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On irreducible convex domains

BY

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Let K be a bounded symmetrical star domain, or short, *a star domain,* in the (x_1, x_2)-plane, i.e. a bounded closed point set of the following kind:

1) The origin $O = (0, 0)$ is an *inner* point of K.
2) When $X = (x_1, x_2)$ belongs to K, then so does the symmetrical point $-X = (-x_1, -x_2)$.
3) The boundary C of K is a JORDAN curve which meets every radius vector from O in just *one* point.

The star domain K is *a convex domain,* if it contains with any two points X and Y also all the points [1])

$$(1-t)\,X + tY, \qquad 0 \leqslant t \leqslant 1,$$

of the line segment XY joining these two points.

A lattice Λ of basis

$$X_1 = (x_{11}, x_{12}), \qquad X_2 = (x_{21}, x_{22})$$

and of determinant

$$d(\Lambda) = |x_{11}\,x_{22} - x_{12}\,x_{21}| > 0$$

consists of all points

$$P = u_1\,X_1 + u_2\,X_2 \qquad (u_1, u_2 = 0, \mp 1, \mp 2, \ldots).$$

Λ is called *K-admissible* if no lattice point except O is an *inner* point of K. The lower bound

$$\Delta(K) = \text{l. b. } d(\Lambda)$$

of $d(\Lambda)$ extended over all K-admissible lattices Λ is a finite positive number. There exists at least one *critical lattice of* K, i.e. a K-admissible lattice Λ such that

$$d(\Lambda) = \Delta(K).$$

It is easily seen that

$$\Delta(K) \geqslant \Delta(H)$$

if K contains H. We say that *K is irreducible* if the stronger inequality

$$\Delta(H) < \Delta(K)$$

holds for all star domains H contained in, but different from, K. I do not

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[1]) We use the notation of sums and scalar products of points or vectors as usual in linear algebra or vector analysis.

know whether every star domain K contains an irreducible star domain K' (not necessarily different from K) such that

$$\Delta(K') = \Delta(K).$$

In this note, I show that this is true at least for convex domains, and, in fact, prove the slightly stronger result:

Theorem 1: *Every convex domain K contains an irreducible convex domain K'* (not necessarily different from K) *such that*

$$\Delta(K') = \Delta(K).$$

§ 1. The parallelogram.

To prove Theorem 1, a number of simple lemmas are required. We begin with an example of an irreducible domain.

Lemma 1: *Every parallelogram with centre at O is irreducible.*

Proof: By affine invariance, it suffices to prove the assertion for the unit square,

$$K_0: \qquad |x_1| \leqslant 1, \; |x_2| \leqslant 1,$$

for which, by MINKOWSKI's theorem on linear forms,

$$\Delta(K_0) = 1.$$

Let H be any star domain contained in, but different from, K_0. Then at least one point P_0 on the boundary C_0 of K_0 lies outside H; without loss of generality, this point P_0 belongs to the line segment

$$P_0 = (\xi, 1), \qquad 0 < \xi < 1,$$

of C_0. Denote by ϑ the number satisfying

$$0 < \vartheta < 1$$

for which ϑP_0 lies on the boundary of H, and by Λ_0 the lattice of basis

$$P_1 = \vartheta P_0 = (\vartheta\xi, \vartheta), \qquad P_2 = (1, \vartheta - 1).$$

This lattice is of determinant

$$d(\Lambda_0) = \begin{vmatrix} 1 & \vartheta - 1 \\ \vartheta\xi & \vartheta \end{vmatrix} = 1 - (1-\vartheta)(1-\vartheta\xi) < 1 = \Delta(K_0).$$

It is H-admissible since $\pm\,\vartheta P_0$ are its only points which are inner points of K_0. Hence

$$\Delta(H) \leqslant d(\Lambda_0) < \Delta(K_0),$$

as asserted.

Corollary: *Theorem 1 holds for all parallelograms with centre at O.*

§ 2. The parallelograms with three vertices on C.

Let K be a convex domain, and let P_1 be any point on its boundary. The line L_1 through O and P_1 divides the (x_1, x_2)-plane into two semiplanes, P_+ and P_-, say. Denote by C_+ the half of C in P_+, by $C + P_1$, and

$C_+ + P_1$, the sets of all points $P + P_1$ where P runs over C, and C_+, respectively. The two points $-P_1 + P_1 = O$ and $P_1 + P_1 = 2P_1$ lie on $C + P_1$, but are situated on different sides of C. Hence C and $C + P_1$ intersect at least once, and so, by symmetry, C_+ and $C_+ + P_1$ also have a non-empty intersection, $I_2(P_1)$ say. If P_2 describes $I_2(P_1)$, then $P_3 = P_2 - P_1$ runs over a second set, $I_3(P_1)$ say. Since $P_3 + P_1 = P_2$ belongs to $C_+ + P_1$, P_3 lies on C_+. *Hence both sets $I_2(P_1)$ and $I_3(P_1)$ are subsets of C_+.*

Lemma 2: *All points of $I_2(P_1)$ and $I_3(P_1)$ lie at the same distance from the line L_1.*

Proof: Denote by $\delta(P)$ the distance of P from L_1, so that

$$\delta(P_2) = \delta(P_3)$$

for corresponding points P_2 and $P_3 = P_2 - P_1$ of $I_2(P_1)$ and $I_3(P_1)$. Let the assertion be false. There exist then two points P_2' and P_2'' of $I_2(P_1)$ and the corresponding points $P_3' = P_2' - P_1$ and $P_3'' = P_2'' - P_1$ of $I_3(P_1)$ such that

$$\delta(P_2') = \delta(P_3') < \delta(P_2'') = \delta(P_3'').$$

Since C_+ is a convex arc, $\delta(P)$ increases on C_+ from the value 0 at $P = P_1$ to a certain maximum value, and then decreases again to the value 0 at $P = -P_1$. Hence P_2'', P_3'' lie on the arc of C_+ bounded by P_2', P_3', while P_1, $-P_1$ lie outside this arc. From the construction, the two lines L_2 through P_2', P_2'', and L_3 through P_3', P_3'', are parallel. Therefore, by the convexity of C_+, P_1 and $-P_1$ lie in the parallel strip bounded by L_2 and L_3. This is, however, impossible because the line segment P_1, $-P_1$ is parallel to, but twice as long as, the line segments from P_2' to P_3', or from P_2'' to P_3''.

Since $\delta(P)$ is constant in $I_2(P_1)$ and $I_3(P_1)$, only the following two cases arise.

a) $I_2(P_1)$ consists of a single point P_2, $I_3(P_1)$ of a single point $P_3 = P_2 - P_1$. The line segment P_2P_3 is parallel and of equal length to the segment OP_1, and so $OP_1P_2P_3$ is a parallelogram. The line segment may possibly form part of C_+, but then no larger segment containing it has this property.

b) $I_2(P_1)$ contains at least two different points P_2', P_2'', and $I_3(P_1)$ contains at least the corresponding points $P_3' = P_2' - P_1$, $P_3'' = P_2'' - P_1$ All four points P_2', P_2'', P_3', P_3'' lie on *one* line L parallel to L_1; let Σ^* be the smallest line segment on L containing them. By the convexity of C_+, Σ^* is a subset of this arc. There exists then a longest line segment Σ contained in C_+ and itself containing Σ^*. This segment Σ is of greater length than OP_1 since Σ^* has this property. It is now clear that $I_2(P_1)$ consists of all points P_2 of Σ for which $P_3 = P_2 - P_1$ also lies on Σ, and that $I_3(P_1) = I_2(P_1) - P_1$. If P_2, $P_3 = P_2 - P_1$ is any pair of corresponding points in $I_2(P_1)$ and $I_3(P_1)$, then $OP_1P_2P_3$ is a parallelogram. By what has already been proved, the area of this parallelogram depends on P_1, but not on P_2 and P_3; denote it by $A(P_1)$. Write further $\Lambda(P_1)$ for any lattice of basis P_1, P_2 where P_2 belongs to $I_2(P_1)$. Then

$$d(\Lambda(P_1)) = A(P_1).$$

Example: Let K_0 be again the unit square

$$|x_1| \leqslant 1, \quad |x_2| \leqslant 1.$$

It suffices, by symmetry, to consider points

$$P_1 = (1, \eta), \quad 0 \leqslant \eta \leqslant 1,$$

which lie on the side $x_1 = 1$ of K_0. Choose for P_+ the semiplane $y \geq \eta x$. Then, if $\eta \neq 0$, $I_2(P_1)$ consists of the single point $P_2 = (0, 1)$, and $I_3(P_1)$ of the single point $P_3 = (-1, 1-\eta) = P_2 - P_1$. If, however, $\eta = 0$, then $I_2(P_1)$ is the line segment of all points $P_2 = (\xi, 1)$ where $0 \leq \xi \leq 1$, and $I_3(P_1)$ is the adjoining line segment of all points $P_3 = (\xi', 1)$ where $-1 \leq \xi' \leq 0$. In both cases,

$$d(\Lambda(P_1)) = A(P_1) = 1,$$

independent of the choice of P_1.

§ 3. The critical lattices of K.

By MINKOWSKI [2]), the following result holds:

Lemma 3: *Let Λ be any critical lattice of the convex domain K. Then Λ contains three points P_1, P_2, P_3 on C such that (i) P_1, P_2 is a basis of Λ, and (ii) $OP_1P_2P_3$ is a parallelogram of area $d(\Lambda) = \triangle(K)$. Conversely, if P_1, P_2, P_3 are three points on C such that $OP_1P_2P_3$ is a parallelogram, then the area of this parallelogram is not less than $\triangle(K)$, and it is equal to $\triangle(K)$ if and only if the lattice of basis P_1, P_2 is critical.*

This lemma, together with the results of last paragraph, leads immediately to the following construction of the critical lattices of K.

Lemma 4: *Denote by Π the set of all points P_1 on C for which $A(P_1)$ assumes its smallest value, A say [3]). Let further $\{\Lambda\}$ be the set of all lattices $\Lambda(P_1)$ where P_1 runs over Π. Then $\{\Lambda\}$ is identical with the set of all critical lattices of K.*

Example: Let K_0 be again the unit square

$$|x_1| \leqslant 1, \quad |x_2| \leqslant 1.$$

Then $\Pi = C$, and *all* lattices $\Lambda(P_1)$, where P_1 is an arbitrary point on C, are critical. We shall see that a similar result holds for all irreducible convex domains.

For parallelograms with centre at O, Theorem 1 has already been proved

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[2]) Diophantische Approximationen, § 4. See also my paper Proc. London Math. Soc. (2) **49** 137, 158—159 (1946).

[3]) That such a minimum value is attained, follows immediately from Lemma 3 and the existence of critical lattices.

in the corollary to Lemma 1. In the next three lemmas, this case is excluded, and it is assumed that K *is a convex domain which is not a parallelogram.*

Lemma 5: *Let* P_1 *be any point on* C *such that there exists a critical lattice* Λ *of* K *containing* P_1. *Denote by* P_2 *and* P_3 *two further points of* Λ *on* C *such that* $OP_1P_2P_3$ *is a parallelogram of area* $\triangle(K)$. *Then all points of the line segment* P_2P_3 *different from* P_2 *and* P_3 *are inner points of* K, *and so* Λ *is the only critical lattice of* K *containing* P_1.

Proof: Denote by L^* any tac line of C at P_1, by L^{**} the line through P_2 and P_3, by $-L^*$ and $-L^{**}$ the lines symmetrical to L^* and L^{**} in O, and by K^* the parallelogram bounded by the four lines L^*, L^{**}, $-L^*$, $-L^{**}$. If at least one inner point of the segment P_2P_3 lies on L^{**}, then L^{**} is a tac line of C, and so K is contained in K^* as a subset. Now Λ is K^*-admissible, hence a critical lattice of K^*, whence

$$\triangle(K^*) = \triangle(K).$$

Therefore, by Lemma 1, K^* coincides with K, contrary to hypothesis. This proves the first part of the assertion. The second part also holds since $I_2(P_1)$ reduces to the single point P_2.

Corollary: *Every critical lattice* Λ *of* K *has just six points*

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3$$

on C. *These points divide* C *into six arcs* $A_1, \ldots, A_6$, *none of which is a line segment.*

Lemma 6: *Let* Λ *and* Λ^* *be two different critical lattices of* K; *let* $P_1, \ldots, P_6$ *be the six points of* Λ *on* C; *and let* $A_1, \ldots, A_6$ *be the six arcs into which these points divide* C. *Then* Λ^* *has just one point* P_l^* *on each arc* A_l.

Proof: Write $\{X, Y\} = x_1y_2 - x_2y_1$ for the determinant of any two points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, and choose the indices such that, if C is described in positive direction, the points of Λ on C are met in the order

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3,$$

or in a cyclical permutation therefrom. Further denote by $A_1, \ldots, A_6$ the six arcs

$$\widehat{P_1 P_2}\ ,\ \widehat{P_2 P_3}, \ldots, \widehat{P_5 P_6}\ ,\ \widehat{P_6 P_1}$$

of C bounded by these points. By Lemma 3, P_1 and P_2 form a basis of Λ, and so

(a): $$\{P_1, P_2\} = \triangle(K),$$

since the determinant on the left is positive.

One may assume that Λ^* contains a point P_1^* of A_1; this must be an *inner* point of A_1, since Λ^* is different from Λ (Lemma 5). None of the three arcs A_6, A_1, A_2 is a line segment; hence, by the convexity of C, P_1^*

is also an *inner* point of the *triangle* with vertices at P_1, $P_1 + P_2$, P_2. Therefore, if P_1^* is written as

$$P_1^* = sP_1 + tP_2,$$

then s and t satisfy the inequalities,

$$0 < s < 1,\ 0 < t < 1,\ s + t > 1,$$

whence, by (a),

$$\begin{aligned} 0 < \{P_1^*, P_2\} &= s\,\Delta(K) < \Delta(K), \\ \{P_1^*, P_3\} &= (s+t)\,\Delta(K) > (K), \\ 0 < \{P_1^*, P_4\} &= t\,\Delta(K) < (K). \end{aligned}$$

Hence the line

(b): $$\{P_1^*, X\} = \Delta(K)$$

intersects C in at least one point P_2^* of A_2 and at least one points P_3^* of A_3, and both points are inner points of these arcs. There cannot be more than *one* such point of intersection on each arc A_2 and A_3, since, by Lemma 5, Λ^* has just two points on C satisfying (b).

Lemma 7: *Let Λ, Λ^*, Λ^{**} be three different critical lattices of K, and let P_l, P_l^*, P_l^{**}* ($l = 1, 2, \ldots, 6$) *be their points on C, the indices being chosen such that* (i) *the points P_l follow one another in their natural order if C is described in positive direction, and* (ii) *the two points P_l^* and P_l^{**} lie on the arc A_l bounded by P_l and P_{l+1}* (P_7 is to mean the same as P_1). *If P_1^* separates P_1 from P_1^{**} on A_1, then, for $l = 2, \ldots, 6$, P_l^* likewise separates P_l from P_l^{**} on A_l.*

Proof: Assume the assertion is false. Let then λ with $2 \leq \lambda \leq 6$ be the smallest index for which P_λ^* does not separate P_λ from P_λ^{**} on A_λ. Then $P_{\lambda-1}^{**}$ and P_λ^{**} are two consecutive points of Λ^{**} on C such that the arc joining them contains no point of Λ^*, in contradiction to the last lemma.

§ 4. The critical lattices of an irreducible convex domain.

The last lemmas lead to a particularly simple result if K is irreducible. I have proved elsewhere [4]) the following result:

Lemma 8: *If K is an irreducible star domain, and if P is any point on C, then there exists at least one critical lattice of K containing P.*

On combining this result with the Lemmas 5—7, one finds:

Lemma 9: *Let K be an irreducible convex domain which is not a parallelogram. Then to every point P_1 on C, there exists a unique critical lattice $\Lambda = \Lambda(P_1)$ containing P_1. This lattice has just six points $P_l = P_l(P_1)$* ($l = 1, \ldots, 6$) *on C. Let $A_1, \ldots, A_6$ be the six arcs into which these points divide C; denote further by P_1^* a variable point on A_1, and by $P_l^* = P_l(P_1^*)$ for $l = 2, \ldots, 6$ the other five points of $\Lambda(P_1^*)$ on C. If P_1^**

[4]) Proc. Royal Acad. Amsterdam, **49**, 331—343 (1946), Theorem C.

describes A_1 continuously in positive direction, then P_l^*, for $l = 2, \ldots, 6$, describes A_l in the same manner.

Proof: Choose the indices in the same way as in the last proofs. The line,

$$L(P_1^*): \qquad \{P_1^*, X\} = \Delta(K),$$

has two, and by Lemma 5 only two, points of intersection with C, namely P_2^* and P_3^*. When P_1^* describes A_1 continuously, then this line changes in a continuous manner, and so the same is true for P_2^* and P_3^*. Further, if P_1^* runs over A_1 in positive direction, then, by Lemma 7, P_2^* and P_3^* do the same on A_2 and A_3.

By means of this lemma, one can construct all irreducible convex domains. In a further note, I shall apply this construction.

§ 5. Lemmas on reducible domains.

In the paper already mentioned, I proved the following results:

Lemma 10: *Let K be a reducible star domain and P a point on C such that no critical lattice of K contains P. Then there exists a star domain H contained in K, but not containing the two points $\pm P$ symmetrical in O, for which $\Delta(H) = \Delta(K)$* [5].

Lemma 11: *Let K be a star domain, and let Π be the set of all points on C which belong to at least one critical lattice of K. Then Π is a closed set. The set Π^* of all points of C which do not belong to Π is therefore open and consists of an enumerable set of open arcs on C* [6].

Lemma 12: *If K is a convex domain, and if every point of C belongs to Π, then K is irreducible* [7].

§ 6. The main lemma.

The following lemma forms the basis for the proof of Theorem 1:

Lemma 13: *Let K be a reducible convex domain. Then there exists a* **convex** *domain H contained in, but different from, K such that $\Delta(H) = \Delta(K)$.*

Proof: Divide the points P on C into two classes A and B, according as to whether P is, or is not, an *inner* point of a *line segment contained in C.*

By the hypothesis and by the Lemmas 11 and 12, Π^* is not the null set and so contains at least one arc of C. Assume, firstly, that at least one point P of Π^* is of class B. By Lemma 10, there exists a star domain H' contained in K, but not containing the two points $\pm P$, such that $\Delta(H') = \Delta(K)$. Draw a line L' through P which has no point in common with H' and is not a tac line of C at P. Then a line L parallel to L', and separating L' from O, can be chosen which also has no points in common

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5) l.c. 4), proof of Theorem C.

6) l.c. 4), special case of Theorem B.

7) l.c. 4), Theorems D and E.

with H'; denote by $-L$ the line symmetrical to L in O, and by H the part of K between L and $-L$. Then H is a convex domain containing H', and is itself a proper subset of K. Evidently $\triangle(H') \leq \triangle(H) \leq \triangle(K)$, whence $\triangle(H) = \triangle(K)$.

Secondly, let all points of Π^* be of class A, and let P be one of these points. The tac line to C at P is unique, and its intersection with C is a line segment, Γ say. Then P is an inner point of Γ, and the two endpoints of Γ are of class B, hence belong to Π. There exists therefore a largest sub-segment Γ_1 of Γ such that, (*i*) P is an inner point of Γ_1; (*ii*) all inner points of Γ_1 belong to Π^*; (*iii*) the two endpoints P_1, P'_1 of Γ_1 belong to Π.

Denote by Λ, Λ' the two critical lattices of K containing P_1 and P'_1, respectively, and by P_l, P'_l $(l = 2, \ldots, 6)$ the other points of these two lattices on C. Let the notation be again such that if C is described in positive direction, then the points P_l, and similarly the points P'_l, follow one another in the order of their indices; denote further by $A_1, \ldots, A_6$ the arcs

$$\widehat{P_1 P_2}\ ,\ \widehat{P_2 P_3}, \ldots, \widehat{P_5 P_6}\ ,\ \widehat{P_6 P_1}$$

of C. By the hypothesis about Γ_1, P'_1 belongs either to A_1 or to A_6; assume, without loss of generality, that P'_1 lies on A_1 and so is an inner point of this arc. By Lemma 6, P'_l is then, for $l = 2, \ldots, 6$, an inner point of A_l, and so the arc

$$\Gamma_l = \widehat{P_l P'_l} \qquad (l = 2, \ldots, 6)$$

is a subarc of A_l.

By construction, the endpoints of all arcs Γ_l, where $l = 1, 2, \ldots, 6$, belong to Π. On the other hand, it is immediately clear from Lemma 7 that the inner points of these arcs belong to Π^*; for this is the case for Γ_1. But then the inner points of all arcs Γ_l are of class A, and so all six arcs Γ_l are line segments.

One shows now, just as in the proof of Lemma 6, that P'_1, P'_2 and P'_3 are inner points of the triangles with vertices at $P_1, P_1 + P_2, P_2$, at P_2, $P_2 + P_3, P_3$, and at P_3, $-P_1 + P_3$, $-P_1$, respectively. The three points

$$Q_1 = P'_1 - P_1,\ Q_2 = P'_2 - P_2,\ Q_3 = P'_3 - P_3$$

are therefore inner points of the triangles with vertices at O, P_2, P_3, at O, P_3, $-P_1$, and at O, $-P_1$, $-P_2$, respectively. By the assumed choice of indices, the radius vector from O to Q_1 changes therefore into that from O to Q_3 by a rotation in the *positive* sense of *less* than 180°, and so finally,

$$\{Q_1, Q_3\} > 0.$$

Next, the general point P_l^* of the arc Γ_l, where $l = 1, 2, 3$, is of the form

$$P_l^* = t_l P'_l + (1 - t_l) P_l = P_l + t_l Q_l, \qquad \text{where } 0 \leqslant t_l \leqslant 1.$$

Choose, in particular, $t_1 = t_2 = t_3, = t$ say. Then

$$P_1^* + P_3^* = P_2^*,$$

and so $OP_1^* P_2^* P_3^*$ is a parallelogram with three vertices on C. By the hypothesis and by Lemma 3, this parallelogram is of area

$$\{P_1^*, P_3^*\} > \Delta(K) \qquad \text{for } 0 < t < 1.$$

Now,

$$\{P_1^*, P_3^*\} = \{P_1 + tQ_1, P_3 + tQ_3\}, \qquad = f(t) \text{ say},$$

can be written as

$$f(t) = a + bt + ct^2, \quad \text{where} \quad c = \{Q_1, Q_3\}.$$

Hence, by the construction of the points P_l and P'_l, and by (a),

$$f(0) = f(1) = \Delta(K), \; f''(t) = 2c > 0,$$

and so there exists a number τ with $0 < \tau < 1$ such that $f(t)$ assumes a minimum value at $t = \tau$ satisfying

$$f(\tau) < \Delta(K).$$

The lattice of basis $P_1 + \tau Q_1$, $P_3 + \tau Q_3$, which is K-admissible, is therefore of determinant less than $\Delta(K)$, which is impossible.

This proves that all points of Π^* cannot be of class A, and so completes the proof.

§ 7. **Proof of Theorem 1.**

Let K be a reducible convex domain, hence, by Lemma 1, not a parallelogram. By the last lemma, the set P of all convex domains H contained in, but different from, K and satisfying $\Delta(H) = \Delta(K)$, is not the null set. The elements H of P are all of area greater than $\Delta(K)$. For if $\tilde{\omega}$ is any parallelogram $OP_1P_2P_3$ with three vertices of the boundary of H, then $\tilde{\omega}$ is a subset of H, and so by Lemma 3,

$$V(H) \geqq V(\tilde{\omega}) \geqq \Delta(K),$$

where $V(J)$ denotes the area of J.

Let now

$$v = \underset{H \text{ in } P}{\text{fin inf}} \; V(H)$$

be the lower bound of $V(H)$ extended over all elements of P; evidently

$$v \geqslant \Delta(K) > 0.$$

An infinite sequence of elements $H_1, H_2, H_3, \ldots,$ of P not necessarily all different, can be chosen such that

$$\lim_{n \to \infty} V(H_n) = v.$$

All elements of this sequence are convex domains contained in K; hence, by a selection theorem of W. BLASCHKE [8]), a suitable subsequence,

$$K_1 = H_{n_1},\ K_2 = H_{n_2},\ K_3 = H_{n_3}, \ldots \qquad (n_1 < n_2 < n_3 < \ldots)$$

tends of a convex domain, K' say. As we show now, this convex domain has the required properties.

For firstly [9])

$$V(K') = \lim_{n \to \infty} V(K_n) = v.$$

Secondly, also [10])

$$\Delta(K') = \lim_{n \to \infty} \Delta(K_n) = \Delta(K).$$

Thirdly, K' is a subset of K. I assert that K' is irreducible. If this were false, then, by Lemma 13, there would exist a convex domain K'' contained in, but different from, K' such that $\Delta(K'') = \Delta(K') = \Delta(K)$. This is, however, impossible, since by the construction,

$$V(K'') < V(K') = v,$$

contrary to the definition of v. This completes the proof.

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[8]) KREIS und KUGEL (Leipzig 1916), 62.

[9]) l.c. [8]), 61.

[10]) Theorem 9 of my paper "Lattice points in n-dimensional star bodies I". Proc. Royal Society, A, **187** (1946), 151—187.