ON REDUCED POSITIVE DEFINITE QUATERNARY QUADRATIC FORMS

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According to MINKOWSKI's definition [1]), a positive definite quadratic form in n variables with real coëfficients

$$f(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k$$

is called reduced, if for $h = 1, 2, \ldots, n$

$f(x) \geqq a_{hh}$ for all integers $x_1, \ldots, x_n$ such that $(x_h, x_{h+1}, \ldots, x_n) = 1$, and also

$$a_{12} \geqq 0,\ a_{23} \geqq 0,\ \ldots,\ a_{n-1,n} \geqq 0.$$

MINKOWSKI, using a method of Hermite, proved that there is a constant $c_n > 0$ depending only on n, such that for reduced forms [2])

$$a_{11} a_{22} \ldots a_{nn} \geqq c_n \mathrm{D},$$

where D is the determinant of $f(x)$. For the lowest values of n, the smallest value of this constant is

$$c = \tfrac{4}{3},\ c_3 = 2,\ c_4 = 4.$$

The first result is classic, the second one due to Gauss (who proved it for Seeber's definition of a reduced ternary form, which is nearly identical with the case $n = 3$ of MINKOWSKI's definition) [3]). I prove here the formula $c_4 = 4$, which seems to be new. *)

My proof is derived from one of MINKOWSKI for $c_3 = 2$ [4]),

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[1]) Ges. Abh. II, 53—100.

[2]) l.c. [1]); see also my note, Quart. Journ. 9 (1938), 259—262.

[3]) Gauss, Werke II.

*) Addition September 1946. Compare the paper by R. Remak, Proc. Royal Acad. Amsterdam, **44**, (1931), 1071—1076, where a similar method is used to study pseudoreduction of quadratic forms.

[4]) Ges. Abh. II.

and is based on the following theorem of KORKINE and ZOLOTAREFF [5]):

"For every positive definite quaternary quadratic form $f(x)$ of determinant D_0, there exists a lattice point $(x) \neq 0$ such that

$$f(x) \leqq \sqrt[4]{4D}$$

with equality if and only if $f(x)$ is equivalent to the form

$\sqrt[4]{4D}\,\varphi_0(x)$, where $\varphi_0(x)=x_1^2+x_2^2+x_3^2+x_4^2+(x_1+x_2+x_3)x_4$. (1)

Proof of the inequality $a_{11}a_{22}a_{33}a_{44} \leqq 4D$ for $n = 4$.

I use the vector notation; lower indices denote the different coordinates, upper ones different points.

Let

$$x^{(k)} = (x_1^{(k)},\ x_2^{(k)},\ x_3^{(k)},\ x_4^{(k)}) \qquad (k = 1, 2, 3, 4)$$

be four lattice points such that $x^{(1)} \neq 0$, and $f(x^{(1)}) = A_{11}$ is the minimum of $f(x)$ in all lattice points $x \neq 0$; and such that for $k = 2$, 3, and 4, $x^{(k)}$ is linearly independent of

$$x^{(1)}, \ldots, x^{(k-1)}, \text{ and } f(x^{(k)}) = A_{kk}$$

is the minimum of $f(x)$ for all lattice points x which are linearly independent of $x^{(1)}, \ldots, x^{(k-1)}$. The four points $x^{(k)}$ are therefore linearly independent, and their determinant

$$d = |\, x_h^{(k)} \,|_{h,k=1,2,3,4}$$

is a non-vanishing integer.

An arbitrary point $x = (x_1, x_2, x_3, x_4)$ can be written as

$$x = \sum_{k=1}^{4} X_k x^{(k)},$$

where the X_k are real numbers; let $X = (X_1, X_2, X_3, X_4)$ be the point with these numbers as its coordinates. The change of x into X is an integral linear transformation of determinant d, namely

$$x_h = \sum_{k=1}^{4} x_h^{(k)} X_k \qquad (h = 1, 2, 3, 4). \quad (2)$$

5) Oeuvres de Zolotareff, Vol. 1.

If X is a lattice point, then so is x; the converse need not hold. The transformation (2) changes $f(x)$ into a new quadratic form

$$F(X) = f\left(\sum_{k=1}^{4} X_k x^{(k)}\right) = \sum_{h,k=1}^{4} A_{hk} X_h X_k,$$

where the A_{kk} are the numbers as defined before. Since, if necessary, we may replace $x^{(k)}$ by $-x^{(k)}$, we can assume that

$$A_{12} \geqq 0,\ A_{23} \geqq 0,\ A_{34} \geqq 0. \qquad (3)$$

By the definition of the lattice points $x^{(k)}$,

$$f(x) \geqq \begin{cases} A_{11} & \\ A_{22} & \\ A_{33} & \\ A_{44} & \end{cases} \text{ for all lattice points } x \text{ such that } \begin{cases} \sum_1^4 X_k^2 > 0, \\ \sum_2^4 X_k^2 > 0, \\ \sum_3^4 X_k^2 > 0, \\ X_4^2 > 0. \end{cases} \qquad (4)$$

The same inequalities with $F(X)$ instead of $f(x)$ hold if X is a lattice point; hence $F(X)$ is a reduced form.

We can write $F(X)$ as a sum of squares of linear forms,

$$F(X) = \Xi_1^2 + \Xi_2^2 + \Xi_3^2 + \Xi_4^2, \qquad (5)$$

such that Ξ_k contains only $X_k, \ldots, X_4$; except for changes of sign, this representation is unique. If we replace X by x according to (2), then (5) is transformed into the analogous representation

$$f(x) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 \qquad (6)$$

of $f(x)$ as a sum of squares of linear forms in the x's. In this representation,

$$\xi_k \text{ vanishes if } X_k = X_{k+1} = \ldots = X_4 = 0.$$

Let

$$g(x) = \frac{\xi_1^2}{A_{11}} + \frac{\xi_2^2}{A_{22}} + \frac{\xi_3^2}{A_{33}} + \frac{\xi_4^2}{A_{44}} \qquad (7)$$

be a new quadratic form of determinant

$$D' = \frac{D}{A_{11}A_{22}A_{33}A_{44}}. \tag{8}$$

Since by the definition of the minima A_{kk}

$$0 < A_{11} \leqq A_{22} \leqq A_{33} \leqq A_{44},$$

we get from (4) for lattice points x that

$$g(x) = \frac{\xi_1^2}{A_{11}} = \frac{f(x)}{A_{11}} \geqq 1, \text{ if } X_1 \neq 0,\ X_2 = X_3 = X_4 = 0;$$

$$g(x) = \frac{\xi_1^2}{A_{11}} + \frac{\xi_2^2}{A_{22}} \geqq \frac{\xi_1^2 + \xi_2^2}{A_{22}} = \frac{f(x)}{A_{22}} \geqq 1, \text{ if } X_2 \neq 0,\ X_3 = X_4 = 0;$$

$$g(x) = \frac{\xi_1^2}{A_{11}} + \frac{\xi_2^2}{A_{22}} + \frac{\xi_3^2}{A_{33}} \geqq \frac{\xi_1^2+\xi_2^2+\xi_3^2}{A_{33}} = \frac{f(x)}{A_{33}} \geqq 1, \text{ if } X_3 \neq 0,\ X_4 = 0;$$

$$g(x) = \frac{\xi_1^2}{A_{11}} + \frac{\xi_2^2}{A_{22}} + \frac{\xi_3^2}{A_{33}} + \frac{\xi_4^2}{A_{44}} \geqq \frac{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2}{A_{44}} = \frac{f(x)}{A_{44}} \geqq 1, \quad \text{if } X_4 \neq 0. \tag{9}$$

Therefore for every lattice point x,

$$g(x) \geqq 1.$$

By the theorem of KORKINE and ZOLOTAREFF, this implies $1 \leqq \sqrt[4]{D'}$ and therefore by (8),

$$A_{11}A_{22}A_{33}A_{44} \leqq 4D. \tag{10}$$

We consider firstly the case that the sign of equality holds in (10), so that $g(x)$ has the determinant $D' = D/4D = \frac{1}{4}$. By the theorem of KORKINE and ZOLOTAREFF, $g(x)$ must therefore be equivalent to

$$\varphi_0(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 + x_2 + x_3)\, x_4.$$

Hence there are 12 essentially different lattice points [6])

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[6]) The *equation* $\phi_0(x) = 1$ has the twelve solutions (1000), (0100), (0010), (0001), (100 —1), (010 —1), (001 —1), (110 —1), (101 —1), (011 —1), (111 —1), (111 —2), and twelve further one derived from these by changing all signs.

$$p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, p^{(5)} = p^{(1)} - p^{(4)}, p^{(6)} = p^{(2)} - p^{(4)}, p^{(7)} = p^{(3)} - p^{(4)},$$
$$p^{(8)} = p^{(1)} + p^{(2)} - p^{(4)},\ \ p^{(9)} = p^{(1)} + p^{(3)} - p^{(4)},\ \ p^{(10)} = p^{(2)} + p^{(3)} - p^{(4)},$$
$$p^{(11)} = p^{(1)} + p^{(2)} + p^{(3)} - p^{(4)},\ \ p^{(12)} = p^{(1)} + p^{(2)} + p^{(3)} - 2p^{(4)},$$

of which the first four are linearly independent, such that

$$g(p^{(l)}) = 1 \qquad (l = 1, 2, \ldots, 12).$$

Neither of the two linear forms

$$\xi_1 = \xi_1(x) \text{ and } \xi_4 = \xi_4(x)$$

in (6) vanishes identically. Hence at least one of the four numbers

$$\xi_1(p^{(k)}) \qquad (k = 1, 2, 3, 4),$$

say the number $\xi_1(p^{(k_1)})$, and at least one of the four numbers

$$\xi_4(p^{(k)}) \qquad (k = 1, 2, 3, 4),$$

say the number $\xi_4(p^{(k_2)})$, is different from zero. If $k_1 = k_2$, then

$$\xi_1(p^{(k_0)}) \neq 0,\ \xi_4(p^{(k_0)}) \neq 0 \quad \text{for } k_0 = k_1 = k_2.$$

We prove that if there is no index $k = 1, 2, 3$ or 4 such that both $\xi_1(p^{(k)})$ and $\xi_4(p^{(k)})$ are diffent from zero, there is still at least one index k_0 in the interval $1 \leqq k_0 \leqq 12$ such that

$$\xi_1(p^{(k_0)}) \neq 0,\ \xi_4(p^{(k_0)}) \neq 0. \tag{11}$$

For reasons of symmetry, it obviously suffices to consider the cases that $k_1 = 1,\ k_2 = 4$, or that $k_1 = 1,\ k_2 = 2$. In the first case

$$\xi_1(p^{(1)}) \neq 0,\ \xi_4(p^{(1)}) = 0; \quad \xi_1(p^{(4)}) = 0,\ \xi_4(p^{(4)}) \neq 0,$$

and therefore

$$\xi_1(p^{(5)}) = \xi_1(p^{(1)}) - \xi_1(p^{(4)}) \neq 0,\ \xi_4(p^{(5)}) = \xi_4(p^{(1)}) - \xi_4(p^{(4)}) \neq 0.$$

In the second case

$$\xi_1(p^{(1)}) \neq 0,\ \xi_4(p^{(1)}) = 0; \quad \xi_1(p^{(2)}) = 0,\ \xi_4(p^{(2)}) \neq 0,$$

and furthermore without loss of generality

$$\xi_1(p^{(4)}) = \xi_4(p^{(4)}) = 0;$$

hence

$$\xi_1(p^{(8)}) = \xi_1(p^{(1)}) + \xi_1(p^{(2)}) - \xi_1(p^{(4)}) \neq 0,$$
$$\xi_4(p^{(8)}) = \xi_4(p^{(1)}) + \xi_4(p^{(2)}) - \xi_4(p^{(4)}) \neq 0.$$

The lattice point $p^{(k_0)}$ in (11) satisfies the further inequality

$$X_4 = X_4(p^{(k_0)}) \neq 0,$$

since $\frac{X_4}{\xi_4}$ is a non-vanishing constant. Hence by (9)

$$g(p^{(k_0)}) = 1 = \frac{\xi_1(p^{(k_0)})^2}{A_{11}} + \frac{\xi_2(p^{(k_0)})^2}{A_{22}} + \frac{\xi_3(p^{(k_0)})^2}{A_{33}} + \frac{\xi_4(p^{(k_0)})^2}{A_{44}} \geqq$$
$$\geqq \frac{\xi_1(p^{(k_0)})^2 + \xi_2(p^{(k_0)})^2 + \xi_3(p^{(k_0)})^2 + \xi_4(p^{(k_0)})^2}{A_{44}} = \frac{f(p^{(k_0)})}{A_{44}} \geqq 1,$$

and since $0 < A_{11} \leqq A_{22} \leqq A_{33} \leqq A_{44}$, we must have

$$A_{11} = A_{22} = A_{33} = A_{44} = \sqrt[4]{4D}.$$

Therefore $f(x)$ is equivalent to the form

$$\sqrt[4]{4D}\,\varphi_0(x).$$

Hence, if $f(x)$ itself is reduced, then [7]

$$a_{11} = a_{22} = a_{33} = a_{44} = \sqrt[4]{4D},$$

and the assertion is proved ....

Secondly, let (10) be true with the sign "<". The form $F(X)$ has the determinant Dd^2; therefore by a well known property of positive definite quadratic forms

$$Dd^2 \leqq A_{11}A_{22}A_{33}A_{44},$$

and by (10),

$$Dd^2 < 4D, \quad d^2 < 4, \quad d = \mp 1,$$

since d is a non-vanishing integer. Hence now the reduced form $F(X)$ is equivalent to $f(x)$; therefore, if $f(x)$ is also reduced, then the statement follows at once, since [7]

$$a_{11} = A_{11}, \quad a_{22} = A_{22}, \quad a_{33} = A_{33}, \quad a_{44} = A_{44}.$$

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[1]) Two equivalent reduced forms $f(x) = \sum_{h,k=1}^{n} a_{hk} x_h x_k$ and $F(X) = \sum_{h,k=1}^{n} A_{kk} X_h X_k$ satisfy the equations

$$a_{kk} = A_{kk} \qquad (k = 1, 2, \ldots, n),$$

since both are lowest forms.

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