to \infty} | Y_g^{(r)} - Y_g | = 0 \qquad (g = 1, 2, \ldots, n).$$

Then there are integers $u_1^{(r)}, u_2^{(r)}, \ldots, u_n^{(r)}$ not all zero such that

$$P^{(r)} = u_1^{(r)} Y_1^{(r)} + u_2^{(r)} Y_2^{(r)} + \ldots + u_n^{(r)} Y_n^{(r)} \qquad (r = 1, 2, 3, \ldots),$$

and real numbers $u_1, u_2, \ldots, u_n$ not all zero such that

$$P = u_1 Y_1 + u_2 Y_2 + \ldots + u_n Y_n.$$

From the hypothesis, the points

$$P^{(r)}, Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \qquad (r = 1, 2, 3, \ldots)$$

are bounded; further

$$|\{ Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \}| = d(\Lambda^{(r)}) \geqq \Delta(K).$$

Hence, by the equations

$$u_g^{(r)} \{ Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)} \} = \{ Y_1^{(r)}, \ldots, Y_{g-1}^{(r)}, P^{(r)}, Y_{g+1}^{(r)}, \ldots, Y_n^{(r)} \} \qquad (g = 1, 2, \ldots, n).$$

all coefficients $u_1^{(r)}, \ldots, u_n^{(r)}$ are bounded. Hence the left-hand side of the identity

$$(P - P^{(r)}) + u_1^{(r)} (Y_1^{(r)} - Y_1) + \ldots + u_n^{(r)} (Y_n^{(r)} - Y_n) = (u_1 - u_1^{(r)}) Y_1 + \\ + \ldots + (u_n - u_n^{(r)}) Y_n$$

and so also the right-hand side tends to O when r tends to infinity. Since the points $Y_1, Y_2, \ldots, Y_n$ are independent, this implies that

$$\lim_{r \to \infty} u_g^{(r)} = u_g \qquad (g = 1, 2, \ldots, n).$$

Now the coefficients $u_g^{(r)}$ are integers, hence also their limits u_g, and so P

is a point of Λ. As in earlier proofs, it is easy to prove that Λ is K-admissible and therefore critical. This completes the proof.

Theorem C: *Let K: $F(X) \leq 1$ be an irreducible star body of the finite type, and let P be any finite point on the boundary C of K. Then there exists at least one critical lattice of K containing P.*

Proof: Let us assume that on the contrary no critical lattice of K passes through a certain finite point P on C; we shall show that K is then reducible.

By the continuity of $F(X)$, every point X on C sufficiently near to P lies at a bounded distance from O. We assert, firstly, that a positive number σ exists such that no critical lattice of K passes through a point of the set S consisting of all points X on C for which

$$\left| \, |X|^{-1} X - |P|^{-1} P \right| \leqq \sigma.$$

For let this assertion be false. Then there exists an infinite sequence of critical lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

of K with a point P_r in each lattice Λ_r such that

$$\lim_{r \to \infty} |P_r - P| = 0.$$

But then, by Theorem B, at least one critical lattice of K passes through P, contrary to hypothesis.

We assert, secondly, that there is a positive number δ such that

$$d(\Lambda) \geqq (1 + \delta)^n \, \Delta(K)$$

for every K-admissible lattice which contains a point of S. For let this assertion be false. Then an infinite sequence of K-admissible lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

exists such that

$$\lim_{r \to \infty} d(\Lambda_r) = \Delta(K),$$

and that further each lattice Λ_r contains a point P_r of S. Since S is bounded and closed, it is then possible to find an infinite sequence of indices

$$k_1, k_2, k_3, \ldots \qquad (k_1 < k_2 < k_3 < \ldots)$$

such that the corresponding points

$$P_{k_1}, P_{k_2}, P_{k_3}, \ldots$$

tend to a point of S, the point P^* say. Theorem B, if applied to the sequence of lattices

$$\Lambda_{k_1}, \Lambda_{k_2}, \Lambda_{k_3}, \ldots,$$

gives now the existence of a critical lattice of K passing through P^*, contrary to what had just been proved.

Next put

$$G(X) = F(X)\left\{1 + \frac{\delta}{\sigma} \max\ (0, \sigma - \left| |X|^{-1} X - |P|^{-1} P \right|)\right\} \quad \text{if } X \neq O,$$

and

$$G(X) = 0 \text{ if } X = O,$$

so that $G(X)$ is a distance function. Evidently

$$F(X) \leq G(X) \leq (1+\delta)F(X) \qquad \text{for all points } X,$$

and

$$G(X) > F(X) \quad \text{if and only if} \quad X \neq O, \quad \left| |X|^{-1} X - |P|^{-1} P \right| < \sigma.$$

Hence the star body H: $G(X) \leq 1$ satisfies the relation

$$H < K.$$

The theorem is then proved if we can show that even

$$H \prec K.$$

Let this be untrue. Then there exists a critical lattice Λ of H which is not K-admissible, and so the set Σ of all points

$$Q_s \neq O \qquad (s = 1, 2, 3, \ldots)$$

of Λ which are *inner* points of K, is not empty. Every point Q_s satisfies the inequalities

$$F(Q_s) < 1 \leq G(Q_s) \leq (1+\delta)\, F(Q_s),$$

whence

$$\frac{1}{1+\delta} \leq F(Q_s) < 1,$$

and it also satisfies the inequality

$$\left| |Q_s|^{-1} Q_s - |P|^{-1} P \right| < \sigma.$$

By the definition of S, these formulae imply that Q_s belongs to a bounded part of R_n. Hence Σ has only a finite number of elements, the lattice points

$$Q_1, Q_2, \ldots, Q_r$$

say. Put

$$\min_{s=1,2,\ldots,r} F(Q_s) = \frac{1}{\lambda},$$

so that

$$1 < \lambda \leq 1 + \delta.$$

The lattice

$$\Lambda^* = \lambda \Lambda$$

is then K-admissible, and at least one of the points

$$\lambda Q_1, \lambda Q_2, \ldots, \lambda Q_r$$

lies in the subset S on C. Hence, as we showed above,

$$d(\Lambda^*) \geqq (1+\delta)^n \triangle(K).$$

On the other hand,

$$d(\Lambda^*) = \lambda^n d(\Lambda) = \lambda^n \triangle(H),$$

since Λ is a critical lattice of H. Hence

$$\triangle(H) = \frac{1}{\lambda^n} d(\Lambda^*) \geqslant \left(\frac{1+\delta}{\lambda}\right)^n \triangle(K) \geqslant \triangle(K),$$

contrary to the assumption that $\triangle(H) < \triangle(K)$.

The so proved Theorem C applies to all irreducible star bodies irrespective of whether these are bounded or not. The following problem arises now:

Problem 1: *Do there exist unbounded irreducible star bodies?*

I have reasons to believe that the answer is in the negative. However, I cannot prove this generally, but only for a special class of star bodies to be considered later (Theorem H).

§ 4. *A sufficient condition for irreducibility.*

For this reason, the sufficient condition for irreducibility given in this paragraph will apply only to bounded star bodies.

Let K: $F(X) \leqq 1$ be a bounded star body, and Λ any critical lattice of K. This lattice has only a finite number of points on the boundary C of K, say the points

$$\mp P_1, \mp P_2, \ldots, \mp P_m;$$

here $m \geqq n$ by Theorem 11. Denote by $Y_1, \ldots, Y_n$ a reduced basis of Λ, and write the points P_k in the form

$$P_k = u_1^{(k)} Y_1 + \ldots + u_n^{(k)} Y_n \qquad (k = 1, 2, \ldots, m),$$

so that the coefficients $u_g^{(k)}$ are integers. Let $\varepsilon > 0$. Any n points $Y_1^*, \ldots, Y_n^*$ satisfying

$$| Y_g^* - Y_g | < \varepsilon \qquad (g = 1, 2, \ldots, n)$$

generate a neighbouring lattice Λ^*; we say that Λ^* lies in a ε-neighbourhood of Λ. The points

$$P_k^* = u_1^{(k)} Y_1^* + \ldots + u_n^{(k)} Y_n^* \qquad (k = 1, 2, \ldots, m)$$

are then the only points of Λ^* which lie on, or near to, the boundary C of K, if ε sufficiently small.

Definition B: *The critical lattice Λ of K is called a free lattice if to every index $k = 1, 2, \ldots, m$ and to every $\varepsilon > 0$, there exists a lattice Λ^* with the following properties:*

(a): *Λ^* lies in an ε-neighbourhood of Λ.*

(b): $$d(\Lambda^*) < d(\Lambda).$$

(c): *Λ^* contains no inner points of K except O and $\mp P_k$.*

For instance, every singular lattice Λ of K (i.e. a critical lattice with $m = n$) is free. For let $\mp P_1, \ldots, \mp P_n$ be the points of Λ on C. There exists then a neighbouring lattice Λ^* such that the points P^*_g of Λ^* corresponding to $P_1, \ldots, P_n$ are given by

$$P^*_g = \begin{cases} P_g & \text{if } g = 1, \ldots, k-1, k+1, \ldots, n, \\ (1-\varepsilon) P_k & \text{if } g = k. \end{cases}$$

Since

$$d(\Lambda^*) = (1-\varepsilon)\, d(\Lambda) < d(\Lambda),$$

the conditions (a), (b), (c) are satisfied.

Other examples of free lattices are the critical lattices of the unit circle

$$x_1^2 + x_2^2 \leqslant 1$$

in R_2 ($m = 3$, $n = 2$), and the critical lattices of the unit sphere

$$x_1^2 + x_2^2 + x_3^2 \leqslant 1$$

in R_3 ($m = 6$, $n = 3$). See the next paragraph.

Not every critical lattice is free, as is seen by the following example in R_2 which is easily extended to more dimensions:

Denote by K the non-convex hexagon in R_2 of vertices

$$(1, 0),\ (2, 1),\ (-2, 1),\ (-1, 0),\ (-2, -1),\ (2, -1).$$

This hexagon contains the square Q,

$$|x_1| \leq 1, \quad |x_2| \leq 1$$

as a subset; moreover, the critical lattice

$$\Lambda: \qquad x_1 = u_1, \quad x_2 = u_2 \qquad (u_1, u_2 = 0, \mp 1, \mp 2, \ldots)$$

of Q is K-admissible. Hence $K > Q$, and Λ is also a critical lattice of K. Denote by

$$\Lambda^*: \ x_1 = (1+\alpha) u_1 + \beta u_2,\ x_2 = \gamma u_1 + (1+\delta) u_2 \ (u_1, u_2, = 0, \mp 1, \mp 2, \ldots)$$

a neighbouring lattice; thus

$$|\alpha| + |\beta| + |\gamma| + |\delta|$$

is very small. Then it is impossible that the point

$$(\beta, 1+\delta)$$

of Λ^* is an inner point of K, while the two other points

$$(1+\alpha+\beta,\ 1+\gamma+\delta),\ (-1-\alpha+\beta,\ 1-\gamma+\delta)$$

of Λ^* lie on the boundary or outside of K, since the three inequalities

$$1+\delta < 1,\ 1+\gamma+\delta \geq 1,\ 1-\gamma+\delta \geq 1$$

are mutually contradictory. Hence Λ is not free.

In this example, K is reducible. We may then ask:

Problem 2: *Does there exist a bounded irreducible star body with at least one critical lattice which is not free?*

I have so far not been able to solve this question.

Theorem D: *Let K be a bounded star body, and Σ a set of points on its boundary C which is everywhere dense. If further P is any point of Σ, let there be a free lattice of K containing P. Then K is irreducible.*

Proof: Assume there is a star body H such that $H \prec K$. There exists then a point Q on C which lies outside H; hence $\varepsilon > 0$ can be chosen so small that all points X with

$$|X - Q| \leq \varepsilon$$

lie also outside H. By the hypothesis, we can find a point P of Σ such that

$$|P - Q| \leqq \frac{\varepsilon}{2};$$

let Λ be the free lattice through P. Then, by Definition B, there is a neighbouring lattice Λ^* of determinant

$$d(\Lambda^*) < d(\Lambda)$$

and containing no inner points of K except O and two symmetrical points $\mp P^*$ such that

$$|P^* - P| \leqq \frac{\varepsilon}{2} \qquad \text{that is,} \qquad |P^* - Q| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence Λ^* is H-admissible, and so

$$\triangle(H) \leq d(\Lambda^*) < d(\Lambda) = \triangle(K),$$

contrary to hypothesis.

Remark: The same proof shows that if the hypothesis of Theorem D is satisfied, then at least one free lattice of K passes through every point on C.

§ 5. *Examples of irreducible convex star bodies.*

By means of Theorem D, any number of irreducible star bodies can be constructed. We first show that some of the two- and three-dimensional convex regions considered already by MINKOWSKI [1]) are irreducible. To this end, we use the following results of MINKOWSKI:

(A) Every critical lattice Λ of a convex star domain K in R_2 contains at least *six* points on the boundary C of K. If Λ contains only six such points, then a system of parallel coordinates ξ_1, ξ_2 can be chosen in which these six points are of coordinates [2])

$$(1, 0), \quad (0, 1), \quad (-1, 1).$$

1) Diophantische Approximationen, 24—28, 54—55, 67—75, 75—77, 105—111. For the case of the cylinder, see my note: "On lattice points in a cylinder", which is to appear in the Quarterly Journal.

2) Diophantische Approximationen, 51—54.

(B) Every critical lattice Λ of a convex star body K in R_3 contains at least *twelve* points on the boundary C of K. If Λ contains only twelve such points, then a system of parallel coordinates ξ_1, ξ_2, ξ_3 can be chosen in which these twelve points are either of coordinates

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, 1, -1), \quad (-1, 0, 1), \quad (1, -1, 0),$$

or of coordinates [1])

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0).$$

Theorem E: *Let K be a convex star domain in R_2, and Λ a critical lattice of K with just six points on C. Then Λ is a free lattice.*

Proof: It is clear from (A) that no three of the lattice points on C are collinear. Hence there exists a line L which separates any chosen point among these six from the five other ones. Let $-L$ be the line symmetrical to L in O, and let K^* be the set of all points of K which are not separated from O by either L or $-L$; then K^* is also a convex star domain. Since only four points of Λ lie on the boundary of K^*, there exists a neighbouring lattice Λ^* which is K^*-admissible and of determinant $d(\Lambda^*) < d(\Lambda)$; this lattice satisfies the conditions of definition B.

Theorem F: *Let K be a convex star body in R_3, and Λ a critical lattice of K with just twelve points on C. Then Λ is a free lattice.*

Proof: We first show that each one of the twelve lattice points on C can be separated from the eleven others by a plane L. For if the twelve lattice points are

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, 1, -1), \quad (-1, 0, 1), \quad (1, -1, 0),$$

then $(1, 0, 0)$ is separated from the other points by the plane

$$4\xi_1 + 2\xi_2 + 2\xi_3 - 3 = 0,$$

and $(0, 1, -1)$ is separated by the plane

$$2\xi_2 - 2\xi_3 - 3 = 0.$$

If, however, the twelve lattice points are

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0),$$

then $(1, 0, 0)$ is separated from the other points by the plane

$$6\xi_1 - 2\xi_2 - 2\xi_3 - 5 = 0,$$

and $(0, 1, 1)$ is separated by the plane

$$2\xi_1 - 2\xi_2 - 2\xi_3 + 3 = 0.$$

By a cyclical permutation of the coordinates ξ_1, ξ_2, ξ_3 and by changing ξ_1, ξ_2, ξ_3 into $-\xi_1, -\xi_2, -\xi_3$, we obtain the equations of separating planes belonging to the other points. Having thus fixed L, let $-L$ be the plane symmetrical to L in O, and K^* the set of all points of K which are

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1) Gesammelte Abhandlungen, II.

not separated from O by either L or $-L$; then K^* is also a convex star body. Since Λ has only ten points on the boundary of K^*, there exists a neighbouring lattice Λ^* which is K^*-admissible and of determinant $d(\Lambda^*) < d(\Lambda)$, hence satisfies the conditions of Definition B.

From Theorems E and F, we obtain now the following examples of irreducible convex star bodies:

(a) The square K_1,

$$\max(|x_1|, |x_2|) \leq 1$$

in R_2 of determinant $\triangle(K_1) = 1$. For if $0 < |\xi| < 1$, then the critical lattice of basis

$$(1, 0), \ (\xi, 1)$$

passes through the boundary point $(\xi, 1)$ and the critical lattice of basis

$$(1, \xi), \ (0, 1)$$

passes through the boundary point $(1, \xi)$; further both types of lattice have just six points on the boundary of K_1.

(b) The circle K_2,

$$+\sqrt{(x_1^2 + x_2^2)} \leq 1$$

in R_2 of determinant $\triangle(K_2) = \frac{1}{2}\sqrt{3}$. For if $(\cos\theta, \sin\theta)$ is any point on C, then the critical lattice of basis

$$(\cos\theta, \sin\theta), \quad \left(\cos\left\{\theta + \frac{\pi}{3}\right\}, \sin\left\{\theta + \frac{\pi}{3}\right\}\right)$$

passes through this point and has just six points on the boundary.

(c) The cube K_3,

$$\max(|x_1|, |x_2|, |x_3|) \leq 1,$$

in R_3 of determinant $\triangle(K_3) = 1$. For if $0 < |\xi_1| < 1$, $0 < |\xi_2| < 1$, then the critical lattice of basis

$$(1, 0, 0), \ (\tfrac{1}{2}, 1, 0), \ (\xi_1, \xi_2, 1)$$

passes through the boundary point $(\xi_1, \xi_2, 1)$, and similar lattices pass through the points $(\xi_1, 1, \xi_2)$ and $(1, \xi_1, \xi_2)$. Moreover, these three types of critical lattices have just twelve points on C.

(d) The sphere K_4,

$$+\sqrt{(x_1^2 + x_2^2 + x_3^2)} \leq 1,$$

in R_3 of determinant $\triangle(K_4) = \sqrt{\frac{1}{2}}$. For all critical lattices are obtained from the lattice of basis

$$(1, 0, 0), (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, 0), (0, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{2}{3}})$$

by a rotation about O. They each contain twelve points on C, and they pass through every point on this boundary.

(e) The cylinder K_5,

$$\max(+\sqrt{(x_1^2 + x_2^2)}, |x_3|) \leq 1,$$

in R_3 of determinant $\triangle(K_5) = \sqrt{\frac{3}{4}}$. For one type of critical lattice is obtained from the lattice of basis

$$(1, 0, 0), (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, 0), (\xi_1, \xi_2\ 1)$$

by all rotations about the x_3-axis; if

$$0 < \xi_1^2 + \xi_2^2 < 1, \quad 0 < (\xi_1 - 1)^2 + \xi_2^2 < 1, \quad 0 < (\xi_1 - \tfrac{1}{2})^2 + (\xi_2 - \sqrt{\tfrac{3}{4}})^2 < 1,$$

then this type of lattice has just twelve points on C, and it passes through every point (x_1, x_2, x_3) on C for which

$$0 < x_1^2 + x_2^2 < 1, \qquad x_3 = \mp 1.$$

Further a second type of critical lattice is obtained from the lattice of basis

$$(1, 0, \xi_1), \ (\tfrac{1}{2}, \sqrt{\tfrac{3}{4}}, \xi_2), \ (0, 0, 1)$$

by all rotations about the x_3-axis; if

$$0 < |\xi_1| < \tfrac{1}{2}, \ 0 < |\xi_2| < \tfrac{1}{2},$$

then also this type of lattice contains just twelve points on C, and it passes through every point on C for which

$$x_1^2 + x_2^2 = 1, \qquad 0 < |x_3| < 1.$$

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Second communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of March 30, 1946.)

§ 6. *Irreducible convex star domains in R_2.*

Theorem G: *A convex star domain K in R_2 is irreducible if and only if all parallelograms with one vertex at O and the other three vertices on the boundary C of K are of equal areas.*

Proof: Every critical lattice of K has at least six and at most eight points on C. If it has eight points on C, then K is a parallelogram, hence irreducible; in this case, the inscribed parallelograms clearly satisfy the assertion. Assume next that every critical lattice of K has just six points on C, and let P_1 be any given point on C. There exists then on C at least one pair of points P_2, P_3 such that

$$P_2 = P_1 + P_3, \quad \{P_1, P_2\} > 0;$$

i.e., $OP_1P_2P_3$ is a parallelogram with its vertices described in this order in *positive* direction. In general, only one such parallelogram exists for given P_1. If, however, C contains a line segment parallel to OP_1 and of greater length than this vector, then there are an infinity of such parallelograms, and all are of equal areas. Select one such parallelogram $OP_1P_2P_3$ and call its area $A(P_1)$. Then the lattice Λ of basis P_1, P_2 is of determinant $d(\Lambda) = A(P_1)$ and is K-admissible. As is easily seen,

$$\triangle(K) = \min_{P_1 \text{ on } C} A(P_1).$$

The assertion follows therefore from the theorems, C, D, and E.

Theorem G enables us to construct any number of irreducible convex star domains in R_2. Take any three points Q_1, Q_2, Q_3 in R_2 such that

$$Q_2 = Q_1 + Q_3, \quad \{Q_1, Q_2\} > 0.$$

Denote by T_1, T_2, T_3 the three triangles of vertices

$$Q_1, Q_1 + Q_2, Q_2 \text{ or } Q_2, Q_2 + Q_3, Q_3 \text{ or } Q_3, Q_3 - Q_1, -Q_1$$

and by A_1 and A_2 two continuous arcs of the following kind:

(a) A_1 connects Q_1 with Q_2, and A_2 connects Q_2 with Q_3.

(b) A_1 lies in T_1, and A_2 lies in T_2.

(c) Neither A_1 nor A_2 contains a line segment [4]).

(d) The region bounded by A_1, A_2, and the two line segments OQ_1 and OQ_3 is convex.

---

[4]) This condition is not essential. If it is dropped, then P_2 need not be a single-valued function of P_1.

To every point P_1 on A_1, there is then a unique point P_2 on A_2 such that

$$\{P_1, P_2\} = \{Q_1, Q_2\}.$$

Denote by A_3 the arc of all points $P_3 = P_2 - P_1$, where P_1 runs over A_1; as is easily shown, A_3 lies in T_3 and connects Q_3 with $-Q_1$. Denote by $-A_1$, $-A_2$, $-A_3$ the arcs symmetrical to A_1, A_2, A_3 in O, and by K the region bounded by the six arcs $A_1, A_2, A_3, -A_1, -A_2, -A_3$. If K is convex, then K is irreducible.

As an example, let

$$Q_1 = (\tfrac{1}{2}, -1), \quad Q_2 = (1, 0), \quad Q_3 = (\tfrac{1}{2}, 1),$$

and let A_1 and A_2 be the arcs Q_1Q_2 and Q_2Q_3 of the parabola

$$x = 1 - \tfrac{1}{2} y^2.$$

A simple calculation shows that A_3 is the arc Q_3, $-Q_1$ defined by

$$4x^2 = y^4 - 8y^2 + 8y = y(y-2)(y^2 + 2y - 4);$$

hence

$$-\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \qquad 1 \leqq y \leqq \sqrt{(5)} - 1$$

for all points on A_3. This arc is symmetrical in the y-axis. Hence its convexity is proved if we can show that

$$\frac{d^2x}{dy^2} < 0 \text{ if } 0 \leqq x \leqq \tfrac{1}{2}, 1 \leqq y \leqq \sqrt{(5)} - 1.$$

Now

$$2x \frac{dx}{dy} = y^3 - 4y + 2 = (y-1)(y^2 + y - 3) - 1,$$

and

$$y - 1 \geqq 0, y^2 + y - 3 \leqq (\sqrt{(5)} - 1)^2 + \sqrt{(5)} - 3)^2 - 3 =$$
$$= 2 - \sqrt{5} < 0, \quad 0 \leqq 2x \leqq 1,$$

hence

$$\frac{dx}{dy} \leqq -1, \qquad \left(\frac{dx}{dy}\right)^2 \geqq 1.$$

Further

$$2x \frac{d^2x}{dy^2} + 2\left(\frac{dx}{dy}\right)^2 = 3y^2 - 4,$$

whence

$$2x \frac{d^2x}{dy^2} \leqq 3(\sqrt{(5)} - 1)^2 - 4 - 2 \times 1 = 12 - 6\sqrt{5} < 0, \ \frac{d^2x}{dy^2} < 0,$$

whence the assertion. Both A_2 and A_3 have the gradient

$$\frac{dy}{dx} = -1$$

at the point Q_3. Hence the set K bounded by the six arcs $A_1, A_2, A_3, -A_1, -A_2, -A_3$ is therefore an irreducible convex star domain of determinant

$$\triangle(K) = \{ Q_1, Q_2 \} = 1;$$

its boundary has everywhere a continuous tangent.

Irreducible convex star domains in R_2 can also be obtained by the following construction:

Denote by $a_1(t), b_1(t), a_2(t), b_2(t)$ four continuous functions of t of period 1/6 which satisfy the identity

$$a_1(t)\, b_2(t) - a_2(t)\, b_1(t) = \triangle,$$

where $\triangle$ is a positive constant. Let then C be the closed curve in R_2 consisting of all points $P(t) = (x_1(t), x_2(t))$ where

$$x_1(t) = a_1(t) \cos 2\pi t + b_1(t) \sin 2\pi t,$$

$$x_2(t) = a_2(t) \cos 2\pi t + b_2(t) \sin 2\pi t,$$

and where t runs over any interval of unit length. It is easily verified that

$$P(t+\tfrac{1}{6}) = P(t) + P(t+\tfrac{1}{3}), \qquad \{ P(t), P(t+\tfrac{1}{6}) \} = \triangle.$$

Hence C forms the boundary of an irreducible convex star domain K provided it is a convex curve. In the special case that $a_1(t), b_1(t), a_2(t), b_2(t)$ are constants, C is an ellipse. But there are an infinity of other permissible choices, and it is, in particular, possible to find algebraic curves C different from ellipses and forming the boundaries of irreducible convex star domains.

§ 7. *Further examples of irreducible star domains.*

In my note, Proc. Cambridge Phil. Soc., 40, part 2 (1944), 107—116, I gave the first example of a *non-convex* irreducible star domain in R_2, namely the domain K,

$$|x_1 x_2| \leqslant 1, \qquad |x_1 + x_2| \leqslant \sqrt{5},$$

of determinant $\triangle(K) = \sqrt{5}$. I shall prove in a separate paper that the following non-convex star bodies in R_2 are likewise irreducible:

(1) The domain K_1,

$$|x_1| + |x_2| \leqslant 1, \qquad |x_2| \leqslant \max(c, 1 - \{x^2 + 1 - 2c\}^{1/2}), \qquad (0 < c < \tfrac{1}{2}),$$

of determinant $\triangle(K_1) = c$.

(2) The domain K_2,

$$x_1^2 + x_2^2 \leqslant 1, \qquad |x_2| \leqslant \max(\sin c, \{2 - 2\cos c - x_1^2\}^{1/2}), \qquad \left(0 < c < \frac{\pi}{6}\right),$$

of determinant $\triangle(K_2) = \sin c$.

(3) The domain K_3,

$$|x_1| \leqslant |x_2| + 1, \qquad |x_2| \leqslant \min(c, \{x^2 + 1 + 2c\}^{1/2} - 1), \quad (0 < c < \tfrac{1}{2}),$$

of determinant $\triangle(K_3) = c$.

(4) The domain K_4,

$$-1 \leqslant x_1 x_2 \leqslant \frac{1}{c}(1-c)^2, \qquad |x_1 + x_2| \leqslant \frac{1}{c} - c, \qquad \left(\frac{3-\sqrt{5}}{2} \leqslant c < 1\right)$$

of determinant $\triangle(K_4) = \frac{1}{c} - c$.

There is no difficulty in constructing an infinity of other examples in R_2. On the other hand, it is much more difficult to construct irreducible star domains in R_3; I hope, however, to discuss also some examples of this kind in the paper referred to.

§ 8. *The concavity coefficient of a star body.*

Let K: $F(X) \leq 1$ be a bounded star body. Then $F(X_1 + X_2)$ is a continuous function of X_1 and X_2 on the closed bounded set $F(X_1) + F(X_2) = 1$, and so assumes a maximum value, ω_K say, on this set. Hence, by homogeneity,

$$F(X_1 + X_2) \leqslant \omega_K (F(X_1) + F(X_2)) \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for any two points X_1 and X_2. We call ω_K the *concavity coefficient* of K. This coefficient is evidently an affine invariant. On putting $X_2 = O$ in (a), we see that $\omega_K \geq 1$; the equality sign holds if and only if K is a convex body [5]). On applying (a) repeatedly, one obtains the inequality

$$F(X_1 + \ldots + X_n) \leqslant \omega_K^{n^*} (F(X_1) + \ldots + F(X_n)) \quad . \quad . \quad . \quad (b)$$

where n^* denotes the integer defined by

$$2^{n^*-1} < n \leqslant 2^{n^*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

As an example, if K is the star body in R_2 defined by

$$|x_1 x_2| \leqslant 1, \qquad |x_1 + x_2| \leqslant \sqrt{5},$$

i.e. if $F(X)$ is the distance function

$$F(X) = \max\left(|x_1 x_2|^{1/2}, \frac{1}{\sqrt{5}} |x_1 + x_2|\right),$$

then a simple discussion gives $\omega_K = 3/2$.

Let K be of volume $V(K)$, and let Λ be any lattice of determinant $\triangle(K)$. Then by a theorem of L. J. MORDELL [6]), Λ contains at least one point $P \neq O$ such that

$$F(P) \leqslant 2\omega_K \left(\frac{\triangle(K)}{V(K)}\right)^{1/n}.$$

---

[5]) If K is not bounded, but of the finite type, then (a) does not hold for all points X_1, X_2, however large ω^K is taken.

[6]) Comp. Math. **1**, 248—253 (1935), in particular pp. 248 and 251.

Let now Λ be a critical lattice of K. Then $F(P) \geq 1$, and so [7])

$$V(K) \leqslant (2\,\omega_K)^n \,\triangle\,(K). \qquad \text{(I)}$$

Next let Λ be a K-admissible lattice of determinant $d(\Lambda)$ with n independent points $P_1, \ldots, P_n$ on the boundary C of K,

$$F(P_1) = \ldots = F(P_n) = 1. \qquad (d)$$

The determinant

$$D = |\{P_1, \ldots, P_n\}|$$

of these n points is a positive integral multiple

$$D = N\,d(\Lambda)$$

of $d(\Lambda)$; we call

$$N = \text{ind}\,(P_1, \ldots, P_n)$$

the index of the n lattice points $P_1, \ldots, P_n$. An upper bound for this index is obtained in the following way [8]):

A basis $R_1, \ldots, R_n$ of Λ can be chosen such that

$$R_1 = P_1,$$

$$R_g = \frac{a_{g1}\,P_1 + \ldots + a_{g,g-1}\,P_{g-1} + P_g}{a_g} \qquad (g = 2, 3, \ldots, n);$$

where the a's are integers, and

$$a_2 \geqslant 1,\, a_3 \geqslant 1, \ldots, a_n \geqslant 1,\, a_2\,a_3 \ldots a_n = N. \qquad (e)$$

Every point P of Λ can be written as

$$P = u_1\,R_1 + \ldots + u_n\,R_n$$

with integral coefficients $u_1, \ldots, u_n$. On replacing the R's by the P's, this gives

$$P = v_1\,P_1 + \ldots + v_n\,P_n,$$

where

$$v_1 = u_1 + \frac{a_{21}}{a_2}\,u_2 + \ldots + \frac{a_{n1}}{a_n}\,u_n;$$

$$v_2 = \frac{1}{a_2}\,u_2 + \frac{a_{32}}{a_3}\,u_3 + \ldots + \frac{a_{n2}}{a_n}\,u_n; \ldots; \; v_n = \frac{1}{a_n}\,u_n.$$

By a theorem of MINKOWSKI [9]), integers $u_1, \ldots, u_n$ not all zero can be chosen such that

$$|v_1| + \ldots + |v_n| \leqslant \left(\frac{n!}{N}\right)^{1/n}.$$

---

[7]) By a theorem of MINKOWSKI and HLAWKA (Math. Zeitschr. **49**, 285—312 (1943), in particular pp. 288—299), there is also a lower bound for $V(K)$, namely

$$V(K) \geqslant 2\,\zeta(n)\,\triangle\,(K).$$

[8]) MINKOWSKI, Geometrie der Zahlen, 173—176 and 187—189.

[9]) Geometrie der Zahlen, p. 122. Put $p = 1$, $r = n$, $s = 0$, and use (e).

Therefore by (b) and (d),

$$1 \leqslant F(P) \leqslant \omega_K^{n*} \{F(v_1 P_1) + \ldots + F(v_n P_n)\} =$$
$$= \omega_K^{n*} (|v_1| + \ldots + |v_n|) \leqslant \omega_K^{n*} \left(\frac{n!}{N}\right)^{1/n},$$

whence

$$\text{ind}(P_1, P_2, \ldots, P_n) \leqslant n!\, \omega_K^{n n*}. \qquad \text{(II)}$$

Finally, let again Λ be K-admissible, let $P_1, \ldots, P_m$ be the only points of Λ on C, and put $P_0 = O$. In the basis $R_1, \ldots, R_n$ of Λ, these points can be written as

$$P_\mu = u_1^{(\mu)} R_1 + \ldots + u_n^{(\mu)} R_n \quad (\mu = 0, 1, \ldots, m)$$

with integral coefficients $u_g^{(\mu)}$. Denote by q the integer for which

$$2\,\omega_K < q \leqslant 2\,\omega_K + 1.$$

Then

$$m \leqslant q^n - 1 \leqslant (2\,\omega_K + 1)^n - 1. \qquad \text{(III)}$$

For let this assertion be false, i.e. let $m \geq q^n$. Then two of the $m+1$ points $P_0, P_1, \ldots, P_m$, the points P_μ and P_ν say, satisfy the congruences

$$u_g^{(\mu)} \equiv u_g^{(\nu)} \pmod{q} \qquad (g = 1, 2, \ldots, n).$$

Hence

$$P = \frac{1}{q}(P_\mu - P_\nu)$$

is again a point of Λ, and $P \neq O$ since $\mu = \nu$. But then

$$1 \leqslant F(P) \leqslant \omega_K \left\{ F\left(\frac{1}{q} P_\mu\right) + F\left(-\frac{1}{q} P_\nu\right) \right\} \leqslant \omega_K \left(\frac{1}{q} + \frac{1}{q}\right) < 1,$$

a contradiction [10]).

The two inequalities (II) and (III) apply, in particular, to the critical lattices of K. They show that these critical lattices are essentially only of a finite number of different types, depending alone on the value of the concavity coefficient ω_K.

§ 9. *Some unsolved problems.*

Special results suggest that each one of the following four problems has an affirmative answer, though I have not succeeded in obtaining proofs. We assume always that *K is a bounded irreducible star body in R_n*, that Λ is a critical lattice of K, and that $V(K)$, ind $(P_1, \ldots, P_n)$, and m have the same meaning as in the last paragraph:

Problem 3: *To decide whether, to every dimension n, there exists a positive constant a_n such that for all K,*

$$\omega_K \leqslant a_n.$$

---

10) Compare MINKOWSKI, Geometrie der Zahlen, 77—80.

Problem 4: *To decide whether, to every dimension n, there exists a positive constant b_n such that for all K*

$$V(K) \leq b_n \Delta(K).$$

Problem 5: *To decide whether, to every dimension n, there exists a positive constant c_n such that for every K and for every critical lattice Λ of K,*

$$\text{ind } (P_1, \ldots, P_n) \leq c_n.$$

Problem 6: *To decide whether, to every dimension n, there exists a positive constant d_n such that for every K and for every critical lattice Λ of K,*

$$m \leq d_n.$$

It is clear from the last paragraph that if the first one of these four problems has an affirmative anwer, then the same is true for the three other ones; by (I), (II), (III), we may then, in fact, put

$$b_n = (2\, a_n)^n, \qquad c_n = n!\, a_n^{nn*}, \qquad d_n = (2\, a_n + 1)^n - 1.$$

But it is, of course, possible that no a_n, but at least one of the three numbers b_n, c_n, d_n exists.

While the last problems deal with properties of given irreducible star bodies, the main existence problem, as follows, refers to reducible star bodies:

Problem 7: *To decide whether every bounded reducible star body contains at least one irreducible star body of equal determinant.*

It is highly probable that the answer is in the affirmative, and that even a continuous infinity of irreducible star bodies of the wanted kind exists; but I have not succeeded in proving this. One reason for this failure is the following fact: If $H, K, K_1, K_2, \ldots$ are star bodies such that

$$H < K_r < K \qquad (r = 1,\ 2,\ 3, \ldots),$$
$$K_1 > K_2 > K_3 > \ldots,$$

then the star bodies K_r tend to a limiting set, namely their intersection, but this set is not necessarily a star body. Presumably, a proof will be constructive and will consist of a finite number of steps. — If Problem 7 has an affirmative answer, then only irreducible star bodies need be considered for most purposes, in so far as bounded star bodies are concerned. — The analogous problem for unbounded star bodies has probably a negative answer; but again, I have not so far succeeded in proving this.

§ 10. *A general principle.*

We consider in the following paragraphs non-trivial examples of unbounded reducible star bodies, and begin with a simple principle on star bodies with automorphisms.

Denote by $F(X)$ and $G(X)$ two distance functions in R_n, and by $\varphi(x, y)$ a distance function in R_2, satisfying the following conditions:

(a): The star body K: $F(X) \leq 1$ is unbounded and of the finite type, and admits a group Γ of automorphisms Ω.

(b): If $s > 0$ is arbitrary, then there exists an element Ω of Γ such that if $|X| \leq s$, then $G(\Omega^{-1} X) \leq 1$.

(c): $\varphi(1, 0) = 1$, and $\varphi(1, y) \geq 1$ for all real numbers y.

Hence

$$F^*(X) = \varphi(F(X), G(X))$$

is also a distance function. By (c),

$$F^*(X) \geq F(X) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

for all points X; the star body K^*: $F^*(X) \leq 1$ is therefore contained in K.

Theorem H: $\triangle(K^*) = \triangle(K)$.

Proof: We first prove a simple inequality for φ. Denote by c the maximum of $\varphi(x, y)$ *if* $|x| + |y| = 1$; this maximum exists since φ is a continuous function. By homogeneity,

$$\varphi(x, y) \leq c(|x| + |y|). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Choose further $\varepsilon > 0$ arbitrarily small, and choose $\delta > 0$ so small that

$$2c\delta \leq 1,$$

and that further

$$\varphi(1, y) \leq 1 + \varepsilon \qquad \text{if } |y| \leq 2c\delta;$$

this is possible by (c). Then

$$\varphi(x, y) = |x| \varphi\left(1, \frac{y}{x}\right) \leq 1 + \varepsilon \qquad \text{if } \frac{1}{2c} \leq |x| \leq 1, \ |y| \leq \delta,$$

since $|y/x| \leq 2c\delta$; and by (e),

$$\varphi(x, y) \leq c(|x| + |y|) \leq 1 \qquad \text{if } |x| \leq \frac{1}{2c}, \ |y| \leq \delta,$$

since $\delta \leq 1/2c$. On combining these two inequalities,

$$\varphi(x, y) \leq 1 + \varepsilon \qquad \text{if } |x| \leq 1, \ |y| \leq \delta. \quad . \quad . \quad . \quad . \quad . \quad (f)$$

The proof proceeds now as follows: Denote by r an arbitrarily large positive number, put $s = r/\delta$, and choose the automorphism $\Omega = \Omega_r$ in Γ such that

$$\text{if } |X| \leq s, \text{ then } G(\Omega_r^{-1} X) \leq 1,$$

and so, by homogeneity,

$$\text{if } |X| \leq r, \text{ then } G(\Omega_r^{-1} X) \leq \delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

The star body $F^*(\Omega_r^{-1}X) \leq 1$ is identical with $\Omega_r K$. We saw that K^* is a subset of K; hence, by the invariance of K,

$$\Omega_r K^* \textit{ is contained in } K. \qquad (h)$$

Next, let K_r be the set of all points X satisfying

$$F(X) \leqslant 1, \quad |X| \leqslant r,$$

hence by (g),

$$G(\Omega_r^{-1} X) \leqslant \delta.$$

Then by (f),

$$F^*(\Omega_r^{-1} X) = \varphi(F(X), G(\Omega_r^{-1} X)) \leqslant 1 + \varepsilon,$$

which means that

$$K_r \textit{ is contained in } (1+\varepsilon)\, \Omega_r K^*. \qquad (i)$$

The two relations (h) and (i) imply, by Theorem 9 of Part I, that, as $\varepsilon \to 0$ and $r \to \infty$,

$$\lim \triangle(\Omega_r K^*) = \triangle(K).$$

But, by Theorem 17 of Part I, all automorphisms Ω_r are of determinants ∓ 1. Hence by Theorem 16 of Part I,

$$\triangle(\Omega_r K^*) = \triangle(K^*),$$

whence finally

$$\triangle(K^*) = \triangle(K),$$

as asserted.

Remark: The restriction (a) that K is of the *finite* type, is essential, as the following example in R_2 shows. Take

$$F(X) = |x_1^2 x_2|^{1/3}, \quad G(X) = |x_2|,$$

$$\varphi(x, y) = \max(|x|, |y|).$$

The star domain K: $F(X) \leq 1$ admits the automorphisms

$$x_1 = t^{-1} x_1', \; x_2 = t^2 x_2'$$

of arbitrary determinant $t \neq 0$, hence is of the infinite type. On the other hand, the star domain

$$K^*: \qquad |x_1^2 x_2| \leqslant 1, \; |x_2| \leqslant 1$$

is of the finite type since it is contained in the star domain

$$H: \qquad |x_1 x_2| \leqslant 1$$

of determinant $\triangle(H) = \sqrt{5}$.

We see also that the more general star domain

$$K_\tau^*: \qquad |x_1^2 x_2| \leqslant 1, \; |x_2| \leqslant \tau$$

is of determinant

$$\triangle(K_\tau^*) = \triangle(K^*)\sqrt{\tau},$$

an expression which tends to infinity with τ.

§ 11. *Applications of the last theorem.*

Let us assume that, in the last theorem, $G(X)$ and $\varphi(x, y)$ satisfy the additional conditions

$$G(X) \neq 0 \text{ if } F(X) \neq 0,$$

and

$$\varphi(x, 0) = |x|, \quad \varphi(x, y) > |x| \text{ if } y \neq 0.$$

Then

$$F^*(X) = \varphi(F(X), G(X)) > F(X) \text{ if } F(X) \neq 0,$$

and so every point of K^*: $F^*(X) \leq 1$ is an *inner* point of K: $F(X) \leq 1$. Hence the critical lattices of K, which by $\Delta(K^*) = \Delta(K)$ are also critical lattices of K^*, have no points on the boundary of K^*. The simplest n-dimensional star domain K^* of this kind is obtained for

$$F(X) = |x_1 \ldots x_n|^{1/n},$$

$$G(X) = \{(x_1^2 + \ldots + x_{n-1}^2)\, x_1^2 \ldots, x_{n-1}^2\}^{\frac{1}{2n}}, \ \varphi(x, y) = (x^{2n} + y^{2n})^{\frac{1}{2n}}.$$

For the star body K: $F(X) \leq 1$ admits the automorphisms

$$\Omega: \qquad x_1 = t\, x_1', \ldots, x_{n-1} = t\, x_{n-1}', \ x_n = t^{-(n-1)}\, x_n',$$

and so, if X is restricted by a condition $|X| \leq r$, then a number $t > 0$ depending only on r can be found such that $G(\Omega^{-1} X) \leq 1$. The star body

$$K^*: \qquad x_1^2 x_2^2 \ldots x_{n-1}^2 (x_1^2 + x_2^2 + \ldots + x_n^2) \leqslant 1$$

has therefore critical lattices with no points on its boundary [11]).

As a second example, choose

$$F(X) = \max(|x_1^{n-1} x_n|^{1/n}, \ldots, |x_{n-1}^{n-1} x_n|^{1/n}),$$

take for $G(X)$ either of the two distance functions

$$G_1(X) = \max\left(\left|\frac{x_1}{\varepsilon_1}\right|, \ldots, \left|\frac{x_{n-1}}{\varepsilon_{n-1}}\right|\right) \text{ or } G_2(X) = \left|\frac{x_n}{\varepsilon_n}\right|,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are arbitrary positive numbers, and put

$$\varphi(x, y) = \max(|x|, |y|).$$

The star body K: $F(X) \leq 1$ is of the finite type and admits the automorphisms

$$\Omega: \qquad x_1 = t\, x_1', \ldots, x_{n-1} = t\, x_{n-1}', \ x_n = t^{-(n-1)}\, x_n'.$$

Therefore, if X is restricted by a condition $|X| \leq r$, then numbers $t > 0$ depending only on r can be found such that $G_1(\Omega^{-1} X) \leq 1$, or such that $G_2(\Omega^{-1} X) \leq 1$. Hence, by Theorem H, both star bodies

$$K_1^*: \qquad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leqslant 1, \ |x_1| \leqslant \varepsilon_1, \ldots, |x_{n-1}| \leqslant \varepsilon_{n-1}$$

---

11) Compare a similar example in Theorem 15 of Part I which was proved in a far more complicated way.

and

$$K_2^*: \qquad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leq 1, \ |x_n| \leq \varepsilon_n$$

are of the same determinant

$$\triangle(K_1^*) = \triangle(K_2^*) = \triangle(K), \qquad = D \text{ say}$$

as

$$K: \qquad |x_n|^{\frac{1}{n-1}} \max(|x_1|, \ldots, |x_{n-1}|) \leq 1.$$

Hence, if $\varepsilon > 0$ and $d(\Lambda) = D$, then at least one point $P \neq O$ of Λ belongs to $(1+\varepsilon) K_1^*$, and at least one such point belongs to $(1+\varepsilon) K_2^*$.

Let now $\alpha_1, \ldots, \alpha_{n-1}$ be $n-1$ real numbers at least one of which is irrational; and let $\beta_1, \ldots, \beta_{n-1}, 1$ be n real numbers which are linearly independent over the rational field. Both lattices

$$\Lambda_1: \ x_1 = u_1 - \alpha_1 u_n, \ldots, \ x_{n-1} = u_{n-1} - \alpha_{n-1} u_n, \ x_n = D u_n \qquad (u_1, \ldots, u_n = 0, \mp 1, \mp 2, \ldots),$$

$$\Lambda_2: \ x_1 = v_1, \ldots, \ x_{n-1} = v_{n-1}, \ x_n = D(\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n) \qquad (v_1, \ldots, v_n = 0, \mp 1, \mp 2, \ldots),$$

are of determinant D. Hence, however small $\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are chosen, there exist integers $u_1, \ldots, u_n$ not all zero such that

$$(A): \ |D u_n|^{\frac{1}{n-1}} \max(|u_1 - \alpha_1 u_n|, \ldots, |u_{n-1} - \alpha_{n-1} u_n|) < 1 + \varepsilon,$$
$$|u_1 - \alpha_1 u_n| \leq \varepsilon_1, \ldots, \ |u_{n-1} - \alpha_{n-1} u_n| \leq \varepsilon_{n-1},$$

and however small ε and ε_n are chosen, there exist integers $v_1, \ldots, v_n$ not all zero such that

$$(B): \ |D(\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n)|^{\frac{1}{n-1}} \max(|v_1|, \ldots, |v_{n-1}|) < 1 + \varepsilon,$$
$$|\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n| \leq v_n.$$

Let now $\varepsilon_1, \ldots, \varepsilon_n$ tend to zero. Then, from the hypothesis, both $|u_n|$ and $\max(|v_1|, \ldots, |v_{n-1}|)$ tend to infinity. Hence, by (A) and (B), there exist an infinity of systems of n integers $u_1, \ldots, u_n$ such that

$$(C): \ |u_1 - \alpha_1 u_n| < \left|\frac{1+\varepsilon}{D u_n}\right|^{\frac{1}{n-1}}, \ldots, \ |u_{n-1} - \alpha_{n-1} u_n| < \left|\frac{1+\varepsilon}{D u_n}\right|^{\frac{1}{n-1}}, \ |u_n| \to \infty,$$

and an infinity of systems of n integers $v_1, \ldots, v_n$ such that

$$(D): \ |\beta_1 v_1 + \ldots + \beta_{n-1} v_{n-1} + v_n| < \frac{1+\varepsilon}{D} \max(|v_1|, \ldots, |v_{n-1}|)^{-(n-1)},$$
$$\max(|v_1|, \ldots, |v_{n-1}|) \to \infty.$$

Connected with this, the following problems seem of interest:

Problem 8: *To evaluate* $D = \triangle(K)$.

Problem 9: *To decide whether the constant factors* $D^{-\frac{1}{n-1}}$ *and* D^{-1} *in* (C) *and* (D) *are the best possible ones.*

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Third communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 27, 1946.)

§ 12. *Boundedly irreducible and reducible star bodies.*

In the case of unbounded star bodies of the finite type, the following definition seems to be of interest:

Definition C: *The unbounded star body K of the finite type is called boundedly reducible if there exists a bounded star body H such that $H \prec K$, and it is called boundedly irreducible if no bounded star body H with $H \prec K$ exists.*

Theorem J: *For every dimension n, there exists a boundedly irreducible star body K in R_n.*

Proof: We choose for K the star body

$$K^*: \qquad x_1^2 x_2^2 \dots x_{n-1}^2 (x_1^2 + x_2^2 + \dots x_n^2) \leqslant 1$$

considered already in the last paragraph, and for H any bounded star body contained in K. As we saw, K^* is contained in

$$K_0: \qquad |x_1 x_2 \dots x_n| \leqslant 1$$

and of the same determinant $\Delta(K^*) = \Delta(K_0)$; moreover, all boundary points of K^* are *inner* points of K_0. Hence the boundary points of H are likewise *inner* points of K_0; there exists then a constant θ with $0 < \theta < 1$ such that

$$|x_1 x_2 \dots x_n| \leqslant \theta$$

for all points of H. But this implies that

$$\Delta(H) \leqslant \theta \Delta(K_0) < \Delta(K^*),$$

and so it is not true that $H \prec K$, whence the assertion.

If K is any star body, then, as in Part I, we denote by K^t the set of all points X of K for which $|X| \leq t$.

Theorem K: *If the star body K: $F(X) \leq 1$ is boundedly irreducible, then there exists to every $t > 0$ a critical lattice Λ of K and an infinite sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \dots$, with the following properties:*

(a): *All lattices Λ_r are K^t-admissible.*

(b): $$d(\Lambda_r) < \Delta(K) \qquad (r = 1, 2, 3, \dots).$$

(c): *The lattices Λ_r tend to the lattice Λ.*

Proof: Denote, for $r = 1, 2, 3, \ldots$, by $\Lambda^{(r)}$ any critical lattice of K^{r+t}; all these lattices are K^t-admissible. Since K^{r+t} is a bounded subset of K, from the hypothesis,

$$d(\Lambda^{(r)}) = \Delta(K^{r+t}) < \Delta(K) \qquad (r = 1, 2, 3, \ldots).$$

Further, by the corollary to Theorem 10 of Part I,

$$\lim_{r\to\infty} d(\Lambda^{(r)}) = \lim_{r\to\infty} \Delta(K^{r+t}) = \Delta(K).$$

The lattices $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \ldots$ form therefore a *bounded* sequence, and so, by Theorem 2 of Part I, there exists an infinite subsequence

$$\Lambda_1 = \Lambda^{(k_1)}, \quad \Lambda_2 = \Lambda^{(k_2)}, \quad \Lambda_3 = \Lambda^{(k_3)}, \ldots \quad (1 \leqslant k_1 < k_2 < k_3 < \ldots),$$

which converges to a limiting lattice, Λ say. It is clear that the so defined lattices Λ_r and Λ satisfy the assertions (a), (b), and (c) of the theorem; but there remains to prove that Λ is a critical lattice of K.

We show firstly that Λ is *K-admissible*. Let $P \neq O$ be any point of Λ. There is then in each lattice Λ_r a point $P_r \neq O$ such that

$$\lim_{r\to\infty} |P_r - P| = 0.$$

Further, if r is sufficiently large,

$$|P_r| < t + k_r.$$

Since Λ_r is K^{t+k_r}-admissible, this means that

$$F(P_r) \geqslant 1,$$

whence by the continuity of $F(X)$,

$$F(P) = \lim_{r\to\infty} F(P_r) \geqslant 1,$$

i.e. Λ is K-admissible.

Secondly, Λ is even critical since

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda_r) = \Delta(K).$$

This completes the proof.

Definition D: *Let K be an infinite star body of the finite type. Then a critical lattice Λ of K is called strongly critical if there exists a bounded star body K^* contained in K such that*

$$d(\Lambda^*) \geqslant d(\Lambda)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ [12]).

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12) We say that Λ^* is near to Λ if there exist reduced bases

$$Y_1, Y_2, \ldots, Y_n \text{ and } Y_1^*, Y_2^*, \ldots, Y_n^*$$

of Λ and Λ^* such that all numbers

$$|Y_g - Y_g^*| \qquad (g = 1, 2, \ldots, n)$$

are less than a prescribed constant.

It is clear from this definition and from Theorem K that if K is boundedly irreducible, then at least one critical lattice of K is not strongly critical. Hence the following theorem follows at once:

Theorem L: *Let K be an infinite star body of the finite type, and let further every critical lattice of K be strongly critical. Then K is boundedly reducible.*

Proof Assume that, on the contrary, K is boundedly irreducible, and denote by K^* any *bounded* star body contained in K. There exists then a positive number t such that $|X| \leq t$ for every point X of K^*. If Λ is now the critical lattice of K given for this value of t by Theorem K, then Λ is clearly *not* strongly critical.

Theorem L allows in many cases to decide whether a given unbounded star body is boundedly reducible. A few such cases are discussed in the next paragraphs.

§ 13. *Examples of boundedly reducible star domains in R_2.*

In his work on binary cubic forms [13], L. J. MORDELL showed that the two star domains

$$K_1: \qquad |x_1 x_2 (x_1 + x_2)| \leq 1$$

and

$$K_2: \qquad |x_1^3 + x_2^3| \leq 1$$

are of determinants

$$\Delta(K_1) = \sqrt[3]{7} \text{ and } \Delta(K_2) = \sqrt[6]{\tfrac{23}{27}}.$$

It is of interest that his proof gave, incidentally, the result that both star domains are *boundedly reducible*; they were the first non-trivial examples of this kind. I later gave an even simpler example,

$$K_3: \qquad |x_1 x_2| \leq 1, \qquad \text{with } \Delta(K_3) = \sqrt{5},$$

of a boundedly reducible star domain, and made some applications of this property of K_3 [14].

By means of Theorem L, independent proofs that K_1, K_2, and K_3 are boundedly reducible, may be easily obtained. To this purpose, one uses considerations analogous to those in the next paragraphs.

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13) Since his latest proof has not yet appeared, I refer to two articles Journal Lond. Math. **18**, 201—210 and 210—217 (1943), where the two affine-equivalent regions

$$|x_1^3 + x_1^2 x_2 - 2 x_1 x_2^2 - x_2^3| \leq 1 \text{ and } |x_1^3 - x_1 x_2^2 - x_2^3| \leq 1$$

are considered.

14) Proc. Cambr. Phil. Soc. **40**, 108—116, 116—120 (1943), and Journ. Lond. Math. Soc. **18**, 233—238 (1943).

§ 14. *The star body* $|x_1 x_2 x_3| \leq 1$ *in* R_3.

By a theorem of H. DAVENPORT [15]), the star body

$$K: \qquad |x_1 x_2 x_3| \leq 1$$

is of determinant

$$\triangle(K) = 7.$$

Let

$$\theta = 2\cos\frac{2\pi}{7}, \quad \varphi = 2\cos\frac{4\pi}{7}, \quad \psi = 2\cos\frac{6\pi}{7}$$

be the three roots of

$$t^3 + t^2 - 2t - 1 = 0.$$

Then

$$\Lambda_0: x_1 = \theta u_1 + \varphi u_2 + \psi u_3, \; x_2 = \varphi u_1 + \psi u_2 + \theta u_3, \; x_3 = \psi u_1 + \theta u_2 + \varphi u_3, \quad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots)$$

is a critical lattice of K, and every other critical lattice of K is of the form $\Lambda = \Omega \Lambda_0$ where Ω is one of the automorphisms

$$\Omega: \qquad x_1 = t_1 x'_\alpha, \quad x_2 = t_2 x'_\beta, \quad x_3 = t_3 x'_\gamma$$

of K; here t_1, t_2, t_3 are real numbers satisfying

$$t_1 t_2 t_3 = \mp 1,$$

and α, β, γ is any permutation of 1, 2, 3.

Theorem M: *The star body* K: $|x_1 x_2 x_3| \leq 1$ *in* R_3 *is boundedly reducible.*

Proof: It suffices to show that Λ_0 *is a strongly critical lattice of* K because, by affine invariance, the same is then true for all critical lattices of K, and so the assertion follows immediately from Theorem L.

By definition, the lattice Λ_0 is strongly critical if there exists a bounded star body $K^* < K$ such that

$$d(\Lambda^*) \geqq d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* sufficiently near to Λ_0. Such a lattice Λ^* near to Λ_0 contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 obtained for $u_1 = 1$, $u_2 = 0$, $u_3 = 0$. There exists then an automorphism

$$\Omega^*: \qquad x_1 = t_1^* x_1^*, \quad x_2 = t_2^* x_2^*, \quad x_3 = t_3^* x_3^* \qquad (t_1^* t_2^* t_3^* = 1)$$

[15]) Proc. Lond. Math. Soc. **44**, 412—431 (1938).

of K which changes P^* into a point $\Omega^* P^*$ collinear with O and P_0:

$$t_1^* \theta^* : t_2^* \varphi^* : t_3^* \psi^* = \theta : \varphi : \psi.$$

Hence, by affine invariance, it suffices to show that

$$d(\Lambda^*) \geqslant d(\Lambda_0)$$

for every K^*-admissible lattice Λ^* which is (i) sufficiently near to Λ_0, and which (ii) contains a point

$$P^* = (\theta^*, \varphi^*, \psi^*)$$

arbitrarily near to the point

$$P_0 = (\theta, \varphi, \psi)$$

of Λ_0 such that O, P_0, P^* are collinear.

Now every lattice Λ^* near to Λ_0 can be written in the form

$$\Lambda^* : x_1 = \theta v_1 + \varphi v_2 + \psi v_3, \quad x_2 = \varphi v_1 + \psi v_2 + \theta v_3, \quad x_3 = \psi v_1 + \theta v_2 + \varphi v_3$$

with

$$\left.\begin{aligned} v_1 &= u_1 + (a_{11} u_1 + u_{12} u_2 + u_{13} u_3), \\ v_2 &= u_2 + (a_{21} u_1 + u_{22} u_2 + u_{23} u_3), \\ v_3 &= u_3 + (a_{31} u_1 + a_{32} u_3 + a_{33} u_3), \end{aligned}\right\} (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$

where the coefficients a_{hk} are real numbers such that

$$\mathrm{a} = \max_{h, k = 1,2,3} |a_{hk}|$$

is less than any given constant. The point P^* of Λ^* corresponding to P_0 is

$$P^* = ((1+a_{11})\theta + a_{21}\varphi + a_{31}\psi, (1+a_{11})\varphi + a_{21}\psi + a_{31}\theta, (1+a_{11})\psi + a_{21}\theta + a_{31}\varphi)$$

and is collinear with O and P_0 if and only if

(a) : $$a_{21} = a_{31} = 0,$$

because the three points

$$P_0 = (\theta, \varphi, \psi), \quad P_1 = (\varphi, \psi, \theta), \quad P_2 = (\psi, \theta, \varphi)$$

are linearly independent. We consider from now on only lattices Λ^* for which the condition (a) is satisfied.

Put for shortness,

$$S(U) = (\theta u_1 + \varphi u_2 + \psi u_3)(\varphi u_1 + \psi u_2 + \theta u_3)(\psi u_1 + \theta u_2 + \varphi u_3) =$$
$$= (u_1^3 + u_2^3 + u_3^3) - 4(u_2 u_3^2 + u_3 u_1^2 + u_1 u_2^2) + 3(u_2^2 u_3 + u_3^2 u_1 + u_1^2 u_2) - u_1 u_2 u_3,$$

so that

$$x_1 x_2 x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. Similarly

$$x_1 x_2 x_3 = S(V)$$

for the point of Λ^* belonging to $V = (v_1, v_2, v_3)$, or, on replacing V by its value in U,

$$x_1 x_2 x_3 = S(U) + T(U);$$

here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^2) + (B_1 u_2 u_3^2 + B_2 u_3 u_1^2 + B_3 u_1 u_2^2) + \\ + (C_1 u_2^2 u_3 + C_2 u_3^2 u_1 + C_3 u_1^2 u_2) + D u_1 u_2 u_3,$$

with the coefficients

$$\begin{aligned}
A_1 &= 3a_{11} + O(a^2),\\
A_2 &= 3a_{22} - 4a_{12} + 3a_{32} + O(a^2),\\
A_3 &= 3a_{33} - 4a_{23} + 3a_{13} + O(a^2),\\
B_1 &= -4a_{22} - 8a_{33} + 3a_{12} + 6a_{23} + 3a_{32} - a_{13} + O(a^2),\\
B_2 &= -8a_{11} - 4a_{33} + 3a_{23} + 3a_{13} + O(a^2),\\
B_3 &= -4a_{11} - 8a_{22} + 6a_{12} - a_{32} + O(a^2),\\
C_1 &= 6a_{22} + 3a_{33} - a_{12} + 3a_{23} - 8a_{32} - 4a_{13} + O(a^2),\\
C_2 &= 3a_{11} + 6a_{33} - a_{23} - 8a_{13} + O(a^2),\\
C_3 &= 6a_{11} + 3a_{22} + 3a_{12} - 4a_{32} + O(a^2),\\
D &= -a_{11} - a_{22} - a_{33} - 8a_{12} - 8a_{23} + 6a_{32} + 6a_{13} + O(a^2),
\end{aligned}$$

where, in all cases, the O-term consists of the products of two or three of the a_{hk}. These formulae imply, in particular, that when a tends to zero, then the maximum

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|)$$

satisfies the inequality

$$A = O(a).$$

On solving for the coefficients a_{hk}, we find further that

$$\begin{aligned}
3a_{11} &= A_1 + O(a^2),\\
105a_{22} &= -70A_1 - 15A_2 + 30A_3 + 18B_1 - 12B_2 - 27B_3 - 6D + O(a^2),\\
105a_{33} &= 65A_1 + 45A_2 - 18B_1 + 12B_2 + 27B_3 - 9D + O(a^2),\\
35a_{12} &= -25A_1 - 5A_2 + 15A_3 + 9B_1 - 6B_2 - 6B_3 - 3D + O(a^2),\\
35a_{23} &= 35A_1 + 15A_2 - 5A_3 - 6B_1 + 9B_2 + 9B_3 - 3D + O(a^2),\\
35a_{32} &= -10A_1 + 10A_2 + 10A_3 + 6B_1 - 4B_2 + B_3 - 2D + O(a^2),\\
35a_{13} &= 25A_1 + 5A_2 + 5A_3 - 2B_1 + 8B_2 + 3B_3 - D + O(a^2),
\end{aligned}$$

and we also obtain the three identities,

$$\begin{aligned}
5C_1 &= 5A_1 - 5A_2 - 10A_3 - 7B_1 + 3B_2 - 2B_3 - D + O(a^2),\\
5C_2 &= -10A_1 + 5A_2 - 5A_3 - 2B_1 - 7B_2 + 3B_3 - D + O(a^2),\\
5C_3 &= -5A_1 - 10A_2 + 5A_3 + 3B_1 - 2B_2 - 7B_3 - D + O(a^2),
\end{aligned}$$

and the inequality

$$a = O(A), \qquad O(a^2) = O(A^2).$$

So far, the star body K^* has not yet been defined; nor have we yet used that Λ^* is K^*-admissible. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqslant 3, \quad |u_2| \leqslant 3, \quad |u_3| \leqslant 3,$$

belong to K^t. Then the ten points of Λ_0 given by

$$U = (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0),$$
$$(0, -1, -2), (-2, 0, -1), (-1, -2, 0), (-1, -1, -1),$$

satisfy the equation,

$$S(U) = 1.$$

The points of Λ^* belonging to the same U cannot be *inner* of $K^* = K^t$ since Λ^* is K^*-admissible. The numbers

$$\alpha_1, \alpha_2, \alpha_3, \quad \beta_1, \beta_2, \beta_3, \quad \gamma_1, \gamma_2, \gamma_3, \quad \delta$$

defined by

$$T(1, 0, 0) = \alpha_1, \; T(0, 1, 1) = \beta_1, \; T(0, -1, -2) = \gamma_1,$$
$$T(0, 1, 0) = \alpha_2, \; T(1, 0, 1) = \beta_2, \; T(-2, 0, -1) = \gamma_2, \; T(-1, -1, -1) = \delta,$$
$$T(0, 0, 1) = \alpha_3, \; T(1, 1, 0) = \beta_3, \; T(-1, -2, 0) = \gamma_3,$$

are therefore non-negative because

$$x_1 x_2 x_3 = S(U) + T(U) = 1 + T(U) \geqslant 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{aligned}
\alpha_1 &= A_1, \\
\alpha_2 &= A_2, \\
\alpha_3 &= A_3, \\
\beta_1 &= A_2 + A_3 + B_1 + C_1, \\
\beta_2 &= A_1 + A_3 + B_2 + C_2, \\
\beta_3 &= A_1 + A_2 + B_3 + C_3, \\
\gamma_1 &= -A_2 - 8A_3 - 4B_1 - 2C_1, \\
\gamma_2 &= -8A_1 - A_3 - 4B_2 - 2C_2, \\
\gamma_3 &= -A_1 - 8A_2 - 4B_3 - 2C_3, \\
\delta &= -A_1 - A_2 - A_3 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3 - D,
\end{aligned}$$

and conversely,

$$\begin{aligned}
A_1 &= \alpha_1,\\
A_2 &= \alpha_2,\\
A_3 &= \alpha_3,\\
B_1 &= \tfrac{1}{2}\alpha_2 - 3\alpha_3 - \beta_1 - \tfrac{1}{2}\gamma_1,\\
B_2 &= -3\alpha_1 + \tfrac{1}{2}\alpha_3 - \beta_2 - \tfrac{1}{2}\gamma_2,\\
B_3 &= \tfrac{1}{2}\alpha_1 - 3\alpha_2 - \beta_3 - \tfrac{1}{2}\gamma_3,\\
C_1 &= -\tfrac{3}{2}\alpha_2 + 2\alpha_3 + 2\beta_1 + \tfrac{1}{2}\gamma_1,\\
C_2 &= 2\alpha_1 - \tfrac{3}{2}\alpha_3 + 2\beta_2 + \tfrac{1}{2}\gamma_2,\\
C_3 &= -\tfrac{3}{2}\alpha_1 + 2\alpha_2 + 2\beta_3 + \tfrac{1}{2}\gamma_3,\\
D &= \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 - \beta_3 - \delta.
\end{aligned}$$

From these formulae, we deduce that identically,

$$(b):\quad \begin{aligned}
\gamma_1 &= 2\alpha_1 + \alpha_2 - 2\alpha_3 + 2\beta_2 + 2\delta + O(\mathbf{a}^2),\\
\gamma_2 &= -2\alpha_1 + 2\alpha_2 + \alpha_3 + 2\beta_3 + 2\delta + O(\mathbf{a}^2),\\
\gamma_3 &= \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\beta_1 + 2\delta + O(\mathbf{a}^2).
\end{aligned}$$

If further

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then, by these formulae, all three numbers a, A, α are of the same order,

$$\alpha = O(\mathrm{a}) = O(A),\quad O(\mathrm{a}^2) = O(\alpha^2),\quad O(A^2) = O(\alpha^2).$$

The proof of the theorem proceeds now as follows:
The lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0)\begin{vmatrix} 1 + a_{11} & a_{12} & a_{13}\\ 0 & 1 + a_{22} & a_{23}\\ 0 & a_{32} & 1 + a_{33}\end{vmatrix} = d(\Lambda_0)(1 + \sigma),$$

where

$$(c):\quad \begin{aligned}
\sigma &= a_{11} + a_{22} + a_{33} + O(\mathbf{a}^2),\\
&= \tfrac{2}{7}(A_1 + A_2 + A_3) - \tfrac{1}{7}D + O(A^2),\\
&= \tfrac{1}{7}(\alpha_1 + \alpha_2 + \alpha_3) + \tfrac{1}{7}(\beta_1 + \beta_2 + \beta_3) + \tfrac{1}{7}\delta + O(\alpha^2).
\end{aligned}$$

Assume now that Λ^* is so near to Λ_0 that a, hence also A and α, are sufficiently small. Then

either

$$\sigma > 0,\quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0,\quad d(\Lambda^*) = d(\Lambda_0).$$

By (c), the second case cannot hold unless

$$(d):\quad \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = \delta = 0,$$

hence also

(e): $$\gamma_1 = \gamma_2 = \gamma_3 = 0,$$

because by (b),

$$\max(|\gamma_1|, |\gamma_2|, |\gamma_3|)$$

is of at most the same order as

$$\max(|a_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\delta|).$$

The equations (d) and (e) imply next that

$$A_1 = A_2 = A_3 = B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D = 0,$$

hence also

$$a_{11} = a_{22} = a_{33} = a_{12} = a_{23} = a_{32} = a_{13} = 0;$$

and so Λ^* coincides with Λ_0. This concludes the proof.

Although the proof just given is a pure *existence proof*, it can easily be altered so as to lead to the construction of a bounded star body K^* satisfying $K^* < K$.

The next theorems are all proved in a manner similar to that of Theorem M.

for every K^*-admissible lattice Λ^* which is (i) *sufficiently near to Λ_0, and which* (ii) *contains a point P^* arbitrarily near to P_0 and collinear with O and P_0.*

Now every latice Λ^* near to Λ_0 can be written in the form

$$\Lambda^*: \quad x_1 = v_1 + \frac{\alpha+\beta}{2} v_2 + \frac{\alpha^2+\beta^2}{2} v_3, \ x_2 = \frac{\alpha-\beta}{2i} v_2 + \frac{\alpha^2-\beta^2}{2i} v_3,$$
$$x_3 = v_1 + \gamma v_2 + \gamma^2 v_3,$$

with

$$v_1 = u_1 + (a_{11} u_1 + a_{12} u_2 + a_{13} u_3),$$
$$v_2 = u_2 + (a_{21} u_1 + a_{22} u_2 + a_{23} u_3), \qquad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots),$$
$$v_3 = u_3 + (a_{31} u_1 + a_{32} u_2 + a_{33} u_3),$$

where the coefficients a_{hk} are real numbers such that

$$a = \max_{h,k=1,2,3} |a_{hk}|$$

is less than any given constant. The point P^* corresponding to P_0 is $P^* = (x_1^*, x_2^*, x_3^*)$ where

$$x_1^* = 1 + a_{11} + \frac{\alpha+\beta}{2} a_{21} + \frac{\alpha^2+\beta^2}{2} a_{31}, \ x_2 = \frac{\alpha-\beta}{2i} a_{21} + \frac{\alpha^2-\beta^2}{2i} a_{31},$$
$$x_3 = 1 + a_{11} + \gamma a_{21} + \gamma^2 a_{31},$$

and so P^* is collinear with O and P_0 if and only if

(a): $$a_{21} = a_{31} = 0,$$

because the three points

$$(1, 0, 1), \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2i}, \gamma\right), \left(\frac{\alpha^2+\beta^2}{2}, \frac{\alpha^2-\beta^2}{2i}, \gamma^2\right)$$

are linearly independent. We consider from now on only lattices satisfying (a).

Put for shortness,

$$S(U) = (u_1 + \alpha u_2 + \alpha^2 u_3)(u_1 + \beta u_2 + \beta^2 u_3)(u_1 + \gamma u_2 + \gamma^2 u_3) =$$
$$= (u_1^3 + u_2^3 + u_3^3) + u_3^2 u_1 + (-u_2 u_3^2 + 2 u_3 u_1^2 - u_1 u_2^2) - 3 u_1 u_2 u_3,$$

so that

$$(x_1^2 + x_2^2) x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. For the corresponding point of Λ^*,

$$(x_1^2 + x_2^2) x_3 = S(V),$$

or on replacing $V = (v_1, v_2, v_3)$ by its expression in U,

$$(x_1^2 + x_2^2) x_3 = S(U) + T(U).$$

Here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^3) + (B_1 u_2^2 u_3 + B_2 u_3^2 u_1 + B_3 u_1^2 u_2) + \\ + (C_1 u_2 u_3^2 + C_2 u_3 u_1^2 + C_3 u_1 u_2^2) + D u_1 u_2 u_3,$$

with the coefficients,

$$\begin{aligned}
A_1 &= 3 a_{11} + O(a^2),\\
A_2 &= 3 a_{22} - a_{12} + O(a^2),\\
A_3 &= 3 a_{33} - a_{23} + a_{13} + O(a^2),\\
B_1 &= -3 a_{12} + 3 a_{23} - 2 a_{32} - a_{13} + O(a^2),\\
B_2 &= a_{11} + 2 a_{33} - 3 a_{23} + 4 a_{13} + O(a^2).\\
B_3 &= 3 a_{12} + 3 a_{32} + O(a^2),\\
C_1 &= - a_{22} - 2 a_{33} + a_{12} + 3 a_{32} - 3 a_{13} + O(a^2),\\
C_2 &= 4 a_{11} + 2 a_{33} + 3 a_{13} + O(a^2),\\
C_3 &= - a_{11} - 2 a_{22} - 3 a_{32} + O(a^2),\\
D &= -3 a_{11} - 3 a_{22} - 3 a_{33} + 4 a_{12} - 2 a_{23} + 2 a_{32} + O(a^2).
\end{aligned}$$

In all these formulae, the O-term consists of the products of two or three of the a_{hk}. If

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|),$$

then these formulae imply, in particular, that

$$A = O(a).$$

On solving for the coefficients a_{hk}, we find that

$$\begin{aligned}
3 a_{11} &= A_1 + O(a^2),\\
69 a_{22} &= 2 A_1 + 27 A_2 + 9 B_3 + 6 C_3 + O(a^2),\\
69 a_{33} &= - A_1 + 27 A_3 - 9 B_2 + 3 C_2 + O(a^2),\\
23 a_{12} &= 2 A_1 + 4 A_2 + 9 B_3 + 6 C_3 + O(a^2),\\
23 a_{23} &= -11 A_1 - 2 A_3 - 7 B_2 + 10 C_2 + O(a^2),\\
23 a_{32} &= -3 A_1 - 6 A_2 - 2 B_3 - 9 C_3 + O(a^2),\\
23 a_{13} &= -10 A_1 - 6 A_3 + 2 B_2 + 7 C_2 + O(a^2),
\end{aligned}$$

and also obtain the three identities,

$$\begin{aligned}
B_1 &= - A_1 - B_2 - B_3 + C_2 + O(a^2),\\
C_1 &= A_1 - A_2 - C_2 - C_3 + O(a^2),\\
D &= - A_2 - A_3 + B_2 + B_3 - C_2 + O(a^2),
\end{aligned}$$

and the inequality,

$$a = O(A), \quad O(a^2) = O(A^2).$$

So far, the star body K^* has not yet been defined. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqslant 3, \quad |u_2| \leqslant 3, \quad |u_3| \leqslant 3$$

belong to K^t. Then the ten points of Λ_0 given by

$$U = (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (0, 1, 1),\ (-1, 0, 1),\ (1, 1, 0),$$
$$(0, -1, 1),\ (2, 0, -1),\ (-1, 1, 0),\ (1, 1, 1),$$

satisfy the equation,

$$S(U) = 1.$$

If Λ^* is K^*-admissible, then the points of Λ^* belonging to the same U cannot be *inner* points of $K^* = K^t$. The numbers

$$\alpha_1, \alpha_2, \alpha_3,\ \beta_1, \beta_2, \beta_3,\ \gamma_1, \gamma_2, \gamma_3,\ \delta$$

defined by

$$\begin{array}{llll}
T(1,0,0) = \alpha_1, & T(0,1,1) = \beta_1, & T(0,-1,1) = \gamma_1, & \\
T(0,1,0) = \alpha_2, & T(-1,0,1) = \beta_2, & T(2,0,-1) = \gamma_2, & T(1,1,1) = \delta, \\
T(0,0,1) = \alpha_3, & T(1,1,0) = \beta_3, & T(-1,1,0) = \gamma_3, &
\end{array}$$

are then non-negative since

$$(x_1^2 + x_2^2)\, x_3 = S(U) + T(U) = 1 + T(U) \geqq 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{array}{ll}
\alpha_1 = & A_1, \\
\alpha_2 = & A_2, \\
\alpha_3 = & A_3, \\
\beta_1 = & A_2 + A_3 + B_1 + C_1, \\
\beta_2 = & -A_1 + A_3 - B_2 + C_2, \\
\beta_3 = & A_1 + A_2 + B_3 + C_3, \\
\gamma_1 & -A_2 + A_3 + B_1 - C_1, \\
\gamma_2 = & 8A_1 - A_3 + 2B_2 - 4C_2, \\
\gamma_3 = & -A_1 + A_2 + B_3 - C_3, \\
\delta = & A_1 + A_2 + A_3 + B_1 + B_2 + B_3 + C_1 + C_2 + C_3 + D,
\end{array}$$

and conversely,

$$\begin{array}{ll}
A_1 = & \alpha_1, \\
A_2 = & \alpha_2, \\
A_3 = & \alpha_3, \\
B_1 = & -\alpha_3 + \tfrac{1}{2}\beta_1 + \tfrac{1}{2}\gamma_1, \\
B_2 = & 2\alpha_1 + \tfrac{3}{2}\alpha_3 - 2\beta_2 - \tfrac{1}{2}\gamma_2, \\
B_3 = & -\alpha_2 + \tfrac{1}{2}\beta_3 + \tfrac{1}{2}\gamma_3, \\
C_1 = & -a_2 + \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\gamma_1, \\
C_2 = & 3\alpha_1 + \tfrac{1}{2}\alpha_3 - \beta_2 - \tfrac{1}{2}\gamma_2, \\
C_3 = & -\alpha_1 + \tfrac{1}{2}\beta_3 - \tfrac{1}{2}\gamma_3, \\
D = & -5\alpha_1 + \alpha_2 - 2\alpha_3 - \beta_1 + 3\beta_2 - \beta_3 + \gamma_2 + \delta.
\end{array}$$

We deduce from these formulae that

$$\begin{aligned}
\beta_1 &= -\alpha_1 + \alpha_2 - \tfrac{1}{2}\alpha_3 + 2\beta_2 - \beta_3 + \tfrac{1}{2}\gamma_2 + O(a^2),\\
\gamma_1 &= \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\gamma_2 - \gamma_3 + O(a^2),\\
\delta &= 3\alpha_1 - 2\alpha_2 + \tfrac{3}{2}\alpha_3 - 2\beta_2 + \tfrac{1}{2}\beta_3 - \tfrac{1}{2}\gamma_2 + \tfrac{1}{2}\gamma_3 + O(a^2).
\end{aligned}$$

Hence, if

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then all three numbers a, A, α are of the same order,

$$\alpha = O(a) = O(A), \quad O(a^2) = O(\alpha^2), \quad O(A^2) = O(\alpha^2).$$

Finally, the lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0)\begin{vmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+a_{22} & a_{23} \\ 0 & a_{32} & 1+a_{33} \end{vmatrix} = d(\Lambda_0)(1+\sigma),$$

and here

$$\begin{aligned}
\sigma &= a_{11} + a_{22} + a_{33} + O(a^2),\\
&= \tfrac{1}{2\,3}(8A_1 + 9A_2 + 9A_3 - 3B_2 + 3B_3 + C_2 + 2C_3) + O(A^2),\\
&= \tfrac{1}{2\,3}(3\alpha_1 + 6\alpha_2 + 5\alpha_3 + 5\beta_2 + \tfrac{5}{2}\beta_3 + \gamma_2 + \tfrac{1}{2}\gamma_3) + O(\alpha^2).
\end{aligned}$$

We find therefore, just as in the last proof, that either

$$\sigma > 0, \quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0, \quad d(\Lambda^*) = d(\Lambda_0),$$

and that the second case can hold only if $\alpha = a = A = 0$, that is, if Λ^* coincides with Λ_0; whence the assertion.

§ 16. *Some further examples.*

I have applied the method of the last paragraphs to three further star bodies in R_3 and R_4. From the well-known results of A. OPPENHEIM [18]) on the minima of the indefinite quadratic forms in three and four variables, I have so deduced that

the star body K_1: $|x_1^2 + x_2^2 - x_3^2| \leq 1$ in R_3 with $\triangle(K_1) = \sqrt{\tfrac{3}{2}}$,

the star body K_2: $|x_1^2 + x_2^2 + x_3^2 - x_4^2| \leq 1$ in R_4 with $\triangle(K_2) = \sqrt{\tfrac{7}{4}}$,

and the star body K_3: $|x_1^2 + x_2^2 - x_3^2 - x_4^2| \leq 1$ in R_4 with $\triangle(K_3) = \tfrac{3}{2}$,

are each one boundedly reducible. As before, Theorem L is the basis of the proof; since no new ideas are used, I omit this proof.

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18) See L. E. DICKSON, Studies in the theory of numbers (Chicago 1930), chapters 8 and 9.

In all these examples of boundedly reducible star bodies, it would be of great interest to obtain irreducible star bodies of equal determinants contained in them.

§ 17. *Applications.*

The following theorems show that the preceeding results can be useful for other purposes.

Theorem O: *There exists a positive constant c with the following property: If α_1, α_2 are real numbers, and t is a positive number, then there exist integers u_1, u_2, u_3 not all zero such that*

$$\{(u_1-\alpha_1 u_3)^2+(u_2-\alpha_2 u_3)^2\}\,|u_3|\leqslant\frac{2}{\sqrt{23}},$$

$$(u_1-\alpha_1 u_3)^2+(u_2-\alpha_2 u_3)^2\leqslant\frac{c}{t},\quad |u_3|\leqslant t.$$

Hence if α_1, α_2 are irrational, then there are arbitrarily large integers u_1, u_2, u_3 such that [19])

$$\left(\frac{u_1}{u_3}-\alpha_1\right)^2+\left(\frac{u_2}{u_3}-\alpha_2\right)^2\leqslant\frac{2}{\sqrt{23}\,|u_3|^3}.$$

Proof: By Theorem N, a positive number r exists such that

$$K^*:\qquad (x_1^2+x_2^2)\,|x_3|\leqslant 1,\quad x_1^2+x_2^2+x_3^2\leqslant r^2$$

is of the same determinant as

$$K:\qquad (x_1^2+x_2^2)\,|x_3|\leqslant 1,$$

namely $\Delta(K)=\Delta(K^*)=\sqrt{(23)}/2$. On applying the transformation

$$\Omega:\qquad x_1=\tau x_1',\ x_2=\tau x_2',\ x_3=\tau^{-2}x_3'\quad(\tau>0),$$

of K, we find that

$$K_\tau^*:\qquad (x_1^2+x_2^2)\,|x_3|\leqslant 1,\quad \tau^2(x_1^2+x_2^2)+\tau^{-4}x_3^2\leqslant r^2$$

is likewise of determinant $\Delta(K_\tau^*)=\Delta(K)=\sqrt{(23)}/2$. Let Λ be the lattice

$$\Lambda:\ x_1=u_1-\alpha_1 u_3,\ x_2=u_2-\alpha_2 u_3,\ x_3=\frac{\sqrt{23}}{2}u_3\ \ (u_1,u_2,u_3=0,\mp 1,\mp 2,\ldots).$$

Since $d(\Lambda)=\sqrt{(23)}/2$, at least one point $P\neq O$ of Λ lies inside or on the boundary of K_τ^*; let this be the point belonging to the integers u_1, u_2, u_3 not all zero. Then

$$\{(u_1-\alpha_1 u_3)^2+(u_2-\alpha_2 u_3)^2\}\,|u_3|\leqslant\frac{2}{\sqrt{23}},$$

$$\tau^2\{(u_1-\alpha_1 u_3)^2+(u_2-\alpha_2 u_3)^2\}+\frac{23}{4\tau^4}u_3^2\leqslant r^2,$$

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[19]) A slightly less exact result is proved in a joint paper by DAVENPORT and myself, in DUKE Math. Journal **13**, 105—111 (1946).

hence

$$(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2 \leqq \frac{r^2}{\tau^2}, \quad |u_3| \leqq \frac{2\tau^2 r}{\sqrt{23}}.$$

If now

$$c = \frac{2r^3}{\sqrt{23}}, \quad \tau = \left(\frac{23t^2}{4r^2}\right)^{1/4},$$

then

$$(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2 \leqq \frac{c}{t}, \quad |u_3| \leqq t,$$

as asserted. — Assume next that α_1 is irrational and that t tends to infinity. Then u_3 is different from zero for sufficiently large t, and since $|u_1 - \alpha_1 u_3|$ tends to zero, $|u_3|$ tends to infinity.

In a similar way, Theorem M leads to the following result:

Theorem P: *There exists a positive constant γ with the following property: If β_1, β_2 are real numbers and t is a positive number, then integers v_1, v_2, v_3 not all zero exist such that*

$$|v_1 v_2 (\beta_1 v_1 + \beta_2 v_2 + v_3)| \leqq \tfrac{1}{7},$$

$$|v_1| \leqq t, \quad |v_2| \leqq t, \quad |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqq \frac{\gamma}{t^2}.$$

Assume further that β_1, β_2 have the following stronger properties: (i) *$\beta_1, \beta_2, 1$ are linearly independent over the rational field.* (ii) *When the integers v, v', v'' tend to infinity in any way, then*

$$\lim v^2 |\beta_1 v + v'| = \infty, \quad \lim v^2 |\beta_2 v + v''| = \infty.$$

Under these conditions, there exist an infinity of triples of integers v_1, v_2, v_3 all different from zero such that

$$0 < |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqq \frac{1}{7|v_1 v_2|}.$$

The results on boundedly reducible star bodies are also of use for obtaining asymptotic formulae for the determinants of certain star bodies depending on a parameter [20]). For instance, it is easy to deduce from Theorem M that when $\alpha > 0$ tends to zero, then the star body

K_1: $|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha \leqq 1$ **is of determinant** $\Delta(K_1) = \frac{1}{7} e^{-2/\alpha}(1 + O(\alpha))$,

and the star body

K_2: $(|x_1|^\alpha + |x_2|^\alpha)|x_3|^{\alpha/2} \leqq 1$ **is of determinant** $\Delta(K_2) = \frac{1}{7} e^{-1/\alpha}(1 + O(\alpha))$.

I remark, finally, that the just given examples of boundedly reducible star bodies in R_3 and R_4 are all automorphic, and even satisfy the stronger

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[20]) For a special case, see my paper Proc. Cambr. Phil. Soc. **40**, 116—120 (1944), in particular the proof of Theorem 4.

conditions of Theorem 23 of Part I. This suggests that the following problem has an affirmative answer:

Problem 10: *Is it true that every automorphic star body is boundedly reducible if it satisfies the conditions of Theorem 23 of Part I?*

§ 18. *An addition to Theorem 9 of Part I.*

In Theorem 9 of Part I, $\triangle(K)$ was proved to depend continuously on K if K varied in a rather restricted way. To conclude this Part II, we prove a more general continuity property of $\triangle(K)$.

Theorem Q: *Let $F(X)$ and $F_r(X)$ $(r = 1, 2, 3, \ldots)$ be distance functions in R_n such that*

$$\lim_{r\to\infty} F_r(X) = F(X)$$

uniformly in X on the unit sphere $|X| = 1$ [21]*); and let the star bodies*

$$K: \; F(X) \leqslant 1, \quad \text{and} \quad K_r: \; F_r(X) \leqslant 1 \quad (r = 1, 2, 3, \ldots)$$

be of the finite type. Then

$$\liminf_{r\to\infty} \triangle(K_r) \geqslant \triangle(K).$$

Proof: Let $\varepsilon > 0$ be arbitrarily small. By the Corollary to Theorem 10 of Part I, there exists a positive number t such that the determinant of the star body

$$K^t: \qquad F(X) \leqslant 1, \qquad |X| \leqslant t$$

satisfies the inequalities,

$$(1-\varepsilon)\triangle(K) \leqslant \triangle(K^t) \leqslant \triangle(K).$$

There is further an integer $r_0 = r_0(\varepsilon)$ such that

$$F_r(X) \leqslant 1+\varepsilon \text{ for the points } X \text{ of } K^t \qquad \text{if } r \geqslant r_0;$$

hence K^t is contained in $(1+\varepsilon)K_r$ if $r \geqq r_0$. This implies

$$\triangle(K^t) \leqslant (1+\varepsilon)^n \triangle(K_r) \qquad \text{if } r \geqslant r_0,$$

whence

$$\triangle(K_r) \geqslant (1+\varepsilon)^{-n}\triangle(K^t) \geqslant \frac{1-\varepsilon}{(1+\varepsilon)^n}\triangle(K) \qquad \text{if } r \geqslant r_0.$$

For $\varepsilon \to 0$, the assertion is obtained.

In the result

$$\liminf_{r\to\infty} \triangle(K_r) \geqslant \triangle(K)$$

of Theorem Q, the sign "$\geqq$" cannot always be replaced by the equality sign, as the following example shows.

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[21]) This implies the uniform convergence in every bounded set.

Theorem R: *For every dimension n, there exist star bodies K and K_r $(r = 1, 2, 3, \ldots)$ in R_n satisfying the hypothesis of Theorem Q, but such that*

$$\lim_{r \to \infty} \triangle(K_r) \text{ exists and is greater then } \triangle(K).$$

Proof: Denote by $c > 0$ a constant which is so large that the sphere

$$H: \qquad |X| \leqslant c$$

is of greater determinant than the star body

$$K: \qquad F(X) \leqslant 1, \qquad \text{where} \quad F(X) = |x_1 x_2 \ldots x_n|^{1/n};$$

denote further by $r = 1, 2, 3, \ldots$ the sequence of all positive integers. The distance function

$$F_r(X) = \min\left\{F(X), \frac{1}{c}\left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2\right)^{1/2}\right\}$$

defines a star body K_r: $F_r(X) \leq 1$ which contains K and is easily seen to be of the finite type. The automorphisms of K,

$$\Omega_r: \qquad x_1 = r x_1', \ldots, x_{n-1} = r x_{n-1}', \; x_n = r^{-(n-1)} x_n',$$

change K_r into K_1; hence

$$\triangle(K_1) = \triangle(K_2) = \triangle(K_3) = \ldots = \lim_{r \to \infty} \triangle(K_r).$$

On the other hand,

$$\triangle(K_r) \geqslant \triangle(H) > \triangle(K)$$

since K_r contains H; hence

$$\lim_{r \to \infty} \triangle(K_r) > \triangle(K).$$

Consider now $F_r(X)$ on the unit sphere $|X| = 1$. Here

$$F(X) \leqslant 1, \text{ and } \frac{1}{c}\left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2\right)^{1/2} \geqslant \frac{r^{n-1}|x_n|}{c},$$

and so

$$F_r(X) = F(X) \text{ unless } |x_n| \leqslant c\, r^{-(n-1)}.$$

If further

$$|x_n| \leqslant c\, r^{-(n-1)}, \quad |X| = 1,$$

then

$$F(X) \leqslant (c\, r^{-(n-1)})^{1/n}, \quad 0 \leqslant F_r(X) \leqslant F(X),$$

whence

$$|F_r(X) - F(X)| \leqslant F(X) \leqslant (c\, r^{-(n-1)})^{1/n},$$

and here the right-hand side tends to zero uniformly in X, as asserted.

Theorem Q leaves many interesting questions unsolved. For instance, the star domain

$$K_\lambda: \qquad |(x_1^2-x_2^2)(x_1^2-\lambda x_2^2)| \leqq 1$$

is easily proved to be of the finite type; is $\Delta(K_\lambda)$ a continuous function of λ?

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