On the successive minima of a bounded star domain.

Memoria di Kurt Mahler (a Manchester).

Dedicated to Max Dehn.

Sunto. - *È dato nel seguente capoverso.*

Let $F(X)$ be a bounded distance function and Λ an arbitrary lattice in the plane. Let further P, Q run over all pairs of independent points of Λ for which

$$F(P) \leq F(Q).$$

We call

$$\mu_1(\Lambda) = \min F(P), \quad \mu_2(\Lambda) = \min F(Q)$$

the two successive minima of Λ and denote by M the upper bound of $\mu_1(\Lambda)\mu_2(\Lambda)$ extended over all lattices of a fixed given determinant. I prove in this paper that *there exists at least one lattice for which this upper bound is attained.*

§ 1. Points and lattices.

Let (x_1, x_2) be rectangular coordinates in the Euclidean plane. We identify the point $X = (x_1, x_2)$ of these coordinates with the vector X of components x_1, x_2 and use the usual vector notation. Thus if

$$X = (x_1, x_2) \quad \text{and} \quad Y = (y_1, y_2)$$

are any two points, then

$$|X| = \sqrt{x_1^2 + x_2^2}$$

denotes the distance of the point X from the origin

$$O = (0, 0)$$

or the length of the vector X. Further

$$\{X, Y\} = x_1 y_2 - x_2 y_1$$

is the determinant of X and Y, and, for real u, v, $uX + vY$ is the point

$$uX + vY = (ux_1 + vy_1, ux_2 + vy_2).$$

Assume, in particular, that X and Y are independent, i. e. that

$$\{X, Y\} \neq 0.$$

Then the set Λ of all points

$$P = uX + vY, \quad \text{where} \quad u, v = 0, \mp 1, \mp 2, \dots,$$

is a *lattice*, and the positive number

$$d(\Lambda) = |\{X, Y\}|$$

is the *determinant* of this lattice; the points X, Y form a *basis* of, or generate, the lattice.

If $t \neq 0$ is real, then $t\Lambda$ denotes the lattice of all points tP where P runs over Λ. Evidently $t\Lambda$ and $-t\Lambda$ are the same lattice, and

$$d(t\Lambda) = t^2 d(\Lambda).$$

§ 2. Star domains.

Let

$$F(X) = F(x_1, x_2)$$

be a (bounded, symmetrical) *distance function*, i. e. a function of X with the following properties:

(A) $\quad F(O) = 0; \quad F(X) > 0$ if $X \neq O$.

(B) $\quad F(tX) = |t| F(X)$ for all real t and for all points X.

(C) $\quad F(X)$ is a continuous function of X (i. e. of x_1, x_2).

The inequality

$$K: \quad F(X) \leq 1$$

then defines a (bounded, symmetrical) *star domain* K, i. e. a point set K in the plane with the following properties:

(A) K is bounded and closed, and contains O as an inner point.

(B) Every line through O meets K in a finite line segment of which O is the centre.

(C) The boundary C: $F(X) = 1$ of K is a Jordan curve.

The more general inequality,

$$cK: \quad F(X) \leq c,$$

where $c > 0$, defines a star domain cK similar to K; it consists of the points cX where X runs over K.

Since K is a bounded set, there exist *K-admissible* lattices Λ, i. e. lattices which contain no inner points of K except O. Denote by

$$\Delta(K) = l.\ b.\ d(\Lambda)$$

the lower bound of the determinants $d(\Lambda)$ of all K–admissible lattices Λ. Since O is an inner point of K, it is essily proved that

$$\Delta(K) > 0.$$

There exists at least one *critical lattice* of K, i. e. a K–admissible lattice Λ of determinant $d(\Lambda) = \Delta(K)$ (1). *Such a critical lattice always has two independent points on the boundary* C *of* K (2).

Evidently

$$\Delta(cK) = c^2\Delta(K).$$

§ 3, The successive minima of a lattice.

Let K: $F(X) \leq 1$ be a fixed star domain and Λ a variable lattiee; denote by $S(\Lambda)$ the set of all pairs of independent points P, Q of Λ for which

$$\{ P, Q \} > 0, \quad 0 < F(P) \leq F(Q).$$

For every $c > 0$, cK contains at most a finite number of points of Λ. Hence the two minima

$$\mu_1(\Lambda) = \min F(P) \quad \text{and} \quad \mu_2(\Lambda) = \min F(Q),$$

extended over all pairs P, Q of $S(\Lambda)$, are both attained and are positive; they are called the successive minima of Λ. Evidently

$$0 < \mu_1(\Lambda) \leq \mu_2(\Lambda),$$

and by homogeneity,

$$\mu_1(t\Lambda) = | t | \mu_1(\Lambda), \quad \mu_2(t\Lambda) = | t | \mu_2(\Lambda).$$

We therefore norm Λ *and consider these minima from now on only for lattices satisfying*

$$d(\Lambda) = \Delta(K).$$

Put

$$\mu(\Lambda) = \mu_1(\Lambda)\mu_2(\Lambda);$$

this is a positive function of Λ. Further write

$$M(K) = u.\ b.\ \mu(\Lambda)$$

where the upper bound extends over all lattices of determinant $\Delta(K)$.

LEMMA 1. – *For all star domains.*

$$M(K) \geq 1.$$

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(1) See my paper, « Proc. Royal Soc. A », 187, (1946), 151-187, Theorem 8. For shortness, this paper will be quoted as LP.

(2) LP, Theorem 11.

Proof. – Denote by Λ a critical lattice of K. Such a lattice has two independent points P, Q on the boundary C of K. On possibly interchanging these points, we may assume that

$$\{P, Q\} > 0, \quad F(P) = F(Q) = 1;$$

as Λ is admissible, no point of Λ is an *inner* point of K, and so

$$\mu_1(\Lambda) = \mu_2(\Lambda) = 1, \quad \text{hence} \quad \mu(\Lambda) = 1,$$

whence the assertion.

DEFINITION 1. – *The lattice* Λ *of determinant* $\Delta(K)$ *is called an extreme lattice of* K *if*

$$\mu(\Lambda) = M(K).$$

Our problem is to decide whether every star domain possesses at least one extreme lattice. I mention, without proof, that in the case of a convex domain this problem is easily solved; the result is as follows;

« *For every convex domain,*

$$M(K) = 1.$$

If K *is not a parallelogram, then the extreme lattices of* K *are identical with its critical lattices. If, however,* K *is a parallelogram, then there exists an extreme lattice* Λ *with arbitrarily small* $\mu_1(\Lambda)$ ».

§ 4. The function $N(K)$.

The proof of the existence of extreme lattices uses a function $N(K)$ defined as follows.

Let P be a variable point on the boundary C of K. Since K is a bounded closed set, there exists a second point $Q = Q(P)$ on C such that

$$\{P, Q\} > 0$$

and that

$$|\{P, X\}| \leq \{P, Q\} \text{ for all points } X \text{ of } K.$$

Put

$$\varphi(P) = \{P, Q\}.$$

It is easily shown that $\varphi(P)$ is a *continuous* function of P. Hence $\varphi(P)$ assumes its *minimum* on C in at least one point P_0 on C. For this minimum value, we write

$$N(K) = \min_{P \text{ on } C} \varphi(P) = \varphi(P_0).$$

From the definition of $\varphi(P)$, there is then a second point Q_0 on C such that

$$N(K) = \{P_0, Q_0\}$$

and that

$$|\{P_0, X\}| \leq \{P_0, Q_0\} = N(K) \text{ for all points } X \text{ of } K.$$

LEMMA 2. – *For every star body* K,

$$N(K) \geq \Delta(K).$$

Proof. – Denote by Λ_0 the lattice of basis P_0, Q_0 where P_0, Q_0 are the points just defined. *This lattice* Λ_0 *is* K*-admissible.* For of the lattice points $P \neq O$ collinear with O and P_0, only $\mp P_0$ belong to K, while for all other lattice points P,

$$|\{P_0, P\}| \geq \{P_0, Q_0\} = d(\Lambda_0) = N(K)$$

so that these points cannot be *inner* points of K. Therefore from the definition of $\Delta(K)$,

$$\Delta(K) \leq d(\Lambda_0) = N(K),$$

as asserted.

§ 5. Reduction of the proof.

Our aim is to show the existence of an extreme lattice of K. If

$$M(K) = 1,$$

then this assertion is clearly true since every critical lattice of K is also extreme, and since there do exist critical lattices. We may therefore from now on assume that K satisfies the inequality

(1) $$M(K) > 1.$$

Next, from the definition of $M(K)$, there exists an infinite sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

such that

$$d(\Lambda_r) = \Delta(K) \quad \text{for} \quad r = 1, 2, 3, \ldots$$

and that

$$\lim_{r \to \infty} \mu(\Lambda_r) = M(K).$$

By the hypothesis (1), it is allowed to assume that

$$\mu(\Lambda_r) \geq 1 \quad \text{for} \quad r = 1, 2, 3, \ldots .$$

Suppose, first, that there exists a positive constant c_1 such that

$$\mu_1(\Lambda_r) \geq c_1 \quad \text{for} \quad r = 1, 2, 3, \ldots .$$

Then a second positive constant c_2 exists such that no point $P \neq O$ of any lattice Λ_r satisfies

$$|P| < c_2;$$

hence the sequence of lattices Λ_r is bounded ([3]). We can then select an infinite subsequence

$$\Lambda_{r_1}, \quad \Lambda_{r_2}, \quad \Lambda_{r_3},$$

([3]) LP, Definition 1.

converging to a limiting lattice, Λ say [4]. Evidently

$$d(\Lambda) = \lim_{k \to \infty} d(\Lambda_{r_k}) = \Delta(K),$$

and by the continuity and boundedness of $F(X)$ also

$$\mu(\Lambda) = \lim_{k \to \infty} \mu(\Lambda_{r_k}) = M(K).$$

The lattice Λ is therefore extreme and the assertion is proved.

The same construction holds if the inequality

$$\mu_1(\Lambda_r) \geq c_1$$

is satisfied for any infinite sequence of indices

$$r = r', \quad r'', \quad r''', \dots \qquad (r' < r'' < r''' < \dots).$$

Hence we may from now on assume, without loss of generality, that

(2) $$\lim_{r \to \infty} \mu_1(\Lambda_r) = 0.$$

§ 6. Conclusion of the proof.

For every index r, select a pair of points P_r, Q_r of Λ_r satisfying both

$$\{P_r, Q_r\} > 0, \quad \text{hence} \quad \geq d(\Lambda_r) = \Delta(K),$$

and

$$F(P_r) = \mu_1(\Lambda_r), \quad F(Q_r) = \mu_2(\Lambda_r).$$

Put

$$P'_r = \mu_1(\Lambda_r)^{-1} P_r, \quad Q'_r = \mu_2(\Lambda_r)^{-1} Q_r,$$

so that

$$F(P'_r) = F(Q'_r) = 1.$$

Since thus all points P'_r, Q'_r are bounded, there exists an infinite sequence of indices

$$r = r_1, \quad r_2, \quad r_3, \dots \qquad (r_1 < r_2 < r_3 < \dots)$$

and a pair of points P', Q' such that

$$\lim_{k \to \infty} P'_{r_k} = P', \quad \lim_{k \to \infty} Q'_{r_k} = Q'.$$

By the continuity of $F(X)$,

$$F(P') = F(Q') = 1.$$

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[4] LP, Theorem 2.

Further P′ *and* Q′ *are different.* For

$$\{P'_{r_k},\ Q'_{r_k}\} = \mu(\Lambda_{r_k})^{-1}\{P_{r_k},\ Q_{r_k}\} \geq \frac{\Delta(K)}{\mu(\Lambda_{r_k})},$$

and by the definition of $M(K)$,

$$\mu(\Lambda_{r_k}) \leq M(K),$$

hence

$$\{P'_{r_k},\ Q'_{r_k}\} \geq \frac{\Delta(K)}{M(K)},$$

whence

$$\{P',\ Q'\} = \lim_{k\to\infty}\{P'_{r_k},\ Q'_{r_k}\} \geq \frac{\Delta(K)}{M(K)} > 0.$$

For all real t,

$$F(tP' + Q') \geq 1.$$

For assume this assertion is false, i. e. let there be a real number τ such that

$$F(\tau P' + Q') < 1.$$

Then

$$\theta = F(\tau P' + Q')$$

satisfies

$$0 < \theta < 1$$

since $\tau P' + Q' \neq O$.

But by hypothesis

$$F(hP_r + Q_r) \geq \mu_2(\Lambda_r),$$

hence also

$$F\left(h\,\frac{\mu_1(\Lambda_r)}{\mu_2(\Lambda_r)}\,P'_r + Q'_r\right) \geq 1,$$

for

$$r = 1,\ 2,\ 3, \ldots, \quad h = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots .$$

Further

$$\lim_{r\to\infty}\mu_1(\Lambda_r) = 0 \quad \text{and} \quad \mu_1(\Lambda_r)\mu_2(\Lambda_r) = \mu(\Lambda_r) \geq 1,$$

and therefore

$$\lim_{r\to\infty}\frac{\mu_1(\Lambda_r)}{\mu_2(\Lambda_r)} = 0.$$

It is then possible to find a sequence of integers

$$h_{r_1},\ h_{r_2},\ h_{r_3}, \ldots$$

such that

$$\lim_{k\to\infty} h_{r_k}\frac{\mu_1(\Lambda_{r_k})}{\mu_2(\Lambda_{r_k})} = \tau,$$

hence by the continuity of $F(X)$,

$$\theta = F(\tau P' + Q') = \lim_{k \to \infty} F\left(h_{r_k} \frac{\mu_1(\Lambda_{r_k})}{\mu_2(\Lambda_{r_k})} P'_{r_k} + Q'_{r_h}\right) \geq 1,$$

contrary to hypothesis.

The inequality

$$F(tP' + Q') \geq 1 \text{ for all real } t$$

implies that

$$|\{P', X\}| \leq \{P', Q'\} \textit{ for all points } X \textit{ of } K.$$

For X can be written as

$$X = tP' + uQ' \quad \text{where} \quad u = \frac{\{P', X\}}{\{P', Q'\}};$$

if now $|u| > 1$, then

$$F(X) = |u| F\left(\frac{t}{u} P' + Q'\right) \geq |u| > 1,$$

and so X does not belong to K.

Therefore in the notation of § 4,

$$\varphi(P') = \{P', Q\},$$

whence by the definition of $N(K)$ and by Lemma 2,

$$\{P', Q'\} \geq N(K) \geq \Delta(K).$$

Further

$$\{P', Q'\} = \lim_{k \to \infty} \{P'_{r_k}, Q'_{r_k}\} = \lim_{k \to \infty} \frac{\{P_{r_k}, Q_{r_k}\}}{\mu(\Lambda_{r_k})}$$

and

$$\lim_{k \to \infty} \mu(\Lambda_{r_k}) = M(K).$$

Hence

$$\lim_{k \to \infty} \{P'_{r_k}, Q'_{r_k}\}, \quad = L \text{ say},$$

also exists. But

$$\{P_{r_k}, Q_{r_k}\} = g_{r_k} d(\Lambda_{r_k}) = g_{r_k} \Delta(K)$$

where g_{r_k} is some positive integer, and so g_{r_k} has a fixed positive integral value,

$$g_{r_k} = \frac{L}{\Delta(K)}, \quad = g \qquad \text{say,}$$

as soon as k is sufficiently large. Therefore

$$\{P', Q'\} = g \frac{\Delta(K)}{M(K)} \geq \Delta(K),$$

whence

(3) $$M(K) \leq g.$$

By (1), this implies that

$$g \geq 2.$$

The points P_{r_k} and Q_{r_k} do not form a basis of Λ_{r_k}, but there exists a point R_{r_k} of Λ_{r_k} such that P_{r_k} and R_{r_k} form a basis of this lattice; moreover, it may be assumed that

$$\{P_{r_k}, R_{r_k}\} > 0.$$

The lattice point Q_{r_k} can be written as

$$Q_{r_h} = f_{r_k} P_{r_k} + g_{r_k} R_{r_k}$$

where f_{r_k} is a certain integer and g_{r_k} has the same meaning as before. Conversely,

$$R_{r_k} = \frac{1}{g_{r_k}}(Q_{r_k} - f_{r_k} P_{r_k}),$$

hence

$$R_{r_k} = \frac{1}{g} Q_{r_k} - \frac{f_{r_k}}{g} P_{r_k}$$

when k is sufficiently large.

All points

$$R_{r_k} + hP_{r_k} = \frac{1}{g} Q_{r_k} + \left(h - \frac{f_{r_k}}{g}\right) P_{r_k} \quad (h = 0, \mp 1, \mp 2, \dots)$$

belong to Λ_{r_k} and are independent of P_{r_h}; hence by the definition of the second minimum,

$$F\left(\frac{1}{g} Q_{r_k} + \left(h - \frac{f_{r_k}}{g}\right) P_{r_k}\right) \geq \mu_2(\Lambda_{r_k}) \quad (h = 0, \mp 1, \mp 2, \dots),$$

whence

$$F\left(\frac{1}{g} Q'_{r_k} + \left(h - \frac{f_{r_k}}{g}\right) \frac{\mu_1(\Lambda_{r_k})}{\mu_2(\Lambda_{r_k})} P'_{r_k}\right) \geq 1 \quad (h = 0, \pm 1, \pm 2, \dots).$$

In this inequality, choose the integer $h = h_{r_k}$ as function of k such that

$$\left| h_{r_k} - \frac{f_{r_k}}{g} \right| \leq \frac{1}{2},$$

hence that

$$\lim_{k \to \infty} \left(h_{r_k} - \frac{f_{r_k}}{g}\right) \frac{\mu_1(\Lambda_{r_k})}{\mu_2(\Lambda_{r_k})} = 0.$$

Therefore, by the continuity of $F(X)$ and by the limit definition of P' and Q',

$$F\left(\frac{1}{g}\,Q'\right) \geq 1,$$

contrary to

$$F\left(\frac{1}{g}\,Q'\right) = \frac{1}{g}\,F(Q') = 1/g \leq 1/2.$$

The assumption at the end of § 5 is therefore excluded and the following theorem has been proved:

THEOREM 1: *Every* (bounded, symmetrical) *star domain possesses at least one extreme lattice.*

§ 7. A star domain K with $M(K) > 1$.

The result just proved would lose its interest if the equation $M(K) = 1$ were satisfied for all star domains K. For then every critical lattice would be extreme, and so there would have been no need to give a long proof for the existence of extreme lattices. The following theorem excludes this possibility.

THEOREM 2. – *There exists a* (bounded, symmetrical) *star domain* K *satisfying* M(K) > 1.

Proof: Denote by K the non-convex polygon of successive vertices

$$(1,\ 1),\ \left(\frac{1}{3},\ \frac{2}{3}\right),\ (0,\ 1),\ \left(-\frac{1}{3},\ \frac{2}{3}\right),\ (-1,\ 1),$$

$$(-1,\ -1),\ \left(-\frac{1}{3},\ -\frac{2}{3}\right),\ (0,\ -1),\ \left(\frac{1}{3},\ -\frac{2}{3}\right),\ (1,\ -1).$$

This polygon is a star domain and contains the rectangle

$$R: \quad |\,x_1\,| \leq 1, \quad |\,x_2\,| \leq 2/3$$

as a subset. Of the boundary points of R on the line $x_2 = 2/3$, all except the two points $(-1/3,\ 2/3)$ and $(1/3,\ 2/3)$ are *inner* points of K, and these two points have a distance less than unity; hence *no critical lattice of* R *is* K-*admissible*, and so

$$\Delta(K) > \Delta(R) = 1 \times 2/3 = 2/3.$$

Put

$$\delta = +\sqrt{\Delta(K)},$$

and denote by Λ the lattice of basis

$$P = (\delta,\ 0), \quad Q = \left(\frac{1}{2}\,\delta,\ \delta\right)$$

and determinant

$$d(\Lambda) = \delta^2 = \Delta(K).$$

Evidently

$$\mu_1(\Lambda) = \delta, \quad \mu_2(\Lambda) = \frac{3}{2}\delta,$$

the first minimum being attained at P and the second one at Q. Therefore

$$\mu(\Lambda) = \delta \times \frac{3}{2}\delta = \frac{3}{2}\Delta(K) > 1,$$

whence

$$M(K) \geq \mu(\Lambda) > 1,$$

as asserted.

In a further paper, I hope to extend the results of this paper to more dimensions.