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ON THE ADMISSIBLE LATTICES OF AUTOMORPHIC STAR BODIES

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Let K: $F(X) \leq 1$ be an n-dimensional star body of the finite type [1] let d_F be the set of all lattices Λ satisfying

$$F(\Lambda) = 1,$$

and let S_F be the set of the determinants $d(\Lambda)$ of the elements Λ of d_F.

Very little is known about this set S_F. Since all critical lattices belong to d_F, it is clear that $\Delta(K)$ is the smallest element of S_F; moreover, if K is bounded, then S_F consists just of all numbers not less than $\Delta(K)$. If, however, K is not bounded, then S_F may be a very complicated set, as is seen in the classical case of the region [2]

$$| x_1 x_2 | \leq 1.$$

It has therefore some interest to find properties of this set. One such property is proved here in the special case that K is an automorphic star body; this case includes some of the most interesting examples in the geometry of numbers.

THEOREM: *If* K *is an automorphic star body, then* S_F *is a closed set.*

Proof: Let $\Lambda_1, \Lambda_2, \Lambda_3, \cdots$ be an infinite sequence of elements of d_F such that the finite limit

$$\delta = \lim_{r \to \infty} d(\Lambda_r)$$

exists; we must show that this limit belongs to S_F.

We can find, for every index r, a point Q_r of Λ_r such that

$$1 \leq F(Q_r) < 1 + \frac{1}{r},$$

and an automorphism Ω_r of K such that

$$P_r = \Omega_r Q_r$$

is different from 0 and lies inside the sphere of centre 0 and radius c; here c is the constant of Definition 9.[1] The transformed lattices

$$\Lambda_r^* = \Omega_r \Lambda_r \qquad (r = 1, 2, 3, \cdots)$$

satisfy the relations

$$F(\Lambda_r^*) = F(\Lambda_r), \quad d(\Lambda_r^*) = d(\Lambda_r),$$

whence

$$F(\Lambda_r^*) = 1, \qquad \lim_{r \to \infty} d(\Lambda_r^*) = \delta;$$

they form therefore a bounded sequence. Hence we can find an infinite subsequence

$$\Lambda^{(1)} = \Lambda_{r_1}, \qquad \Lambda^{(2)} = \Lambda_{r_2}, \qquad \Lambda^{(3)} = \Lambda_{r_3}, \cdots$$

which converges to a lattice, Λ say. Evidently the following statements hold:

(a) The lattices $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \cdots$ converge to Λ and satisfy the relations,

$$F(\Lambda^{(r)}) = 1, \qquad \lim_{r\to\infty} d(\Lambda^{(r)}) = \delta.$$

(b) The points $P^{(1)} = P_{r_1}, \ P^{(2)} = P_{r_2}, \ P^{(3)} = P_{r_3}, \cdots$ have the properties,

$$P^{(r)} \neq O, \qquad |P^{(r)}| \leq c, \qquad P^{(r)} \text{ lies in } \Lambda^{(r)},$$

$$\lim_{r\to\infty} F(P^{(r)}) = \lim_{r\to\infty} F(Q_{n_r}) = 1.$$

Hence, by Theorem 19 [1],

$$F(\Lambda) = 1,$$

and by the continuity of $d(\Lambda)$,

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda^{(r)}) = \delta,$$

as was to be proved.

It would be of interest to decide whether the theorem remains true for non-automorphic star bodies.

References

1. K. Mahler, "On lattice points in n-dimensional star bodies", *Proc. Royal Soc.*, A **187** (1946).
2. F. G. Koksma, *Erg. der Math.* IV 429-33.