ON THE CRITICAL LATTICES OF ARBITRARY POINT SETS

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(Dedicated to J. G. van der Corput)

In this note, I shall establish necessary and sufficient conditions for the existence of critical lattices of an *arbitrary point set*, and I shall construct a non-trivial example of a point set without any critical lattice. In a previous paper,[1] I proved that *every star body of the finite type possesses at least one critical lattice.*

I.

1. Let S be any point set in n-dimensional Euclidean space R_n. A lattice Λ is called *S-admissible* if no point of Λ, except possibly the origin

$$O = (0, 0, \ldots, 0),$$

is an *inner* point of S. Such admissible lattices need not exist, e.g. if S is the whole space R_n; we say in this case that S is of the *infinite type*, and put

$$\Delta(S) = \infty.$$

If there are admissible lattices, S is called of the *finite type*. We then form the lower bound

$$\Delta(S) = \text{l.b. } d(\Lambda)$$

of the determinants $d(\Lambda)$ of all S-admissible lattices, and call this the *minimum determinant of S*. In the special case that

$$\Delta(S) = 0,$$

there exist S-admissible lattices of arbitrarily small determinant, and S is called of the *zero type*; e.g. the null set has this property.

2. A lattice Λ is called a *critical lattice of S* if

(*a*) Λ is S-admissible, and

(*b*) $d(\Lambda) = \Delta(S)$.

It is clear from the definitions just given that S cannot have a critical lattice if it is of the infinite or the zero types. For there are no S-admissible lattices in the first case; in the second case, the lower bound is not attained since every lattice is of positive determinant.

In the remaining case, when

$$0 < \Delta(S) < \infty, \tag{1}$$

the following criterion holds.

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[1] "On lattice points in n-dimensional star bodies, I," Proc. Royal Soc., A, 187 (1946), 151-187. The letters LP will be used to mark references to this paper.

THEOREM 1. *Let S be a point set in R_n satisfying* (1). *Then S possesses at least one critical lattice, if and only if there exists a bounded*[2] *infinite sequence of S-admissible lattices*

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

such that

$$\lim_{r \to \infty} d(\Lambda_r) = \Delta(S). \tag{2}$$

Proof. (i) If there exists a critical lattice Λ of S, then the infinite sequence of lattices

$$\Lambda, \Lambda, \Lambda, \ldots$$

has the required properties.

(ii) Assume that

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is a bounded infinite sequence of S-admissible lattices satisfying (2). We may then select[3] an infinite subsequence

$$\Lambda_{r_1}, \Lambda_{r_2}, \Lambda_{r_3}, \ldots \quad (r_1 < r_2 < r_3 < \ldots)$$

tending to a limit, the lattice Λ say. By the continuity of the determinant,

$$d(\Lambda) = \lim_{k \to \infty} d(\Lambda_{r_k}) = \Delta(S).$$

The assertion is therefore proved if we can show that Λ is S-admissible, hence critical. If Λ were not S-admissible, there would be a point $P \neq O$ of Λ which is an *inner* point of S. There exists then a neighbourhood of P consisting only of inner points of S. Since the lattices Λ_{r_k} tend to Λ, this neighbourhood contains a point of Λ_{r_k} for all sufficiently large indices k, contrary to the assumption that Λ_{r_k} is S-admissible.

3. Two special cases of Theorem 1 are of particular interest.

THEOREM 2. *If the point set S is of the finite type, and if O is an inner point of S, then S possesses at least one critical lattice.*

Proof. Choose an arbitrary infinite sequence of S-admissible lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

satisfying (2). Then this sequence is bounded since none of its points lie in a sufficiently small neighbourhood of O. The assertion follows therefore immediately from Theorem 1.

THEOREM 3. *If the point set S is bounded and not of the zero type, then it possesses at least one critical lattice.*

Proof. Let the assertion be false, i.e. assume that S has no critical lattice. Denote by ϵ an arbitrarily small positive number, and by ρ so large a positive number that S is contained in the sphere

$$|X| < \rho.$$

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[2]A sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$ is said to be *bounded* if (i) the determinants $d(\Lambda_r)$ are bounded, and (ii) no point $P \neq O$ of these lattices lies in a certain neighbourhood of O. (LP, Definition 1, p. 155.)

[3]It is possible to select from any bounded sequence of lattices a subsequence tending to a limiting lattice. (LP, Theorem 2, p. 156.)

Choose further any infinite sequence of S-admissible lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

satisfying (2). By Theorem 1, this sequence cannot be bounded. Hence there is an index k such that Λ_k contains a point $P_1 \neq O$ at a distance less than ϵ from O. There is no loss of generality in assuming that P_1 is of the form

$$P_1 = (\xi_1, 0, \ldots, 0), \text{ where } 0 < \xi_1 < \epsilon,$$

since the coordinate system may be so selected that the x_1-axis passes through P_1. Let now $P_2, P_3, \ldots, P_n$ be the points

$$P_2 = (0, \rho, 0, \ldots, 0), \quad P_3 = (0, 0, \rho, \ldots, 0), \ldots, P_n = (0, 0, 0, \ldots, \rho),$$

and let Λ be the lattice of basis $P_1, P_2, \ldots, P_n$, hence of determinant

$$d(\Lambda) = \xi_1\rho^{n-1} < \epsilon\rho^{n-1}. \tag{3}$$

Then this lattice is S-admissible. For Λ consists of the points

$$P = u_1P_1 + u_2P_2 + \ldots + u_nP_n \qquad (u_1, u_2, \ldots, u_n = 0, \mp 1, \mp 2, \ldots).$$

Of these lattice points, those with

$$\sum_{h=2}^{n} u_h^2 > 0$$

lie at a distance not less than ρ from 0, hence do not belong to S. If, however,

$$u_1 \neq 0, u_2 = u_3 = \ldots = u_n = 0,$$

then P belongs to Λ_k and so cannot be an inner point of S.

Hence

$$\Delta(S) \leq d(\Lambda) < \epsilon\rho^{n-1},$$

whence

$$\Delta(S) = 0$$

since ϵ may be arbitrarily small. Therefore S is of the zero type, contrary to hypothesis.

Theorem 2 contains as a special case my earlier result on the critical lattices of a star body of the finite type.[4]

II

4. The question arises whether Theorem 1 has a non-trivial content, thus whether there do in fact exist point sets satisfying the condition (1), but having no critical lattices. We shall now answer this problem by constructing an example of such a point set. But it will first be necessary to prove a number of simple lemmas.

5. Let

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

be an infinite sequence of positive numbers satisfying

$$\alpha_1 < \alpha_2 < \alpha_3 < \ldots, \quad \lim_{r\to\infty} \alpha_r = \infty,$$

[4] LP, Theorem 8, p. 159.

and such that

$$\frac{\alpha_r}{\alpha_s} \text{ is irrational if } r \neq s.$$

Denote by Σ the set of all products

$$u\alpha_r, \text{ where } r, u = 1, 2, 3, \ldots.$$

Then all elements of Σ are positive; no two elements of Σ are equal; and any finite interval contains at most a finite number of elements of Σ. Hence if the elements of Σ,

$$\xi_1, \xi_2, \xi_3, \ldots \qquad \text{say,}$$

are arranged according to increasing size,

$$\xi_1 < \xi_2 < \xi_3 < \ldots,$$

then

$$\lim_{\mu \to \infty} \xi_\mu = \infty.$$

If t is any positive number, and if ξ_μ, ξ_ν run over all pairs of elements of Σ for which

$$\xi_\mu \neq \xi_\nu \text{ (i.e. } \mu \neq \nu), \ \xi_\nu \leq t,$$

then at most a finite number of the differences

$$| \xi_\mu - \xi_\nu |$$

are less than an arbitrary given constant. Denote by

$$\rho(t) = \min (| \xi_\mu - \xi_\nu |)$$

the smallest of these differences; it clearly defines a positive and non-increasing function of t.

Moreover,

$$\lim_{t \to \infty} \rho(t) = 0.$$

For Σ contains the elements,

$$u\alpha_1, \ v\alpha_2 \qquad (u, v = 1, 2, 3, \ldots),$$

and, as is well known, there are positive integers u, v, for which

$$| u\alpha_1 - v\alpha_2 |$$

is arbitrarily small.

6. From the definition of $\rho(t)$,

$$| \xi_\mu - \xi_\nu | \geq \max (\rho(\xi_\mu), \rho(\xi_\nu)), \qquad \text{if } \mu \neq \nu. \tag{4}$$

This implies that for no real number x both

$$| x - \xi_\mu | \leq \tfrac{1}{3}\rho(\xi_\mu) \quad \text{and} \quad | x - \xi_\nu | \leq \tfrac{1}{3}\rho(\xi_\nu),$$

unless $\mu = \nu$. For if, e.g. $\mu < \nu$, then from these inequalities,

$$| \xi_\mu - \xi_\nu | = | (x - \xi_\nu) - (x - \xi_\mu) | \leq \tfrac{1}{3}\rho(\xi_\mu) + \tfrac{1}{3}\rho(\xi_\nu) \leq \tfrac{2}{3}\rho(\xi_\mu) < \rho(\xi_\mu),$$

contrary to (4).

LEMMA 1. *Let K be the set of all real numbers x satisfying at least one of the inequalities*

$$| x - \xi_\mu | \leq \tfrac{1}{6}\rho(2\xi_\mu) \qquad (\mu = 1, 2, 3, \ldots).$$

If all multiples

$$2^k x \qquad (k = 0, 1, 2, 3, \ldots)$$

of x belong to K, then x is an element of $\sum$.

Proof. From the hypothesis,

$$|\, 2^k x - \xi_{\mu_k} \,| \leq \tfrac{1}{6}\rho(2\xi_{\mu_k}) \qquad (k = 0, 1, 2, 3, \ldots),$$

where the indices μ_k depend on k. Therefore, in particular,

$$|\, 2^{k+1} x - 2\xi_{\mu_k} \,| \leq \tfrac{1}{3}\rho(2\xi_{\mu_k}),$$

$$|\, 2^{k+1} x - \xi_{\mu_{k+1}} \,| \leq \tfrac{1}{6}\rho(2\xi_{\mu_{k+1}}) < \tfrac{1}{3}\rho(\xi_{\mu_{k+1}}),$$

since $\rho(t)$ is a non-increasing function of t. But if ξ_μ belongs to $\sum$, so does $2\xi_\mu$; hence these inequalities imply that

$$\xi_{\mu_{k+1}} = 2\xi_{\mu_k} \qquad (k = 0, 1, 2, 3, \ldots),$$

whence

$$\xi_{\mu_k} = 2^k \xi_{\mu_0}, \; |\, 2^k(x - \xi_{\mu_0}) \,| \leq \tfrac{1}{6}\rho(2^{k+1}\xi_{\mu_0}) \qquad (k = 0, 1, 2, 3, \ldots).$$

On letting k tend to infinity, the right-hand side tends to zero, and we find that

$$x = \xi_{\mu_0},$$

as asserted.

7. We need also the following, rather simpler, result.

LEMMA 2. *Let β be a positive number, and let K' be the set of all real numbers x satisfying at least one of the inequalities*

$$|\, x - u\beta \,| \leq \frac{\beta}{6} \qquad (u = 1, 2, 3, \ldots).$$

If all multiples

$$2^k x \qquad (k = 0, 1, 2, 3, \ldots)$$

belong to K', then x is a positive integral multiple of β.

Proof. By hypothesis,

$$|\, 2^k x - u_k\beta \,| \leq \frac{\beta}{6} \qquad (k = 0, 1, 2, 3, \ldots)$$

with integers u_k depending on k. Therefore, in particular,

$$|\, 2^{k+1} x - 2u_k\beta \,| \leq \frac{\beta}{3},$$

$$|\, 2^{k+1} x - u_{k+1}\beta \,| \leq \frac{\beta}{6},$$

whence

$$|\, (u_{k+1} - 2u_k)\beta \,| = |\, (2^{k+1}x - 2u_k\beta) - (2^{k+1}x - u_{k+1}\beta) \,| \leq \frac{\beta}{3} + \frac{\beta}{6} = \frac{\beta}{2},$$

and therefore

$$|\, u_{k+1} - 2u_k \,| \leq \tfrac{1}{2}, \; u_{k+1} = 2u_k, \; u_k = 2^k u_0 \qquad (k = 0, 1, 2, 3, \ldots),$$

since the u's are integers. Hence,

$$|\, 2^k(x - u_0\beta) \,| \leq \frac{\beta}{6} \qquad (k = 0, 1, 2, 3, \ldots).$$

On allowing k to tend to infinity, we find that

$$x = u_0\beta,$$

as asserted.

8. From the last two lemmas, we deduce a similar result for a special point set in n-dimensional space R_n.

Denote by

$$\alpha_1, \alpha_2, \alpha_3, \ldots \quad \text{and} \quad \beta_1, \beta_2, \beta_3, \ldots$$

two infinite sequences of positive numbers satisfying the following conditions:

(I)
$$\alpha_1 < \alpha_2 < \alpha_3 < \ldots, \qquad \lim_{r\to\infty} \alpha_r = \infty,$$
$$\beta_1 > \beta_2 > \beta_3 > \ldots, \qquad \lim_{r\to\infty} \beta_r = 0,$$
$$\alpha_1\beta_1 > \alpha_2\beta_2 > \alpha_3\beta_3 > \ldots, \qquad \lim_{r\to\infty} \alpha_r\beta_r = 1.$$

(II) If

$$\gamma_1, \gamma_2, \ldots, \gamma_r$$

is any finite system of integers not all zero, then[5]

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \ldots + \alpha_r\gamma_r \neq 0.$$

Let further $u_1, u_2, \ldots, u_n$ and r run over all positive integers, and denote by

$$\Pi^{(r)}(u) = \Pi^{(r)}(u_1, u_2, \ldots, u_n)$$

the parallelepiped of all points

$$X = (x_1, x_2, \ldots, x_n)$$

which satisfy the inequalities

$$|x_1 - \alpha_r u_1| \leq \tfrac{1}{6}\rho(2\alpha_r u_1), \; |x_2 - \beta_r u_2| \leq \frac{\beta_r}{6}, \; |x_h - u_h| \leq \tfrac{1}{6} \quad (h = 3, 4, \ldots, n);$$

here $\rho(t)$ is the function defined in **5**. The centre of $\Pi^{(r)}(u)$ is at the point,

$$P^{(r)}(u) = P^{(r)}(u_1, u_2, \ldots, u_n) = (\alpha_r u_1, \beta_r u_2, u_3, \ldots, u_n).$$

Denote then by

$$\Pi = \bigcup_{u,r} \Pi^{(r)}(u)$$

the sum set of all parallelepipeds $\Pi^{(r)}(u)$, and by

$$\mathrm{P} = \{P^{(r)}(u)\}$$

the set of all points $P^{(r)}(u)$. Since, from (4),

$$\rho(2\xi_\nu) \leq \xi_\nu,$$

because both ξ_ν and $2\xi_\nu$ belong to Σ, the two point sets Π and P lie completely in the octant

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.$$

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[5]The conditions (I) and (II) are satisfied if, e.g.

$$\alpha_r = \left(1 + \frac{1}{r}\right)e^r, \quad \beta_r = \left(1 + \frac{1}{r}\right)e^{-r} \qquad (r = 1, 2, 3, \ldots),$$

as is trivial for (I), and follows for (II) from the transcendency of e.

LEMMA 3. *Let the point* $X = (x_1, x_2, \ldots, x_n)$ *be such that all multiples*

$$2^k X = (2^k x_1, 2^k x_2, \ldots, 2^k x_n) \qquad (k = 0, 1, 2, 3, \ldots)$$

belong to Π. *Then* X *is an element of* P.

Proof. The first coordinate x_1 of X lies in one of the intervals

$$|x_1 - \xi_\nu| \leq \tfrac{1}{6}\rho(2\xi_\nu) \qquad (\nu = 1, 2, 3, \ldots);$$

the second coordinate x_2 lies in one of the intervals

$$|x_2 - \beta_r u_2| \leq \frac{\beta_r}{6} \qquad (r, u_2 = 1, 2, 3, \ldots);$$

and the remaining coordinates x_h $(h = 3, 4, \ldots, n)$ lie in intervals

$$|x_h - u_h| \leq \tfrac{1}{6} \qquad (u_h = 1, 2, 3, \ldots);$$

moreover, analogous conditions are also satisfied by the coordinates of the points

$$2^k X \qquad (k = 1, 2, 3, \ldots).$$

Therefore, by Lemma 1, x_1 belongs to Σ, so that

$$x_1 = \alpha_r u_1 \tag{5}$$

for some pair of positive integers r and u_1. The same index r occurs in the inequalities for the multiples $2^k x_2$ of x_2; by Lemma 2 applied with $\beta = \beta_r$, there is therefore a positive integer u_2 such that

$$x_2 = \beta_r u_2\,. \tag{6}$$

Finally, by the same lemma applied with $\beta = 1$, there exist $n-2$ positive integers $u_3, u_4, \ldots, u_n$ such that

$$x_h = u_h \qquad (h = 3, 4, \ldots, n). \tag{7}$$

The assertion is contained in (5), (6), and (7).

9. We also need the following simple lemma about the bases of a lattice.

LEMMA 4. *For every lattice* Λ, *a basis*

$$Y_1 = (y_{11}, y_{12}, \ldots, y_{1n}),\ Y_2 = (y_{21}, y_{22}, \ldots, y_{2n}), \ldots,\ Y_n = (y_{n1}, y_{n2}, \ldots, y_{nn})$$

can be found such that

$$y_{hk} > 1 \qquad (h, k = 1, 2, \ldots, n). \tag{8}$$

Proof. First choose an arbitrary point $Y_1 = (y_{11}, y_{12}, \ldots, y_{1n})$ of Λ with

$$y_{11} > 1,\ y_{12} > 1, \ldots, y_{1n} > 1$$

such that no inner point of the line segment joining O to Y_1 belongs to Λ. By Minkowski's selection method[6], $n-1$ further lattice points $Y'_2, Y'_3, \ldots, Y'_n$ can be chosen such that the n points

$$Y_1,\ Y'_2,\ Y'_3, \ldots, Y'_n$$

form a basis of Λ. Then the further n points

$$Y_1,\ Y_2 = Y'_2 + v_2 Y_1,\quad Y_3 = Y'_3 + v_3 Y_1, \ldots,\quad Y_n = Y'_n + v_n Y_1$$

where $v_2, v_3, \ldots, v_n$ are $n-1$ arbitrary integers, also form a basis of Λ. We satisfy now the conditions (8) by taking the v's positive and sufficiently large.

[6] *Geometrie der Zahlen*, § 46.

10. As in **8**, we let $u_1, u_2, \ldots, u_n$ and r run over all positive integers, but denote now by

$$\Pi_0^{(r)}(u) = \Pi_0^{(r)}(u_1, u_2, \ldots, u_n)$$

the *open* parallelepiped of all points X satisfying

$$|x_1 - \alpha_r u_1| < \tfrac{1}{6}\rho(2\alpha_r u_1), \ |x_2 - \beta_r u_2| < \frac{\beta_r}{6}, \ |x_h - u_h| < \tfrac{1}{6} \ (h = 3, 4, \ldots, n);$$

its centre is again at the point $P^{(r)}(u)$, and its closure is $\Pi^{(r)}(u)$. Further denote by

$$\Pi_0 = \bigcup_{u,r} \Pi_0^{(r)}(u)$$

the sum of all parallelepipeds $\Pi_0^{(r)}(u)$, and by Ω the point set

$$x_1 \geq 1, x_2 \geq 1, \ldots, x_n \geq 1.$$

The difference set

$$S = \Omega - \Pi_0$$

of all points of Ω which are not in Π_0, is evidently closed, since Π_0, as a sum of open sets, is open, and since Ω is closed because R_n does not contain a point at infinity.

There are at most a finite number of points $P^{(r)}(u)$ in every finite portion of Ω. Therefore every point of S is either an inner point of S, or a boundary point of Ω, or it is a boundary point of one of the closed parallelepipeds $\Pi^{(r)}(u)$, hence belongs to Π.

11. Let now Λ be any S-admissible lattice. Then choose a basis $Y_1, Y_2, \ldots, Y_n$ of Λ satisfying the condition (8) of Lemma 4. These n points, and also the vector sum

$$Y = Y_1 + Y_2 + \ldots + Y_n$$

are not inner points of S, nor are they boundary points of Ω; and the same is true even for the multiples

(9) $2^k Y_1, 2^k Y_2, \ldots, 2^k Y_n, 2^k(Y_1 + Y_2 + \ldots + Y_n) = 2^k Y \ (k = 0, 1, 2, 3, \ldots)$.

Hence all points (9) belong to Π. But then, by Lemma 3, the $n + 1$ points

$$Y_1, Y_2, \ldots, Y_n, Y$$

are elements of P, and so there exist positive integers

$$r_1, r_2, \ldots, r_n, r$$

and

$$u_{hk}, u_k \qquad (h, k = 1, 2, \ldots, n)$$

such that

$$Y_h = (\alpha_{r_h} u_{h1}, \beta_{r_h} u_{h2}, u_{h3}, \ldots, u_{hn}) \qquad (h = 1, 2, \ldots, n),$$

$$Y = Y_1 + Y_2 + \ldots + Y_n = (\alpha_r u_1, \beta_r u_2, u_3, \ldots, u_n).$$

Therefore, in particular,

$$\alpha_{r_1} u_{11} + \alpha_{r_2} u_{21} + \ldots + \alpha_{r_n} u_{n1} = \alpha_r u_1.$$

By the hypothesis (II) of **8**, this equation can hold only if

$$r_1 = r_2 = \ldots = r_n = r, \quad u_{11} + u_{21} + \ldots + u_{n1} = u_1.$$

Hence all basis points Y_h belong to the same value of r, and the basis is of the form

$$Y_h = (\alpha_r u_{h1}, \beta_r u_{h2}, u_{h3}, \ldots, u_{hn}) \qquad (h = 1, 2, \ldots, n).$$

Denote now by Λ_r the lattice of all points

$$P = (\alpha_r g_1, \beta_r g_2, g_3, \ldots, g_n),$$

where the g's run over all integers; this lattice is of determinant

$$d(\Lambda_r) = \alpha_r \beta_r.$$

Since the basis elements Y_h of Λ belong to Λ_r, Λ is either identical with Λ, or it is a sublattice. In either case,

$$d(\Lambda) = gd(\Lambda_r),$$

where g is a positive integer. Hence, by the hypothesis (I) of **8**,

$$d(\Lambda) \geq d(\Lambda_r) > 1, \qquad \text{and } d(\Lambda) > 2 \text{ if } g > 1.$$

In the other direction, from the same hypothesis,

$$\lim_{r \to \infty} d(\Lambda_r) = 1.$$

We find therefore the following result:

THEOREM 4. *The only admissible lattices of the set S are* (i) *the lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots,$ and* (ii) *their sublattices. All S-admissible lattices are of determinant greater than* 1, *but*

$$\lim_{r \to \infty} d(\Lambda_r) = 1.$$

Hence $\Delta(S) = 1$, and there are no critical lattices of S.

12. Theorem 4 implies, in particular, that S has only an *enumerable* set of admissible lattices, a possibility which cannot arise for star bodies. It is further clear that no point of any S-admissible lattice lies on the boundary of S.

The following, somewhat simpler, example of a point set is possibly even more surprising. Denote by T the set of all points X such that

$$\max(|x_1 - u_1|, |x_2 - u_2|, \ldots, |x_n - u_n|) \geq \tfrac{1}{6}$$

for every system of integers $u_1, u_2, \ldots, u_n$. It is not difficult to deduce from Lemma 2, that the only T-admissible lattices are (i) the lattice of all points with integral coordinates, and (ii) all its sublattices. Therefore $\Delta(T) = 1$, and there is just one critical lattice. Every point of this critical lattice lies at a distance $\frac{1}{6}$ from the boundary of T, and the same is true for the points of the T-admissible lattices. This is very different from the position for star bodies; for every critical lattice of a star body has at least one point arbitrarily near to its boundary.

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Mr. C. A. Rogers, having been told of my result, found the following simpler example of a point set without a critical lattice:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1x_2\left(1 - \frac{x_1x_2}{x_1{}^2 + x_2{}^2}\right) \leq 1 \, .$$

This two-dimensional set differs from my example in having a *continuous* infinity of admissible lattices.

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