ON LATTICE POINTS IN A CONVEX DECAGON.

BY

WALTER LEDERMANN and KURT MAHLER

of MANCHESTER.

Let K be a convex domain in the (x, y)-plane symmetrical in the origin $O = (0, 0)$ of the coordinate system. If

$$X_1 = (x_1, y_1) \text{ and } X_2 = (x_2, y_2)$$

are two points not collinear with O, then the set Λ of all points[1]

$$u_1 X_1 + u_2 X_2 \qquad (u_1, u_2 = 0, \mp 1, \mp 2, \ldots)$$

is a lattice, and the positive number

$$d(\Lambda) = |(X_1, X_2)|$$

is the determinant of Λ. We say that Λ is *K-admissible* if no point of Λ except O is an *inner* point of K. Then the lower bound

$$\Delta(K) = \text{l. b. } d(\Lambda)$$

extended over all K-admissible lattices is a positive number and is called the *minimum determinant of* K. There exist critical lattices of K, i. e. lattices Λ which are K-admissible and of determinant

$$d(\Lambda) = \Delta(K).$$

Except when K is a parallelogram, such lattices have just three pairs of points $\mp A$, $\mp B$, $\mp C$ on the boundary of K, and if the notation is chosen suitably, then

$$A + B = C.$$

---

[1] We use vector notation; thus $u_1 X_1 + u_2 X_2 = (u_1 x_1 + u_2 x_2, u_1 y_1 + u_2 y_2)$, and in particular $-X_1 = (-x_1, -y_1)$. The determinant of X_1 and X_2 is denoted by $(X_1, X_2) = x_1 y_2 - x_2 y_1$.

If $V(K)$ is the area of K, then the quotient

$$Q(K) = \frac{V(K)}{\Delta(K)}$$

is invariant under all affine transformations which leave O unchanged. The quotient $Q(K)$ arises also in connection with the densest packing of convex figures. Place domains of half the linear dimensions of K, but with the same orientation, in such a way that their centres are at the points of Λ. Then no two such domains overlap if and only if Λ is K-admissible. Further the ratio of the part of the plane covered by these domains, to the whole plane, is equal to

$$\frac{V(K)}{4\,d(\Lambda)}$$

and therefore the maximum of this ratio, namely

$$\frac{V(K)}{4\,\Delta(K)} = \frac{1}{4}\,Q(K)$$

is attained when Λ is a critical lattice of K. Since this ratio cannot be greater than unity,

$$Q(K) \leq 4,$$

which is Minkowski's classical theorem on convex domains. Here the equality sign holds if and only if K is a parallelogram or a hexagon.

In the other direction, it is not difficult to show that[1]

$$Q(K) \geq \sqrt{12},$$

but the exact lower bound is not known. It was conjectured by Reinhardt[2] that this lower bound is attained for the *smoothed octagon*, but no proof has so far been given. Reinhardt came to his result by showing a result which may be expressed as follows:

---

[1] K. Mahler, The Theorem of Minkowski-Hlawka, Duke Mathematical Journal, 14 (1946), 611—621, Lemma 2.

[2] K. Reinhardt, Über die dichteste gitterförmige Lagerung congruenter Bereiche, und eine besondere Art convexer Curven, Abh. aus dem Math. Seminar der Hamburgischen Univ. 9 (1933), 216—230. With regard to the smoothed octagon, Reinhardt said: »Die Frage nach den Bereichen dünnster dichtester Lagerung läuft offenbar darauf hinaus, diejenige Kurve (oder diejenigen Kurven), der von uns betrachteten Art zu finden, welche bei gegebenem einbeschriebenem etwa regulärem Sechseck eine möglichst kleine Fläche umschliesst. — Bei unseren Bereichen kommt diejenige Figur in Betracht, welche aus einem regelmässigen Achteck entsteht, wenn man jede Ecke durch diejenige Hyperbel abschneidet, die die beiden anstossenden Seiten berührt, und die beiden wieder an diese grenzenden Seiten zu Asymptoten hat.» We call this figure the smoothed octagon.

»*Denote by U_K the set of all hexagons H bounded by three pairs of tac-lines (Stützlinien) of K symmetrical in O. Then*

$$\Delta(K) = \tfrac{1}{4} \underset{H \varepsilon U_k}{\text{l. b.}} \; V(K).»$$

Without knowledge of his paper, one of us[1] recently rediscovered this formula and was lead to the same conjecture about the lower bound

$$Q$$

of $Q(K)$ extended over all convex domains K symmetrical in O. He further studied the lower bound

$$Q_n$$

of $Q(\Pi_n)$ extended over all convex polygons Π_n bounded by n pairs of sides symmetrical in O, and he showed that[2]

$$4 = Q_2 = Q_3 > Q_4 > Q_5 > Q_6 > \cdots,$$

$$\lim_{n \to \infty} Q_n = Q,$$

$$Q_4 = \frac{16}{7}(3 - \sqrt{2}) = 3 \cdot 62465 \ldots .$$

He further proved that each of the lower bounds Q and Q_n is actually attained.

In the present paper, we continue these investigations and determine the lower bound Q_5. While for $n = 4$ the lower bound Q_4 is attained for the regular octagon, we find that for $n = 5$ the bound is attained for a convex decagon of a non-regular type, and that its value is

$$Q_5 = 3 \cdot 62173 \ldots$$

We also determine the value of $Q(D')$ for the smoothed decagon D', i. e. a certain figure bounded by ten line segments and ten hyperbolic arcs, and we find that

$$Q(D') = 3 \cdot 60974 \ldots$$

This value is larger than the corresponding value

$$Q(O') = 3 \cdot 60965 \ldots$$

for the smoothed octagon, a result which seems to support Reinhardt's conjecture.

---

[1] K. MAHLER, On the minimum determinant and the circumscribed hexagons of a convex domain, Proc. Academy Amsterdam 50 (1947), 692—703, p. 694. This paper will henceforth be referred to as *M*.

[2] *M*, p. 698; p. 702.

1. **The configuration.** The five pairs of parallel lines which form a plane symmetrical decagon D, will be denoted by

$$L_i: l_i x + m_i y + n_i = 0, \quad -L_i: -(l_i x + m_i y) + n_i = 0 \qquad (i = 1, 2, 3, 4, 5). \tag{1.1}$$

The vertices

$$P_1, P_2, P_3, P_4, P_5, -P_1, -P_2, -P_3, -P_4, -P_5 \tag{1.2}$$

of D are the intersections of

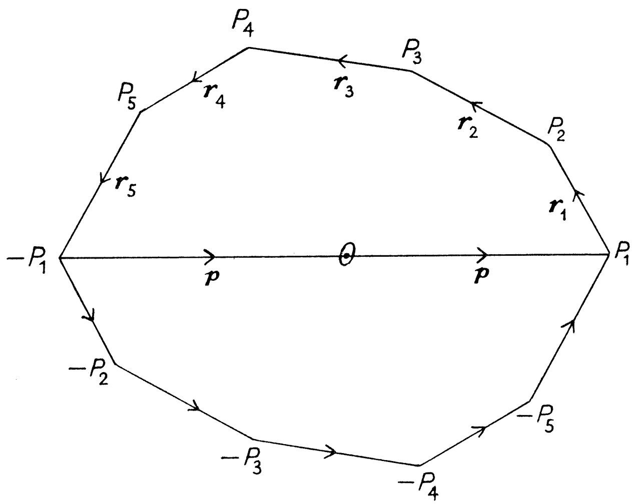


Fig. 1.

$$-L_5 \text{ and } L_1, \; L_1 \text{ and } L_2, \; L_2 \text{ and } L_3, \ldots, -L_4 \text{ and } -L_5$$

respectively.

For many purposes, however, it is more convenient to specify the decagon by the vectors

$$\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5, -\mathbf{r}_1, -\mathbf{r}_2, \ldots, -\mathbf{r}_5, \tag{1.3}$$

which form the sides of the polygon; thus $\mathbf{r}_i = \overrightarrow{P_i P_{i+1}}$. The determinant

$$a_{ij} = (\mathbf{r}_i, \mathbf{r}_j) = -a_{ji} \tag{1.4}$$

represents the area of the parallelogram made by the vectors $\mathbf{r}_i$ aud $\mathbf{r}_j$. It is of course sufficient to let the indices i and j run from 1 to 5, since e.g.

$$a_{17} = (\mathbf{r}_1, -\mathbf{r}_2) = -a_{12}, \quad a_{56} = (\mathbf{r}_5, -\mathbf{r}_1) = a_{15} \text{ etc.}$$

Indeed, the 10 quantities

$$a_{ij} \qquad (i < j,\ i, j = 1, 2, 3, 4, 5)$$

afford a complete analytical description of the configuration we wish to study.

The polygon is *convex* if and only if (Fig. 1)

$$a_{ij} > 0 \qquad (i < j,\ i, j = 1, 2, 3, 4, 5). \tag{1.5}$$

It is important to note that the quantities a_{ij} are not independent. If $i, j, k, l,$ are four distinct numbers out of 1, 2, 3, 4, 5, then

$$a_{ij}\, a_{kl} + a_{jk}\, a_{il} + a_{ki}\, a_{jl} = 0. \tag{1.6}$$

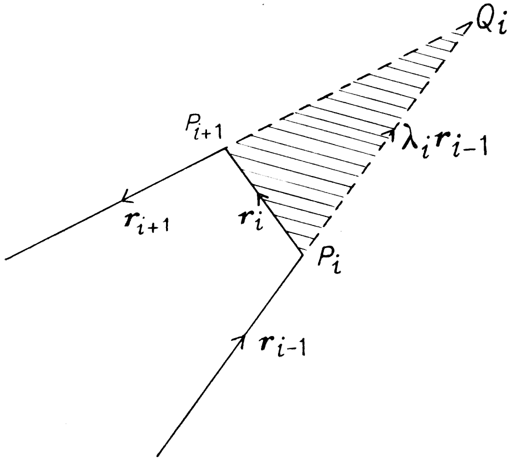


Fig. 2.

For since any three vectors in a plane are linearly dependent,

$$\mathbf{r}_k = \lambda\, \mathbf{r}_i + \mu\, \mathbf{r}_j \qquad (\lambda, \mu \text{ scalars});$$

on forming the outer product with $\mathbf{r}_i$ and $\mathbf{r}_j$, it is found that

$$a_{ik} = \mu\, a_{ij},\ a_{jk} = -\lambda\, a_{ij},$$

and therefore

$$a_{ij}\, \mathbf{r}_k + a_{jk}\, \mathbf{r}_i + a_{ki}\, \mathbf{r}_j = 0,$$

whence, on multiplying by $\mathbf{r}_l$, we obtain (1.6). Making use of the fact that $a_{ij} = -a_{ji}$, we have e. g.

$$\begin{aligned} a_{12}\, a_{34} - a_{13}\, a_{24} + a_{14}\, a_{23} &= 0 \\ a_{12}\, a_{35} - a_{13}\, a_{25} + a_{15}\, a_{23} &= 0 \\ \cdots\cdots\cdots\cdots \end{aligned} \tag{1.7}$$

There are five such PLÜCKER identities, but only three of them are independent. Thus there are seven independent coefficients a_{ij} which determine the configuration apart from affine transformations.

By Fig. 1 the position vector of the vertex P_i is

$$\mathbf{p} + \mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_{i-1}, \qquad (i = 1, 2, 3, 4, 5; \ \mathbf{r}_0 = 0)$$

where

$$\mathbf{p} = -\tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5).$$

Hence

$$\text{area}\,(O\,P_i\,P_{i+1}) = \tfrac{1}{2}(\mathbf{p} + \mathbf{r}_1 + \cdots + \mathbf{r}_{i-1}, \mathbf{r}_i) = \tfrac{1}{2}(\mathbf{p}, \mathbf{r}_i) + \tfrac{1}{2}\sum_{k=1}^{i-1}(\mathbf{r}_k, \mathbf{r}_i),$$

and[1]

$$D = 2\sum_{i=1}^{5} \text{area}\,(O\,P_i\,P_{i+1}) = \left(\mathbf{p}, \sum_{i=1}^{5}\mathbf{r}_i\right) + \sum_{\substack{i,j=1\\ i<j}}^{5}(\mathbf{r}_i, \mathbf{r}_j), \tag{1.8}$$

that is

$$D = \sum_{\substack{i,j=1\\ i<j}}^{5} a_{ij},$$

since $(\mathbf{p}, \sum \mathbf{r}_i) = 0$.

If of the five pairs of sides (1.1) of D one pair, say $\mp L_i$, is omitted, the remaining four pairs form a symmetrical octagon O_i, circumscribed to the original decagon. The points P_i and P_{i+1} do not occur as vertices of this octagon, but are replaced by the single point Q_i, the intersection of L_{i-1} and L_{i+1}. The area of O_i (Fig. 2) exceeds D by

$$\xi_i^2 = \frac{a_{i-1,i}\,a_{i,i+1}}{a_{i-1,\,i+1}} = 2\ \text{area}\,(P_i\,Q_i\,P_{i+1}). \tag{1.9}$$

For

$$2\ \text{area}\,(P_i\,Q_i\,P_{i+1}) = (\overrightarrow{P_i\,Q_i}, \mathbf{r}_i) = \lambda_i(\mathbf{r}_{i-1}, \mathbf{r}_i) = \lambda_i\,a_{i-1,i},$$

where λ_i is a scalar which is determined by the condition that $\lambda_i\,\mathbf{r}_{i-1} - \mathbf{r}_i$ should be parallel to $\mathbf{r}_{i+1}$. Therefore

$$0 = (\lambda\,\mathbf{r}_{i-1} - \mathbf{r}_i, \mathbf{r}_{i+1}) = \lambda_i\,a_{i-1,\,i+1} - a_{i,\,i+1},$$

whence the result follows.

The subsequent argument is chiefly concerned with the symmetrical hexagons that can be circumscribed to D. There are evidently 10 such hexagons, each being obtained by leaving out two pairs of parallel sides, say

$$\mp L_i, \quad \mp L_j,$$

[1] Here, as elsewhere, the same letter is used to denote a plane domain and its area.

from the original configuration. The area of this hexagon will be denoted by

$$H_{ij},$$

the suffixes indicating the sides that have been omitted. We have to distinguish two classes of hexagons H_{ij} according as the two omitted sides are not, or are, adjacent. In this context, $\mathbf{r}_{-5}$ and $\mathbf{r}_1$, or $\mathbf{r}_5$ and $\mathbf{r}_{-1}$ are, of course, adjacent.

(i) *Hexagons of the first class: The sides* $\mathbf{r}_i$ *and* $\mathbf{r}_j$ *are not adjacent.* The area H_{ij} is then obtained by adding to D four triangles based on the sides

$$\mp \mathbf{r}_i, \quad \mp \mathbf{r}_j$$

like the single triangle shown in Fig. 2. Thus,

$$H_{ij} = D + \xi_i^2 + \xi_j^2. \tag{1.10}$$

The quantities

$$E_{ij} = H_{ij} - D = \xi_i^2 + \xi_j^2 \tag{1.11}$$

will be frequently used.

(ii) *Hexagons of the second class: The omitted sides are adjacent, say* $\mathbf{r}_i$ *and* $\mathbf{r}_{i+1}$. The hexagons $H_{i,i+1}$ is obtained from D by the addition of two quadrilaterals, symmetrical in O, one of which viz. $P_i R_i P_{i+2} P_{i+1}$ is shown in Fig. 3. The additional area is given by

$$E_{i,\,i+1} = \frac{(a_{i-1,\,i} + a_{i-1,\,i+1})(a_{i,\,i+2} + a_{i+1,\,i+2})}{a_{i-1,\,i+2}} - a_{i,\,i+1},$$

where the first term is analogous to the expression (1.9), the vector $\mathbf{r}_i$ having been replaced by $\mathbf{r}_i + \mathbf{r}_{i+1}$. On simplifying and applying (1.6) to the indices $i-1$, i, $i+1$, $i+2$ we obtain

$$E_{i,i+1} = \frac{a_{i-1,i}\,a_{i,i+2} + 2\,a_{i-1,i}\,a_{i+1,i+2} + a_{i-1,i+1}\,a_{i+1,i+2}}{a_{i-1,i+2}}. \tag{1.12}$$

2. **The intrinsic variables** ξ_i **and** $\beta_{i,\,i+1}$. The determinants a_{ij} are not the most convenient parameters for defining the configuration. Instead, we shall use as new variables the five expressions (1.9), namely,

$$\xi_1^2 = \frac{a_{15}\,a_{12}}{a_{25}}, \quad \xi_2^2 = \frac{a_{12}\,a_{23}}{a_{13}}, \quad \xi_3^2 = \frac{a_{23}\,a_{34}}{a_{24}}, \quad \xi_4^2 = \frac{a_{34}\,a_{45}}{a_{35}}, \quad \xi_5^2 = \frac{a_{45}\,a_{15}}{a_{14}}, \tag{2.1}$$

together with the five positive quantities

$$\beta_{12} = \sqrt{\frac{a_{13}\,a_{25}}{a_{15}\,a_{23}}},\quad \beta_{23} = \sqrt{\frac{a_{13}\,a_{24}}{a_{12}\,a_{34}}},\quad \beta_{34} = \sqrt{\frac{a_{24}\,a_{35}}{a_{23}\,a_{45}}},$$
$$\beta_{45} = \sqrt{\frac{a_{35}\,a_{14}}{a_{34}\,a_{15}}},\quad \beta_{51} = \sqrt{\frac{a_{25}\,a_{14}}{a_{12}\,a_{45}}}. \tag{2.2}$$

It will presently become clear that only seven of these variables are independent. There can, however, be no identity between the ξ's valid for all symmetrical convex decagons. For if $\mathbf{r}_i$ $(i = 1, 2, 3, 4, 5)$ be the sides of a fixed decagon D, consider a decagon D_θ with sides $\theta_i\,\mathbf{r}_i$ $(i = 1, 2, 3, 4, 5)$, where

$$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$$

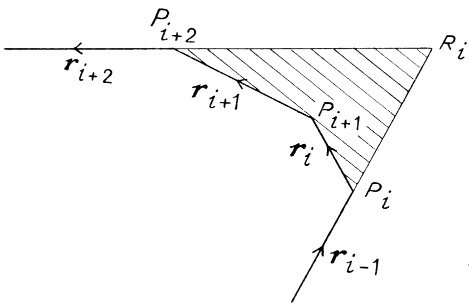


Fig. 3.

are arbitrary positive parameters. The determinants of D_θ are

$$a'_{ij} = \theta_i\,\theta_j\,a_{ij} \tag{2.3}$$

and the ξ's become

$$\xi'_i = \theta_i\,\xi_i \qquad (i = 1, 2, 3, 4, 5). \tag{2.4}$$

Hence the ξ's can be made equal to any five positive numbers.

It is important to note that the β's of D and D_θ are the same, i. e.

$$\beta'_{i,i+1} = \beta_{i,i+1}. \tag{2.5}$$

The β's seem to have no simple geometrical significance.

The equations (2.1) and (2.2) can be solved for the a_{ij}, thus

$$a_{12} = \xi_1\,\xi_2\,\beta_{12},\qquad a_{23} = \xi_2\,\xi_3\,\beta_{23},\qquad a_{34} = \xi_3\,\xi_4\,\beta_{34},\qquad a_{45} = \xi_4\,\xi_5\,\beta_{45},$$
$$a_{15} = \xi_5\,\xi_1\,\beta_{51},$$
$$a_{13} = \xi_1\,\xi_3\,\beta_{12}\,\beta_{23},\quad a_{14} = \xi_4\,\xi_1\,\beta_{45}\,\beta_{51},\quad a_{24} = \xi_2\,\xi_4\,\beta_{23}\,\beta_{34},\quad a_{25} = \xi_5\,\xi_2\,\beta_{51}\,\beta_{12},$$
$$a_{35} = \xi_3\,\xi_5\,\beta_{34}\,\beta_{45}. \tag{2.6}$$

On substituting in (1.8) we find for D the expression

$$D = \beta_{12}\xi_1\xi_2 + \beta_{23}\xi_2\xi_3 + \beta_{34}\xi_3\xi_4 + \beta_{45}\xi_4\xi_5 + \beta_{51}\xi_5\xi_1 \\ + \beta_{12}\beta_{23}\xi_1\xi_3 + \beta_{23}\beta_{34}\xi_2\xi_4 + \beta_{34}\beta_{45}\xi_3\xi_5 + \beta_{45}\beta_{51}\xi_4\xi_1 + \beta_{51}\beta_{12}\xi_5\xi_2. \tag{2.7}$$

The Plücker identities (1.6) imply that the β's are not independent. For example, we have

$$a_{23}a_{45} + a_{34}a_{25} - a_{24}a_{35} = 0,$$

whence

$$1 + \frac{a_{34}a_{25}}{a_{23}a_{45}} = \frac{a_{24}a_{35}}{a_{23}a_{45}} = \beta_{34}^2,$$

i. e.

$$\beta_{34}^2 - 1 = \frac{a_{34}a_{25}}{a_{23}a_{45}}.$$

Similarly

$$\beta_{45}^2 - 1 = \frac{a_{45}a_{13}}{a_{34}a_{15}},$$

and therefore

$$(\beta_{34}^2 - 1)(\beta_{45}^2 - 1) = \frac{a_{25}a_{13}}{a_{23}a_{15}} = \beta_{12}^2.$$

Analogous formulae are obtained by cyclical permutations of the suffixes, thus

$$\begin{aligned} \beta_{12}^2 &= (\beta_{34}^2 - 1)(\beta_{45}^2 - 1), \\ \beta_{23}^2 &= (\beta_{45}^2 - 1)(\beta_{51}^2 - 1), \\ \beta_{34}^2 &= (\beta_{51}^2 - 1)(\beta_{12}^2 - 1), \\ \beta_{45}^2 &= (\beta_{12}^2 - 1)(\beta_{23}^2 - 1), \\ \beta_{51}^2 &= (\beta_{23}^2 - 1)(\beta_{34}^2 - 1). \end{aligned} \tag{2.8}$$

From (2.8), further relations between the β's may be deduced. In particular,

$$\frac{\beta_{51}\beta_{23}}{\beta_{34}\beta_{45}} = \frac{\beta_{12}}{\beta_{12}^2 - 1}, \quad \frac{\beta_{12}\beta_{34}}{\beta_{45}\beta_{51}} = \frac{\beta_{23}}{\beta_{23}^2 - 1}, \quad \frac{\beta_{23}\beta_{45}}{\beta_{51}\beta_{12}} = \frac{\beta_{34}}{\beta_{34}^2 - 1}, \\ \frac{\beta_{34}\beta_{51}}{\beta_{12}\beta_{23}} = \frac{\beta_{45}}{\beta_{45}^2 - 1}, \quad \frac{\beta_{45}\beta_{12}}{\beta_{23}\beta_{34}} = \frac{\beta_{51}}{\beta_{51}^2 - 1}. \tag{2.9}$$

Hence all β's are greater than unity.

These equations between the β's are also not independent. *It is, in fact, possible to express all five β's in terms of the two parameters*

$$s = \frac{\beta_{51}\beta_{12}\beta_{23}}{\beta_{34}\beta_{45}}, \quad t = \frac{\beta_{34}\beta_{45}\beta_{51}}{\beta_{12}\beta_{23}}, \tag{2.10}$$

which, on using the first and the fourth equation (2.9), may also by written as

$$s = \frac{\beta_{12}^2}{\beta_{12}^2 - 1}, \quad t = \frac{\beta_{45}^2}{\beta_{45}^2 - 1}. \tag{2.11}$$

Since the β's are by definition positive, it is easily shown from (2.11), (2.10), (2.8) that

$$\beta_{12} = \sqrt{\frac{s}{s-1}}, \quad \beta_{23} = \sqrt{\frac{st-1}{t-1}}, \quad \beta_{34} = \sqrt{\frac{st-1}{s-1}}, \quad \beta_{45} = \sqrt{\frac{t}{t-1}},$$
$$\beta_{51} = \sqrt{st}. \tag{2.12}$$

In these formulae, s and t may be any real numbers subject to the conditions

$$s > 1, \quad t > 1. \tag{2.13}$$

We next express the area H_{ij} of a circumscribed hexagon, or rather the excess E_{ij} of H_{ij} over D in terms of the new variables ξ_i and $\beta_{i,i+1}$. For hexagons of the first class, this is accomplished by (1.11). As regards hexagons of the second class consider a particular case, say E_{23}. By (1.12)

$$E_{23} = \frac{1}{a_{14}}(a_{12}\,a_{24} + 2\,a_{12}\,a_{34} + a_{13}\,a_{34}).$$

Substituting for the a_{ij} from (2.6), we obtain

$$E_{23} = \frac{\beta_{12}\,\beta_{23}\,\beta_{34}}{\beta_{45}\,\beta_{51}}\left(\xi_2^2 + \frac{2}{\beta_{23}}\,\xi_2\,\xi_3 + \xi_3^2\right),$$

whence by (2.9)

$$E_{23} = \frac{\beta_{23}^2}{\beta_{23}^2 - 1}\left(\xi_2^2 + \frac{2}{\beta_{23}}\,\xi_2\,\xi_3 + \xi_3^2\right) = \xi_2^2 + \xi_3^2 + \xi_{23}^2,$$

where

$$\xi_{23}^2 = \frac{1}{\beta_{23}^2 - 1}(\xi_2^2 + 2\,\beta_{23}\,\xi_2\,\xi_3 + \xi_3^2).$$

Therefore

$$E_{23} = \xi_2^2 + \frac{(\xi_2 + \beta_{23}\,\xi_3)^2}{\beta_{23}^2 - 1} = \xi_3^2 + \frac{(\xi_3 + \beta_{23}\,\xi_2)^2}{\beta_{23}^2 - 1}. \tag{2.14}$$

Four similar formulae are obtained by cyclical permutations of the suffixes.

For reference, we give here a complete list of the 10 quantities E_{ij}:

$$\begin{aligned} E_{52} &= \xi_5^2 + \xi_2^2 \\ E_{13} &= \xi_1^2 + \xi_3^2 \\ E_{24} &= \xi_2^2 + \xi_4^2 \\ E_{35} &= \xi_3^2 + \xi_5^2 \\ E_{41} &= \xi_4^2 + \xi_1^2, \end{aligned} \tag{2.15}$$

$$E_{12} = \xi_1^2 + \xi_2^2 + \xi_{12}^2 = (\beta_{12}^2 \xi_1^2 + 2\beta_{12}\xi_1\xi_2 + \beta_{12}^2 \xi_2^2)/(\beta_{12}^2 - 1)$$
$$E_{23} = \xi_2^2 + \xi_3^2 + \xi_{23}^2 = (\beta_{23}^2 \xi_2^2 + 2\beta_{23}\xi_2\xi_3 + \beta_{23}^2 \xi_3^2)/(\beta_{23}^2 - 1)$$
$$E_{34} = \xi_3^2 + \xi_4^2 + \xi_{34}^2 = (\beta_{34}^2 \xi_3^2 + 2\beta_{34}\xi_3\xi_4 + \beta_{34}^2 \xi_4^2)/(\beta_{34}^2 - 1) \qquad (2.16)$$
$$E_{45} = \xi_4^2 + \xi_5^2 + \xi_{45}^2 = (\beta_{45}^2 \xi_4^2 + 2\beta_{45}\xi_4\xi_5 + \beta_{45}^2 \xi_5^2)/(\beta_{45}^2 - 1)$$
$$E_{51} = \xi_5^2 + \xi_1^2 + \xi_{51}^2 = (\beta_{51}^2 \xi_5^2 + 2\beta_{51}\xi_5\xi_1 + \beta_{51}^2 \xi_1^2)/(\beta_{51}^2 - 1),$$

where

$$\xi_{i,i+1}^2 = (\xi_i^2 + 2\beta_{i,i+1}\xi_i\xi_{i+1} + \xi_{i+1}^2)/(\beta_{i,i+1}^2 - 1). \qquad (2.17)$$

Notice that

$$E_{i,i+1} - \xi_i^2 - \xi_{i+1}^2 = \xi_{i,i+1}^2,$$

hence

$$E_{ij} \geq \xi_i^2 + \xi_j^2, \qquad (2.18)$$

whether or not the suffixes i, j are adjacent.

3. **Critical hexagons.** A symmetrical hexagon circumscribed to the decagon D is said to be *critical* if it is of minimum area. A decagon may, of course, have several critical hexagons and these may be of the first or of the second class.

Theorem I: *If H_{rs} and H_{pq} be critical hexagons of the second class, they have one suffix in common.*

Proof. Assume that, on the contrary, all four suffixes r, s, p, q are distinct. There is no loss of generality in assuming that these suffixes are 1, 2, 3, 4, respectively, i. e. that H_{12} and H_{34} are critical. Therefore, in particular,

$$H_{13} \geq H_{12}, \text{ whence } E_{13} \geq E_{12}$$

i. e.

$$\xi_1^2 + \xi_3^2 \geq \xi_1^2 + \xi_2^2 + \xi_{12}^2,$$

and thus

$$\xi_3^2 > \xi_2^2.$$

Similarly, from

$$H_{24} \geq H_{34}$$

we deduce that

$$\xi_2^2 > \xi_3^2,$$

thus arriving at a contradiction. This proves the theorem.

Corollary: *There cannot be more than two critical hexagons of the second class.*

For two distinct critical hexagons of the second class are necessarily of the form

$$H_{i-1,i}, \quad H_{i,i+1}$$

and a third such hexagon, say $H_{j,j+1}$, cannot have a suffix in common with each of them, unless $j = i$ or $j = i - 1$.

4. **Extreme decagons.** Every decagon possesses one or more critical hexagons. Denote these hexagons by $H_{\alpha\beta}$, $H_{\alpha'\beta'}$, $H_{\alpha''\beta''}$, ... Then

$$H_{\alpha\beta} = H_{\alpha'\beta'} = H_{\alpha''\beta''} = \cdots = \min\{H_{ij}\} = D + E, \tag{4.1}$$

and therefore

$$E = E_{\alpha\beta} = E_{\alpha'\beta'} = E_{\alpha''\beta''} = \cdots = \min\{E_{ij}\}. \tag{4.2}$$

Definition: *A symmetrical convex decagon D is said to be extreme if $Q(D)$ is a minimum, i. e. if*

$$Q(D) \leq Q(D')$$

for every symmetrical convex decagon D'. Here

$$Q(D) = \frac{V(D)}{\Delta(D)} = \frac{D}{\Delta(D)}.$$

As was mentioned on p. 321, it is known that

$$\Delta(D) = \tfrac{1}{4}\min\{H_{ij}\} = \tfrac{1}{4}(D + E)$$

so that

$$Q(D) = \frac{4D}{D+E} = \frac{4}{1 + \left(\frac{D}{E}\right)^{-1}}. \tag{4.3}$$

It follows that *for an extreme decagon the ratio*

$$\phi(D) = \frac{D}{E}$$

takes its smallest value.

Theorem 2: *If D is an extreme decagon, then each of the numbers 1, 2, 3, 4, 5 occurs at least once amongst the suffixes of the critical hexagons of D.*

Proof: If the theorem were false, assume that 5, say, does not occur as a suffix of any critical hexagon of D. Then compare D with the decagon D' defined by the vectors

$$\mathbf{r}_1' = \mathbf{r}_1, \ \mathbf{r}_2' = \mathbf{r}_2, \ \mathbf{r}_3' = \mathbf{r}_3, \ \mathbf{r}_4' = \mathbf{r}_4, \ \mathbf{r}_5 = (1 - \varepsilon)\mathbf{r}_5 \ (\varepsilon > 0).$$

By (2.4) and (2.5),

$$\xi_i' = \xi_i \ (i = 1, 2, 3, 4), \quad \xi_5' = (1 - \varepsilon)\xi_5$$

and

$$\beta_{i,i+1}' = \beta_{i,i+1}, \quad (i = 1, 2, 3, 4, 5)$$

where letters with a prime refer to D'.

Therefore from (2.14), (2.15) and (2.16)

$$E'_{ij} = E_{ij} \qquad (i, j = 1, 2, 3, 4),$$

while the four numbers

$$|E'_{i5} - E_{i5}| \qquad (i = 1, 2, 3, 4)$$

can be made arbitrarily small by choosing ε sufficiently small.

Now, by hypothesis

$$E = \min_{i,j=1,2,3,4} \{E_{ij}\} < \min_{i=1,2,3,4} \{E_{i5}\}. \quad (i \neq j)$$

Hence

$$E' = \min_{i,j=1,2,3,4,5} \{E'_{ij}\} = \min_{i,j=1,2,3,4} \{E'_{ij}\} = E, \quad (i \neq j)$$

since

$$\left| \min_{i=1,2,3,4} \{E'_{i5}\} - \min_{i=1,2,3,4} \{E_{i5}\} \right|$$

is arbitrarily small, and therefore

$$E' < \min_{i=1,2,3,4} \{E'_{i5}\}.$$

On the other hand, by (2.7)

$$D' - D = -\varepsilon(\xi_4 \beta_{45} + \xi_1 \beta_{51} + \xi_3 \beta_{34} \beta_{45} + \xi_2 \beta_{51} \beta_{12}) \xi_5 < 0,$$

whence

$$\phi(D') = \frac{D'}{E'} < \frac{D}{E} = \phi(D),$$

contrary to the assumption that D is extreme.

Theorem 3: *Every extreme decagon possesses at least 3 critical hexagons.*

Proof: This is evident from theorem 2, since the set of the critical hexagons $H_{\alpha\beta}$, $H_{\alpha'\beta'}$, ... involves all five suffixes.

Theorem 4: *If D is an extreme decagon, then at least two of its critical hexagons $H_{\alpha\beta}$, $H_{\alpha'\beta'}$, ... have no suffix in common.*

Proof: Assume that, on the contrary, all critical hexagons involve the suffix 1. This means that each of the critical hexagons is formed from D by omitting the lines $\mp L_1$ and one other pair of lines. Thus a variation of the lines $\mp L_1$ has no effect on the critical hexagons, and consequently leaves the quantity E unaltered. On the other hand, if we move these lines closer to the origin in such a way that the figure remains a symmetrical and convex decagon D', we should have $D' < D$. The new decagon would give rise to a smaller value of the ratio D/E, in contradiction to our hypothesis.

Applying now the corollary of theorem 1 (p. 329), we clearly find that there are just three possible types of extreme decagons, namely,

1st type: The extreme decagon has *no* critical hexagon of the second class.

2nd type: The extreme decagon has exactly *one* critical hexagon of the second class.

3rd type: The extreme decagon has exactly *two* critical hexagons of the second class.

These three types will be discussed separately and it will be shown that the *extreme decagon is, in fact, of the third type.*

5. **Decagons of the first type.** In this section, we shall examine the possibility that the extreme decagon is of the first type, so that all its critical hexagons belong to the set

$$H_{52},\ H_{13},\ H_{24},\ H_{35},\ H_{41}. \tag{5.1}$$

By theorem 3, at least three of these hexagons are of equal minimum area, and it will be necessary to consider separately the cases in which just three, four or five of the hexagons (5.1) are critical.

(a) **Exactly three of the hexagons (5.1) are critical:**

The six suffixes of these three hexagons involve all five suffixes 1, 2, 3, 4, 5 (theorem 2). Hence one of these suffixes occurs twice, say the suffix 3. The critical hexagons of D are then

$$H_{13},\ H_{24},\ H_{35} \tag{5.2}$$

and no others. Thus

$$E = E_{13} = E_{24} = E_{35},$$

whence, by (2.15),

$$E = \xi_1^2 + \xi_3^2 = \xi_2^2 + \xi_4^2 = \xi_3^2 + \xi_5^2. \tag{5.3}$$

The problem is to find the minimum of $Q(D)$, i. e. of

$$D/E = D/(\xi_2^2 + \xi_4^2),$$

subject to the conditions (5.3). Since D is homogeneous and of dimension 2 in the ξ's, the problem is equivalent to finding the minimum of D, subject to the conditions

$$\xi_1^2 + \xi_3^2 = \xi_2^2 + \xi_4^2 = \xi_3^2 + \xi_5^2 = 1,\ \xi_i > 0\ (i = 1, 2, 3, 4, 5). \tag{5.4}$$

We shall show that no such minimum exists. (As the region over which the variables range, is not closed, the minimum is therefore attained on the boundary.)

The conditions (5.4) are satisfied if

$$\xi_1 = \xi_5 = \alpha > 0,\ \xi_3 = \gamma > 0,\ \xi_2 > 0,\ \xi_4 > 0$$

where

$$\alpha^2 + \gamma^2 = 1,\ \xi_2^2 + \xi_4^2 = 1.$$

Substituting these values in (2.7), we can write

$$D = h\,\xi_2\,\xi_4 + g\,\xi_2 + f\,\xi_4 + p,$$

where h, g, f, p are positive quantities depending on α, γ and the β's, but not on ξ_2 and ξ_4. It is sufficient to prove that D, when regarded as a function of ξ_2 and ξ_4, cannot attain a minimum if the variables range over the region

$$\xi_2^2 + \xi_4^2 = 1,\ \xi_2 > 0,\ \xi_4 > 0,$$

i. e. if

$$\xi_2 = \cos\theta,\ \xi_4 = \sin\theta,$$

where θ ranges over the interval $0 < \theta < \pi/2$.

But the function

$$F(\theta) = h\cos\theta\sin\theta + g\cos\theta + f\sin\theta + p$$

cannot attain a minimum for an acute angle θ since

$$F''(\theta) = -2h\sin 2\theta - g\cos\theta - f\sin\theta$$

is negative if $0 < \theta < \pi/2$. This shows that an extreme decagon of the first type cannot have only three critical hexagons.

(b) **Exactly four of the hexagons (5.1) are critical.**

Then one of these hexagons, say H_{52}, is not critical. Thus

$$H_{13} = H_{24} = H_{35} = H_{41} < H_{52},$$

and therefore

$$E = E_{13} = E_{24} = E_{35} = E_{41} < E_{52},$$

i. e.

$$\xi_1^2 + \xi_3^2 = \xi_2^2 + \xi_4^2 = \xi_3^2 + \xi_5^2 = \xi_4^2 + \xi_1^2 < \xi_5^2 + \xi_1^2.$$

Hence we may put

$$u = \xi_1 = \xi_2 = \xi_5,\ v = \xi_3 = \xi_4,$$

where

$$u^2 + v^2 = 1, \; u > 0, \; v > 0. \tag{5.4}$$

The expression (2.7) now becomes

$$D = a u^2 + 2 b u v + c v^2,$$

where a, b, c are certain positive quantities which depend only on the β's, but not on u or v.

The problem is to find the minimum of D, when the variables are subject to the conditions (5.4).

By the method of Lagrange's multipliers any stationary point (u_0, v_0) of D in the set (5.4) satisfies the equations

$$\begin{aligned} (a - \mu) u_0 + b v_0 &= 0 \\ b u_0 + (c - \mu) v_0 &= 0, \end{aligned} \tag{5.5}$$

where

$$\mu = a u_0^2 + 2 b u_0 v_0 + c v_0^2.$$

The stationary point is the minimum, if

$$a u^2 + 2 b u v + c v^2 \geq \mu$$

for every point (u, v) satisfying (5.4). Therefore, in particular, if

$$u = \sqrt{1 - \varepsilon^2}, \; v = \varepsilon, \text{ where } 0 < \varepsilon < 1,$$

then

$$a(1 - \varepsilon^2) + 2 b \varepsilon \sqrt{1 - \varepsilon^2} + c \varepsilon^2 \geq \mu,$$

whence, on passing to the limit $\varepsilon \to 0$,

$$a \geq \mu.$$

Similarly,

$$c \geq \mu.$$

But then (5.5) obviously cannot have a solution in positive numbers u_0, v_0, since $b > 0$, while the other coefficients are non-negative. (Since the determinant is zero, these latter coefficients are, in fact, positive too.) This conclucludes the proof that a decagon of the first type cannot have just four critical hexagons.

(c) All five hexagons (5.1) are critical.

In this case, $\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5, \; = \xi$, say, and the ratio

$$\phi_1 = \frac{D}{E} = \frac{D}{2\xi^2} = \frac{1}{2}(\beta_{12} + \beta_{23} + \beta_{34} + \beta_{45} + \beta_{51} + \beta_{12}\beta_{23} + \beta_{23}\beta_{34} + \beta_{34}\beta_{45} + \beta_{45}\beta_{51} + \beta_{51}\beta_{12})$$

is independent of ξ's. On expressing the β's in terms of s and t according to (2.12), we find that

$$2\phi_1 = \left(\sqrt{\frac{s}{s-1}} + \sqrt{\frac{t}{t-1}}\right)(1+\sqrt{st}) + \sqrt{st-1}\left(\frac{1}{\sqrt{s-1}} + \frac{1}{\sqrt{t-1}}\right) + \sqrt{st} + \frac{st-1}{\sqrt{(s-1)(t-1)}}\left(1 + \frac{\sqrt{s}+\sqrt{t}}{\sqrt{st-1}}\right). \tag{5.6}$$

Introduce the new independent variables

$$u = st, \quad w = \sqrt{(s-1)(t-1)}, \tag{5.7}$$

where, by (2.13)

$$u > 1, \quad w > 0. \tag{5.8}$$

Then (5.6) can be written as

$$2\phi_1 = \frac{1+\sqrt{u}}{w}(u-1+w^2+2w\sqrt{u})^{\frac{1}{2}} + \frac{u-1}{w} + \sqrt{u} + \frac{\sqrt{u-1}}{w}\left\{(u-1-w^2+2w)^{\frac{1}{2}} + (u+1-w^2+2\sqrt{u})^{\frac{1}{2}}\right\}.$$

For any fixed positive value of w, the right-hand side is a strictly increasing function of u. Therefore the minimum of ϕ_1 is attained for the least value of u compatible with this particular value of w. But when

$$st - s - t + 1 = w^2 = \text{const}$$

is given,

$$u = st$$

attains its smallest value if

$$s = t = w + 1.$$

On putting now $t = s$ in (5.6) we find that

$$2\phi_1 = 2(s+1)\sqrt{\frac{s}{s-1}} + 2\sqrt{s+1} + (2s+1) + 2\sqrt{\frac{s(s+1)}{s-1}}.$$

In order to obtain the minimum of this function, it is convenient to introduce the new variable

$$\sqrt{s+1} = z. \tag{5.9}$$

Then

$$\phi_1 = (z^2+z)\left(\sqrt{\frac{z^2-1}{z^2-2}} + 1\right) - \frac{1}{2} \tag{5.10}$$

and the condition

$$\frac{d\,\phi_1}{d\,z} = 0$$

for a stationary value becomes, in a rational form,

$$z^6 + 3\,z^5 - 3\,z^4 - 10\,z^3 + 6\,z + 2 = 0,$$

that is

$$(z^2 - z - 1)\left[z^2 + (2 - \sqrt{2})\,z - \sqrt{2}\right]\left[z^2 + (2 + \sqrt{2})\,z + \sqrt{2}\right] = 0.$$

Hence there are 6 possible stationary values, namely

$$z_1 = \frac{1 + \sqrt{5}}{2},\quad z_2 = \frac{1 - \sqrt{5}}{2},\quad z_3 = \frac{-2 + \sqrt{2} + \sqrt{6}}{2},$$

$$z_4 = \frac{-2 + \sqrt{2} - \sqrt{6}}{2},\quad z_5 = \frac{-2 - \sqrt{2} + \sqrt{6}}{2},\quad z_6 = \frac{-2 - \sqrt{2} - 6}{2}.$$

Since $s > 1$, it follows from (5.9) that only those values of z are admissible which are greater than $\sqrt{2}$. Only the first root

$$z_1 = \frac{1 + \sqrt{5}}{2} = \zeta, \text{ say},$$

fulfils this condition.

However, it does not correspond to a minimum. For since $\zeta^2 = \zeta + 1$, we find that

$$\phi_1 = \tfrac{5}{2}\,(2\,\zeta + 1) = \tfrac{5}{2}\,(2 + \sqrt{5}), \tag{5.11}$$

and consequently

$$Q = \frac{4}{1 + \phi_1^{-1}} = \frac{20}{19}(2\sqrt{5} - 1) = 3\cdot 655\ \ldots. \tag{5.12}$$

But this number is greater than the value

$$Q_4 = 3\cdot 62465\ \ldots$$

for a regular octagon, contrary to the inequality $Q_5 < Q_4$, proved in the general theory (*M.* § 9).

In fact, ζ is the value of z corresponding to the *regular* decagon, for which all β's and all ξ's are evidently equal. The relations (2.8) then become

$$\beta^2 = (\beta^2 - 1)^2,$$

whence

$$\beta^2 - \beta - 1 = 0$$

since $\beta > 1$. It follows that

$$\beta = \zeta = \frac{1 + \sqrt{5}}{2}.$$

Also

$$\xi_1^2 = \xi_2^2 = \xi_3^2 = \xi_4^2 = \xi_5^2 = \tfrac{1}{2} E$$

and

$$\phi_1 = \frac{D}{E} = \tfrac{1}{2}(\beta_{12} + \beta_{23} + \beta_{34} + \beta_{45} + \beta_{51} + \beta_{12}\beta_{23} + \beta_{23}\beta_{34} + \beta_{34}\beta_{45} + \beta_{45}\beta_{51} + \beta_{51}\beta_{12})$$

$$= \tfrac{5}{2}(\beta^2 + \beta) = \tfrac{5}{2}(\zeta^2 + \zeta) = \tfrac{5}{2}(2\zeta + 1),$$

as in (5.11).

6. **Decagons of the second type.** By the result just proved, the extreme decagon cannot be of the first type. In the present section, we shall discuss the question whether it can be of the second type. Accordingly we shall assume that exactly one of the critical hexagons is of the second class, say the hexagon H_{51}. Then

$$E = E_{51} = \xi_5^2 + \xi_1^2 + \xi_{51}^2. \tag{6.1}$$

Since E is the minimum value of the E_{ij}, it follows that, in particular,

$$E_{25} \geq E_{51}, \; E_{14} \geq E_{51},$$

whence, by (2.15) and (2.16),

$$\xi_2^2 \geq \xi_1^2 + \xi_{51}^2, \; \xi_4^2 \geq \xi_5^2 + \xi_{51}^2.$$

On adding these inequalities, we find that

$$\xi_2^2 + \xi_4^2 \geq \xi_5^2 + \xi_1^2 + 2\,\xi_{51}^2 > \xi_5^2 + \xi_1^2 + \xi_{51}^2$$

and so

$$E_{24} > E_{51}.$$

Hence H_{24} cannot be a critical hexagon, and every critical hexagon other than H_{51} belongs to the set

$$H_{13}, \; H_{41}, \; H_{25}, \; H_{35}. \tag{6.2}$$

By theorem 2, the critical hexagons, between them, involve all five suffixes. Hence

$$H_{25} \text{ and } H_{41}$$

are critical hexagons, since otherwise the suffixes 2 and 4 would not occur. Further also at least one of the hexagons H_{13}, H_{35} is critical, since the suffix 3 must occur. We must then distinguish two cases, according as only one, or both, of these two hexagons are critical.

(a) **Only one of H_{13} and H_{35} is critical, say H_{13}.**

Then

$$H_{51},\ H_{13},\ H_{41},\ H_{52}$$

are the only critical hexagons, and in particular

$$H_{13} < H_{35},\ \text{i. e. } E_{13} < E_{35},$$

whence by (2.15)

$$\xi_1 < \xi_5. \tag{6.3}$$

Since

$$E = E_{51} = E_{13} = E_{41} = E_{52},$$

we have by (2.16)

$$E = \frac{\beta_{51}^2 \xi_1^2 + 2\beta_{51}\xi_5\xi_1 + \beta_{51}^2\xi_5^2}{\beta_{51}^2 - 1} = \xi_1^2 + \xi_3^2 = \xi_1^2 + \xi_4^2 = \xi_5^2 + \xi_2^2. \tag{6.4}$$

Thus

$$\xi_5^2 + \frac{(\beta_{51}\xi_1 + \xi_5)^2}{\beta_{51}^2 - 1} = \xi_5^2 + \xi_2^2$$

and

$$\xi_1^2 + \frac{(\xi_1 + \beta_{51}\xi_5)^2}{\beta_{51}^2 - 1} = \xi_1^2 + \xi_3^2 = \xi_1^2 + \xi_4^2,$$

whence

$$\xi_2 = \frac{\beta_{51}\xi_1 + \xi_5}{\sqrt{\beta_{51}^2 - 1}},\quad \xi_3 = \xi_4 = \frac{\xi_1 + \beta_{51}\xi_5}{\sqrt{\beta_{51}^2 - 1}}. \tag{6.5}$$

In order to decide whether $Q(D)$ can attain its minimum for a decagon of this type, assume that the β's are fixed, that ξ_5 and ξ_1 are independent variables, and that ξ_2, ξ_3, ξ_4 are defined as functions of ξ_5 and ξ_1 by (6.5). The expression

$$\begin{aligned} D = {} & \beta_{12}\xi_1\xi_2 + \beta_{23}\xi_2\xi_3 + \beta_{34}\xi_3\xi_4 + \beta_{45}\xi_4\xi_5 + \beta_{51}\xi_5\xi_1 \\ & + \beta_{12}\beta_{23}\xi_1\xi_3 + \beta_{23}\beta_{34}\xi_2\xi_4 + \beta_{34}\beta_{45}\xi_3\xi_5 + \beta_{45}\beta_{51}\xi_4\xi_1 + \beta_{51}\beta_{12}\xi_5\xi_2 \end{aligned} \tag{2.7}$$

then becomes a quadratic form in ξ_5 and ξ_1, say

$$D = A\xi_5^2 + 2B\xi_5\xi_1 + C\xi_1^2, \tag{6.6}$$

where the coefficients A, B, C depend only on the β's. The argument will be based on the fact that

$$A - C > 0. \tag{6.7}$$

In order to prove this inequality we introduce the following notation: if $f(\xi_5, \xi_1)$ is any function of ξ_5 and ξ_1, put

$$[f(\xi_5, \xi_1)] = f(\xi_5, \xi_1) - f(\xi_1, \xi_5).$$

Evidently

$$[af(\xi_5, \xi_1) + bg(\xi_5, \xi_1)] = a[f(\xi_5, \xi_1)] + b[g(\xi_5, \xi_1)], \tag{6.8}$$

if a and b are constants. In particular, in virtue of (6.5),

$$\begin{aligned} -[\xi_1\xi_2] = [\xi_4\xi_5] = [\xi_3\xi_5] &= \frac{\beta_{51}}{\sqrt{\beta_{51}^2 - 1}}(\xi_5^2 - \xi_1^2), \\ -[\xi_1\xi_3] = -[\xi_4\xi_1] = [\xi_5\xi_2] &= \frac{1}{\sqrt{\beta_{51}^2 - 1}}(\xi_5^2 - \xi_1^2), \\ [\xi_3\xi_4] = -(\xi_5^2 - \xi_1^2),\ [\xi_4\xi_2] &= [\xi_2\xi_3] = [\xi_5\xi_1] = 0. \end{aligned} \tag{6.9}$$

Next, we evaluate $[D]$ in two different ways. First, from (6.6) it is obvious that

$$[D]/(\xi_5^2 - \xi_1^2) = A - C. \tag{6.10}$$

Secondly, by (2.7)

$$\begin{aligned} [D] = {} & \beta_{12}[\xi_1\xi_2] + \beta_{23}[\xi_2\xi_3] + \beta_{34}[\xi_3\xi_4] + \beta_{45}[\xi_4\xi_5] + \beta_{51}[\xi_5\xi_1] \\ & + \beta_{12}\beta_{23}[\xi_1\xi_3] + \beta_{23}\beta_{34}[\xi_2\xi_4] + \beta_{34}\beta_{45}[\xi_3\xi_5] + \beta_{45}\beta_{51}[\xi_4\xi_1] + \beta_{51}\beta_{12}[\xi_5\xi_2], \end{aligned}$$

whence, from (6.9)

$$\begin{aligned} [D]/(\xi_5^2 - \xi_1^2) &= \frac{\beta_{51}\beta_{34}\beta_{45} - \beta_{12}\beta_{23}}{\sqrt{\beta_{51}^2 - 1}} + \beta_{34} \\ &= \frac{\beta_{12}\beta_{23}}{\sqrt{\beta_{51}^2 - 1}}\left(\beta_{45}\frac{\beta_{34}\beta_{51}}{\beta_{12}\beta_{23}} - 1\right) + \beta_{34}. \end{aligned}$$

Since, by (2.9),

$$\frac{\beta_{34}\beta_{51}}{\beta_{12}\beta_{23}} = \frac{\beta_{45}}{\beta_{45}^2 - 1},$$

$$[D]/(\xi_5^2 - \xi_1^2) = \frac{\beta_{12}\beta_{23}}{(\beta_{45}^2 - 1)\sqrt{\beta_{51}^2 - 1}} + \beta_{34}, \tag{6.11}$$

which is clearly positive.

Comparing (6.10) with (6.11), we conclude that

$$A - C > 0.$$

We now return to the question whether the function

$$\phi(D) = \frac{D}{E}$$

can attain its minimum for values of ξ_5, ξ_1 satisfying

$$\xi_5 > \xi_1 > 0. \tag{6.12}$$

Since by (6.4) and (6.6)

$$\frac{D}{E} = \frac{\beta_{51}^2 - 1}{\beta_{51}} \frac{A\,\xi_5^2 + 2\,B\,\xi_5\,\xi_1 + C\,\xi_1^2}{\beta_{51}\,\xi_5^2 + 2\,\xi_5\,\xi_1 + \beta_{51}\,\xi_1^2},$$

the problem is equivalent to deciding whether the quadratic form

$$F(\xi_5, \xi_1) = A\,\xi_5^2 + 2\,B\,\xi_5\,\xi_1 + C\,\xi_1^2$$

assumes its minimum if, in addition to (6.12), the variables satisfy the condition

$$\beta_{51}\,\xi_5^2 + 2\,\xi_5\,\xi_1 + \beta_{51}\,\xi_1^2 - 1 = 0. \tag{6.13}$$

By means of Lagrange multipliers, it is found that any such solution,

$$\xi_5 = \tilde{\xi}_5, \ \ \xi_1 = \tilde{\xi}_1 \tag{6.14}$$

say, satisfies the linear equations

$$\begin{aligned} (A - \lambda\,\beta_{51})\,\tilde{\xi}_5 + (B - \lambda)\,\tilde{\xi}_1 &= 0 \\ (B - \lambda)\,\tilde{\xi}_5 + (C - \lambda\,\beta_{51})\,\tilde{\xi}_1 &= 0, \end{aligned} \tag{6.15}$$

where λ is the assumed minimum of F. Thus

$$F(\xi_5, \xi_1) \geq \lambda$$

for any permissible pair of values ξ_5, ξ_1. Let, in particular,

$$\xi_5^{(n)}, \ \xi_1^{(n)} \qquad (n = 1, 2, 3, \ldots)$$

be two sequences of numbers, such that

$$\xi_5^{(n)} > \xi_1^{(n)} > 0$$

$$\beta_{51}\,(\xi_5^{(n)})^2 + 2\,\xi_5^{(n)}\,\xi_1^{(n)} + \beta_{51}\,(\xi_1^{(n)})^2 - 1 = 0$$

and

$$\lim_{n \to \infty} \xi_5^{(n)} = \frac{1}{\sqrt{\beta_{51}}}, \quad \lim_{n \to \infty} \xi_1^{(n)} = 0.$$

Then

$$\lambda \leq \lim_{n \to \infty} F(\xi_5^{(n)}, \xi_1^{(n)}) = F\left(\frac{1}{\sqrt{\beta_{51}}}, 0\right) = \frac{A}{\beta_{51}},$$

i. e.

$$A - \lambda\,\beta_{51} \geq 0.$$

Since $\tilde{\xi}_5$ and $\tilde{\xi}_1$ are positive, we conclude from the first equation (6.15) that

$$B - \lambda \leq 0,$$

and therefore from the second equation (6.15) that

$$C - \lambda\beta_{51} \geq 0.$$

On multiplying the two equations (6.15) by ξ_5 and ξ_1 respectively and subtracting we obtain

$$(A - \lambda\beta_{51})\,\xi_5^2 = (C - \lambda\beta_{51})\,\xi_1^2,$$

whence by (6.3)

$$A - \lambda\beta_{51} < C - \lambda\beta_{51},$$

contrary to (6.7).

Hence the extreme decagon cannot have only the critical hexagons

$$H_{51},\ H_{13},\ H_{41},\ H_{52}.$$

(b) **Both H_{13} and H_{35} are critical.**

Then the complete set of critical hexagons is

$$H_{51},\ H_{13},\ H_{41},\ H_{52},\ H_{35}.$$

Hence

$$\frac{\beta_{51}^2\,\xi_1^2 + 2\,\beta_{51}\,\xi_5\,\xi_1 + \beta_{51}^2\,\xi_5^2}{\beta_{51}^2 - 1} = \xi_1^2 + \xi_3^2 = \xi_1^2 + \xi_4^2 = \xi_5^2 + \xi_2^2 = \xi_3^2 + \xi_5^2,$$

and therefore

$$\xi_1 = \xi_5 = \xi, \text{ say}$$

and

$$\xi_2 = \xi_3 = \xi_4 = \frac{\beta_{51} + 1}{\sqrt{\beta_{51}^2 - 1}}\,\xi.$$

On substituting these values in (2.7), we find that

$$D/\xi^2 = \beta_{51} + (\beta_{12} + \beta_{45} + \beta_{12}\beta_{23} + \beta_{34}\beta_{45} + \beta_{45}\beta_{51} + \beta_{51}\beta_{12})\,\frac{\beta_{51} + 1}{\sqrt{\beta_{51}^2 - 1}}$$
$$+ (\beta_{23} + \beta_{34} + \beta_{23}\beta_{34})\,\frac{(\beta_{51} + 1)^2}{\beta_{51}^2 - 1}.$$

Also

$$E = \xi_1^2 + \xi_3^2 = \frac{2\,\beta_{51}}{\beta_{51} - 1}\,\xi^2.$$

On substituting for the β's in terms of s and t in accordance with (2.12), these expressions become

$$D/\xi^2 = \sqrt{st} + \left\{\left(\sqrt{\frac{s}{s-1}} + \sqrt{\frac{t}{t-1}}\right)(1 + \sqrt{st}) + \frac{\sqrt{st-1}}{\sqrt{(s-1)(t-1)}}(\sqrt{s} + \sqrt{t})\right\}\frac{\sqrt{st}+1}{\sqrt{st-1}}$$
$$+ \left\{\sqrt{\frac{st-1}{t-1}} + \sqrt{\frac{st-1}{s-1}} + \frac{st-1}{\sqrt{(s-1)(t-1)}}\right\}\frac{(\sqrt{st}+1)^2}{st-1}, \qquad (6.16)$$

$$E/\xi^2 = \frac{2\sqrt{st}}{\sqrt{st}-1}. \tag{6.17}$$

As on p. 335 introduce again the variables

$$u = st, \quad w = \sqrt{(s-t)(t-1)}.$$

Then

$$\phi_2 = D/E$$

is given by

$$\begin{aligned} 2\,\phi_2 = (\sqrt{u}-1) + K(u, w)\sqrt{1-\frac{1}{u}} + L(u, w)(\sqrt{u}+1)\sqrt{1-\frac{1}{u}} \\ + \frac{1}{w}(\sqrt{u}+1)^2\left(1-\frac{1}{\sqrt{u}}\right), \end{aligned} \tag{6.18}$$

where, for shortness,

$$\begin{aligned} K(u, w) &= \frac{1}{w}\left\{(1+\sqrt{u})(u-1+w^2+2w\sqrt{u})^{\frac{1}{2}} + (u-1)^{\frac{1}{2}}(u+1-w^2+2\sqrt{u})^{\frac{1}{2}}\right\}, \\ L(u, w) &= \frac{1}{w}(u-1-w^2+2w)^{\frac{1}{2}}. \end{aligned} \tag{6.19}$$

For a fixed value of w, $K(u, w)$ and $L(u, w)$ are strictly increasing functions of u, and so is ϕ_2, by (6.18). Hence, as on p. 335, ϕ_2 can attain its minimum only if

$$s = t = w + 1$$

and therefore

$$u = s^2; \; w = s - 1.$$

The expression for ϕ_2 then becomes

$$\phi_2 = \tfrac{1}{2}(s-1) + \left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)\left\{\left(\sqrt{s+1}+\sqrt{\frac{s+1}{s}}\right) + 1 + \tfrac{1}{2}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)\right\},$$

where the variable s is restricted by the condition

$$s > 1. \tag{6.20}$$

In this range of s, the functions

$$\sqrt{s}+\frac{1}{\sqrt{s}} \text{ and } \sqrt{s+1}+\sqrt{\frac{s+1}{s}}$$

are strictly increasing, since their derivatives

$$\frac{1}{2}\frac{1}{\sqrt{s}}\left(1-\frac{1}{s}\right) \text{ and } \frac{1}{2}\frac{1}{\sqrt{s+1}}\left(1-\frac{1}{s^{3/2}}\right)$$

are always positive. Hence ϕ_2 is also a strictly increasing function of s and therefore cannot assume a minimum in the open range (6.20).

This concludes the proof that the extreme decagon cannot be of the second type.

7. **Decagons of the third type.** As the existence of an extremum is guaranteed by the general theory (M, § 8), there must exist an extreme decagon of the third type, since all other possibilities have already been ruled out.

By theorem 1, a decagon of the third type has two critical hexagons of the form

$$H_{i-1,i}, \; H_{i,i+1}.$$

There is no loss of generality in assuming that $i = 3$, so that

$$H_{23}, \; H_{34}$$

are critical hexagons. The remaining critical hexagons are all of the first class.

Since

$$E = E_{23} = E_{34} = \min\{E_{ij}\},$$

we have

$$E_{24} \geq E_{23}, \; E_{24} \geq E_{34},$$

i. e.

$$\xi_2^2 + \xi_4^2 \geq \xi_2^2 + \xi_3^2 + \xi_{23}^2, \; \xi_2^2 + \xi_4^2 \geq \xi_3^2 + \xi_4^2 + \xi_{34}^2,$$

and so

$$\xi_4 > \xi_3, \; \xi_2 > \xi_3.$$

It follows that

$$\xi_4^2 + \xi_1^2 > \xi_1^2 + \xi_3^2, \; \xi_5^2 + \xi_2^2 > \xi_3^2 + \xi_5^2,$$

or

$$E_{41} > E_{13}, \; E_{52} > E_{35}.$$

Thus H_{41} and H_{52} are certainly not critical, and any further critical hexagons belong to the set

$$H_{13}, \; H_{24}, \; H_{35}.$$

All of these hexagons are in fact critical, H_{13} and H_{35}, because the suffixes 1 and 5 must be represented, and H_{24}, since otherwise each critical hexagon would have 3 as one of its suffixes, contrary to theorem 4 (p. 331).

Hence

$$E = E_{23} = E_{34} = E_{13} = E_{24} = E_{35}. \tag{7.1}$$

The four equations

$$E_{13} = E_{35},\ E_{13} = E_{23},\ E_{24} = E_{34},\ E_{24} = E_{23}$$

allow to express the ratios of the ξ's in terms of the β's, viz.

$$\begin{aligned} \xi_1 = \xi_5 &= \mu(\gamma_{23} + \beta_{23}\beta_{34} + \gamma_{34}), \\ \xi_2 &= \mu(\beta_{23} + \gamma_{23}\beta_{34}), \\ \xi_3 &= \mu(\gamma_{23}\gamma_{34} - 1), \\ \xi_4 &= \mu(\beta_{34} + \beta_{23}\gamma_{34}), \end{aligned} \tag{7.2}$$

where μ is an arbitrary factor, and

$$\gamma_{23} = \sqrt{1 - \beta_{23}^2},\quad \gamma_{34} = \sqrt{1 - \beta_{34}^2}.$$

We again express the β's and γ's in terms of s and t. By (2.12)

$$\gamma_{23} = \sqrt{\frac{t(s-1)}{t-1}},\quad \gamma_{34} = \sqrt{\frac{s(t-1)}{s-1}}.$$

The equations (7.2) then become

$$\begin{aligned} \xi_1 = \xi_5 &= \xi\,\frac{(\sqrt{s}+1)(\sqrt{t}+1)}{\sqrt{st}+1}, \\ \xi_2 &= \xi\,\frac{\sqrt{s-1}\,(\sqrt{t}+1)}{\sqrt{st-1}}, \\ \xi_3 &= \xi\,\frac{\sqrt{(s-1)(t-1)}}{\sqrt{st}+1}, \\ \xi_4 &= \xi\,\frac{\sqrt{t-1}\,(\sqrt{s}+1)}{\sqrt{st-1}}, \end{aligned} \tag{7.3}$$

where

$$\xi = \frac{\mu(st-1)}{\sqrt{(s-1)(t-1)}}$$

is an arbitrary factor. Further

$$E = \xi_2^2 + \xi_4^2 = 2\,\xi^2\,\frac{(\sqrt{s}+1)(\sqrt{t}+1)}{\sqrt{st}+1}.$$

After some elementary calculations, $\phi_3 = D/E$ is obtained in the form

$$\phi_3 = \left(\sqrt{\frac{u}{u-1}} - \frac{1}{2}\,\frac{1}{\sqrt{u+1}} + 1\right)(\sqrt{s}+1)(\sqrt{t}+1) - 1, \tag{7.4}$$

where

$$u = st.$$

When u is fixed, the first factor, viz.

$$\sqrt{\frac{u}{u-1}} - \frac{1}{2}\frac{1}{\sqrt{u}+1} + 1$$

is constant, while

$$(\sqrt{s}+1)(\sqrt{t}+1)$$

assumes its smallest value when

$$s = t.$$

Thus the problem therefore reduces to finding the minimum of

$$\phi_3 = \left(\frac{s}{\sqrt{s^2-1}} - \frac{1}{2}\frac{1}{s+1} + 1\right)(\sqrt{s}+1)^2 - 1, \tag{7.5}$$

when

$$s > 1.$$

The equation

$$\frac{d\phi_3}{ds} = \frac{\sqrt{s}+1}{(s^2-1)^{3/2}}(s^{5/2} - 2s^{1/2} - 1) + \frac{\sqrt{s}+1}{2\sqrt{s}(s+1)^2}\{2(s+1)^2 + \sqrt{s} - 1\} = 0 \tag{7.6}$$

has exactly one positive root, namely

$$\sigma = 1\cdot 43555 \ldots. \tag{7.7}$$

When s passes this value in the positive direction, $d\phi_3/ds$ changes from negative to positive values. Hence the stationary value σ is, in fact, the minimum. On substituting σ for s, we obtain

$$\phi_3 = \phi_3^{(0)} = 9\cdot 574521 \ldots$$

and

$$Q = Q_5 = 3\cdot 62173227 \ldots. \tag{7.8}$$

In agreement with the general theory, this constant is smaller than the corresponding constant for octagons, namely

$$Q_4 = 3\cdot 62465471 \ldots. \tag{7.9}$$

8. **The shape of an extreme decagon.**

We next evaluate the parameters $\beta_{i,i+1}$ for an extreme decagon. We have

$$s = t = \sigma,$$

where σ is the number (7.7). By (2.12),

$$\beta_{12} = \beta_{45} = \sqrt{\frac{\sigma}{\sigma - 1}}, \quad \beta_{23} = \beta_{34} = \sqrt{\sigma + 1}, \quad \beta_{51} = \sigma. \tag{8.1}$$

Next, by (7.3), we obtain the ratios of the ξ_i in the form

$$\xi_1 = \xi_5 = \xi \frac{(\sqrt{\sigma} + 1)^2}{\sigma + 1}, \quad \xi_2 = \xi_4 = \xi \frac{\sqrt{\sigma + 1}}{\sqrt{\sigma} + 1}, \quad \xi_3 = \xi \frac{\sigma - 1}{\sigma + 1}. \tag{8.2}$$

Finally, the ratios of the quantities a_{ij}, are found from (2.6).

Affine-equivalent decagons have the same ratio $Q(D)$. Therefore, two of the vectors (1.3), say $\mathbf{r}_5$ and $\mathbf{r}_1$ can be chosen arbitrarily, as long as the condition of convexity (1.5) is satisfied. Then

$$\mathbf{r}_i = \mu_i \mathbf{r}_5 + \nu_i \mathbf{r}_1 \qquad (i = 1, 2, 3, 4, 5) \tag{8.3}$$

where

$$\mu_i = \frac{a_{1i}}{a_{15}}, \quad \nu_i = \frac{a_{i5}}{a_{15}} \qquad (i = 1, 2, 3, 4, 5) \tag{8.4}$$

$$(a_{11} = a_{55} = 0.)$$

From (2.6), (8.1) and (8.2), we find that

$$\begin{aligned} \mu_2 = \nu_4 &= \frac{1}{\sigma + \sqrt{\sigma}} \sqrt{\frac{\sigma + 1}{\sigma - 1}}, \\ \mu_3 = \nu_3 &= \frac{\sqrt{\sigma^2 - 1}}{(\sqrt{\sigma} + 1)^2 \sqrt{\sigma}}, \\ \mu_4 = \nu_2 &= \frac{\sqrt{\sigma}}{\sigma + \sqrt{\sigma}} \sqrt{\frac{\sigma + 1}{\sigma - 1}}. \end{aligned} \tag{8.5}$$

Also (see Fig. 1)

$$\mathbf{p} = \overrightarrow{OP_1} = -\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5) = -\frac{1}{2} \frac{\sigma - 1 + \sqrt{\sigma^2 - 1}}{\sigma - 1} (\mathbf{r}_5 + \mathbf{r}_1), \tag{8.6}$$

and the remaining vertices are then obtained from (8.3) and (8.4) (Fig. 1).

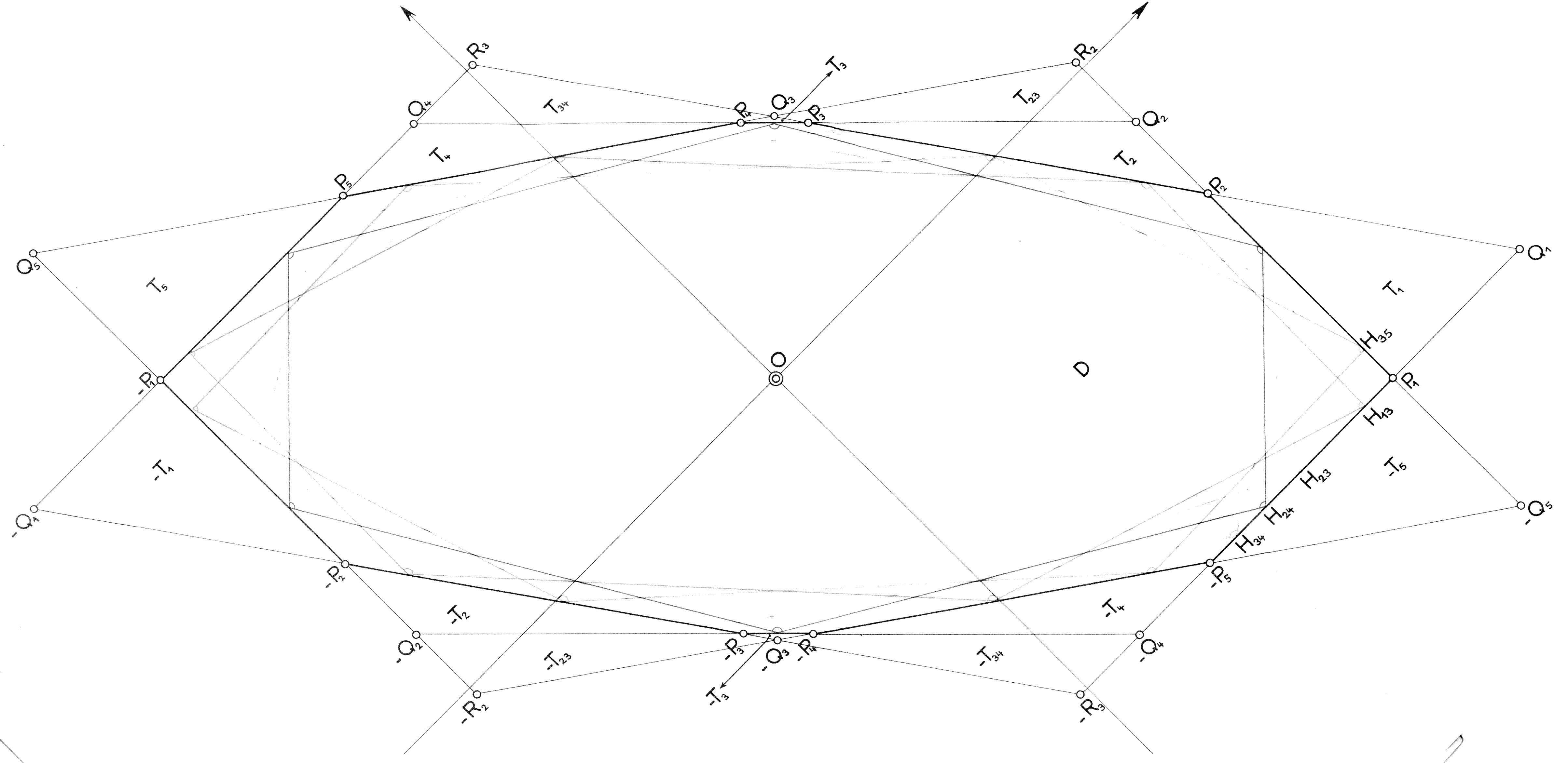
In Fig. 4, we have constructed an extreme decagon where

$$\mathbf{r}_5 = (\sigma - 1 - \sqrt{\sigma^2 - 1}, 0), \quad \mathbf{r}_1 = (0, -\sigma + 1 + \sqrt{\sigma^2 - 1}), \tag{8.7}$$

and therefore

$$\overrightarrow{OP_1} = (1, -1). \tag{8.8}$$

The diagram also shows the intersections Q_i and R_i of non-adjacent sides as indicated in Figs. 2 and 3. The position vectors of these points are given by


R3
R2
T3
Q4
T34
P4
Q3
P3
T23
Q2
T4
T2
P5
P2
Q5
Q1
T5
T1
H35
O
D
-P1
P1
H13
-T1
-T5
H23
H24
-Q1
-Q5
H34
-P2
-P5
-T2
-T4
-Q2
-P3
-Q3
-P4
-Q4
-T23
-T34
-T3
-R2
-R3

$$\overrightarrow{P_i Q_i} = \frac{a_{i,i+1}}{a_{i-1,i+1}} \mathbf{r}_{i-1}, \quad \overrightarrow{P_i R_i} = \frac{a_{i,i+2} + a_{i+1,i+2}}{a_{i-1,i+2}} \mathbf{r}_{i-1}$$

respectively.

The values of the co-ordinates of the 15 points P_i, Q_i, R_i ($i = 1, 2, 3, 4, 5$) are contained in Tables[1] 1—3, where the symbol $[i, j]$ denotes the intersection of the lines L_i and L_j.

Table 1.

	$P_1 = [-5, 1]$	$P_2 = [1, 2]$	$P_3 = [2, 3]$	$P_4 = [3, 4]$	$P_5 = [4, 5]$
x	1	1	0.4663	0.3605	−0.4056
y	−1	−0.4056	0.3605	0.4663	1

Table 2.

	$Q_1 = [-5, 2]$	$Q_2 = [1, 3]$	$Q_3 = [2, 4]$	$Q_4 = [3, 5]$	$Q_5 = [4, -1]$
x	1.4140	1	0.4229	−0.1731	−1
y	−1	−0.1731	0,4229	1	1.4140

Table 3.

	$R_1 = [-5, 3]$	$R_2 = [1, 4]$	$R_3 = [2, 5]$	$R_4 = [3, -1]$	$R_5 = [4, -2]$
x	1.8269	1	0.0209	−1	−2.3646
y	−1	0.0209	1	1.8269	2.3646

9. **The smoothed decagon.** In our notation, the extreme decagon has the 5 critical hexagons

$$H_{13}, \quad H_{35}, \quad H_{34}, \quad H_{24}, \quad H_{23}. \tag{9.1}$$

From the general theory it is known (M, § 4) that the mid-points of the sides of these hexagons define the critical lattices of the decagon. Denote the mid-points of any one of the critical hexagons by

$$\pm A, \quad \pm B, \quad \pm C$$

[1] In order to save space, only four places of decimals are given, but the calculations were actually carried out with greater accuracy.

then, with suitable notation,

$$A + B = C$$

and[1]

$$(A, B) = \Delta(D), \tag{9.2}$$

$\Delta(D)$ being the minimum determinant of the decagon.

In the reference system (8.7), the co-ordinates of the mid-points of the sides of the critical hexagons (9.1) are as follows[2] (see Fig. 4):

(1) H_{13}: $\pm A_1, \ \pm B_1, \ \pm C_1, \quad C_1 = A_1 + B_1,$

where

$$A_1 = -\tfrac{1}{2}(Q_3 + P_5) = (-\cdot 0086, \ -\cdot 7114),$$

$$B_1 = \tfrac{1}{2}(Q_1 + Q_3) = (\cdot 9185, \ -\cdot 2886).$$

(2) H_{35}: $\pm A_2, \ \pm B_2, \ \pm C_2, \quad C_2 = A_2 + B_2,$

where

$$A_2 = -\tfrac{1}{2}(Q_3 + Q_5) = (\cdot 2886, \ -\cdot 9185),$$

$$B_2 = \tfrac{1}{2}(P_2 + Q_3) = (\cdot 7114, \ \cdot 0086).$$

(3) H_{34}: $\pm A_3, \ \pm B_3, \ \pm C_3, \quad C_3 = A_3 + B_3,$

where

$$A_3 = \tfrac{1}{2}(P_1 - R_3) = (\cdot 4896, \ -1),$$

$$B_3 = \tfrac{1}{2}(P_2 + R_3) = (\cdot 5104, \ \cdot 2972).$$

(4) H_{24}: $\pm A_4, \ \pm B_4, \ \pm C_4, \quad C_4 = A_4 + B_4,$

where

$$A_4 = \tfrac{1}{2}(P_1 - Q_4) = (\cdot 5866, \ -1),$$

$$B_4 = \tfrac{1}{2}(Q_2 + Q_4) = (\cdot 4134, \ \cdot 4134).$$

(5) H_{23}: $\pm A_5, \ \pm B_5, \ \pm C_5, \quad C_5 = A_5 + B_5,$

where

$$A_5 = \tfrac{1}{2}(-P_5 + P_1) = (\cdot 7028, \ -1),$$

$$B_5 = \tfrac{1}{2}(R_2 + P_5) = (\cdot 2972, \ \cdot 5104).$$

Note that

$$(A_i, B_i) = \Delta(D) = \cdot 655935 \ldots . \tag{9.3}$$

Just as in the case of the extreme octagon (M, § 12), we can construct an *irreducible* convex sub-domain D' of D, of the same minimum determinant, but of smaller area, and hence satisfying $Q(D') < Q(D)$.

[1] The bracket denotes again the determinant of A and B.

[2] See footnote on p. 347.

The irreducible domain is constructed as follows. Consider, say, the hexagons H_{13} and H_{35}. The mid-points of their sides are

$$\pm A_1, \ \pm B_1, \ \pm C_1$$

and

$$\pm A_2, \ \pm B_2, \ \pm C_2$$

respectively. Let X be a variable point on the line-segment $A_1 A_2$, and let the point Y on the line-segment $B_1 B_2$ be defined by the condition that

$$(X, Y) = \Delta(D).$$

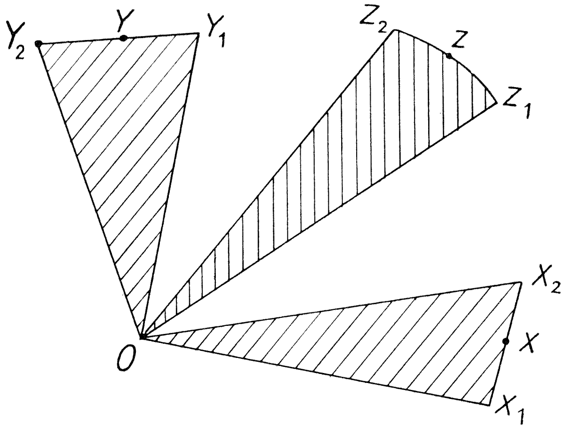


Fig. 5.

The point

$$Z = X + Y$$

then describes a hyperbolic arc which cuts off the vertex P_1 of D and touches the two sides which meet at P_1. We carry out analogous constructions for each of the other vertices by taking other pairs of hexagons. The resulting figure is convex and symmetrical in O.

We now give a brief analytical treatment of this construction (Fig. 5). Suppose

$$X_1, \ X_2, \ Y_1, \ Y_2 \tag{9.4}$$

are four given points such that

$$(X_1, Y_1) = (X_2, Y_2) = \Delta(D), \tag{9.5}$$

and put

$$\alpha = (X_1, Y_2), \quad \beta = (X_2, Y_1). \tag{9.6}$$

Let

$$X = (1 - x) X_1 + x X_2 \qquad (0 \leq x \leq 1)$$

and

$$Y = (1 - y) Y_1 + y Y_2 \qquad (0 \leq y \leq 1)$$

be two points on the line-segments $X_1 X_2$ and $Y_1 Y_2$, respectively, such that

$$(X, Y) = \Delta(D) = \delta, \text{ say.}$$

Then

$$y = -\frac{(\beta - \delta) x}{(2\delta - \alpha - \beta) x - (\delta - \alpha)}. \tag{9.7}$$

When X describes the segment $X_1 X_2$, the point Y moves along the segment $Y_1 Y_2$, but the point

$$Z = X + Y = (1 - x) X_1 + x X_2 + (1 - y) Y_1 + y Y_2 \tag{9.8}$$

describes an arc joining the points

$$Z_1 = X_1 + Y_1 \text{ and } Z_2 = X_2 + Y_2.$$

The parametric equations of this curve are obtained from (9.7) and (9.8) by substituting for y in terms of x in accordance with (9.7).

The area of the sector $O Z_1 Z_2$ is given by

$$\frac{1}{2}\int_0^1 \left(Z, \frac{dZ}{dx}\right) dx = \frac{1}{2}(X_1, X_2) + \frac{1}{2}(Y_1, Y_2) - \frac{(\delta - \alpha)(\beta - \delta)}{2\delta - \alpha - \beta} \log \frac{\delta - \alpha}{\beta - \delta},$$

and the total area, $\frac{1}{2}\Omega$ say, of the shaded part of Fig. 5 is

$$\frac{1}{2}\Omega = (X_1, X_2) + (Y_1, Y_2) - \frac{(\delta - \alpha)(\beta - \delta)}{2\delta - \alpha - \beta} \log \frac{\delta - \alpha}{\beta - \delta}. \tag{9.9}$$

This formula is applied to the five pairs of arcs which cut off the five pairs of opposite vertices of D. The result is summarized in Table 4, where the first entry in each row specifies the hexagons which are moved into each other.

The total area of the smoothed decagon D' is then

$$D' = \Sigma\, \Omega = 2 \cdot 367756 \ldots.$$

As the minimum determinant has not been altered, we finally find that

$$Q_5' = \frac{D'}{\Delta(D')} = \frac{2 \cdot 367756}{\cdot 655935} = 3 \cdot 60974 \ldots. \tag{9.10}$$

Table 4.

	X_1	X_2	Y_1	Y_2	Ω
$H_{13} \to H_{35}$	A_1	A_2	B_1	B_2	.604134
$H_{23} \to H_{13}$	A_5	C_1	B_5	$-A_1$	·579743
$H_{34} \to H_{24}$	C_3	C_4	$-A_3$	$-A_4$	.302068
$H_{24} \to H_{23}$	C_4	C_5	$-A_4$	$-A_5$	.302068
$H_{35} \to H_{34}$	B_2	B_3	$-C_2$	$-C_3$	·579743
					2.367756

This value[1] is, of course, smaller than the number Q_4 obtained in (7.8), but it is slightly *greater* than the corresponding ratio for the smoothed octagon, which is (M, § 12)

$$Q_4' = 3 \cdot 609656737 \ldots .$$

This fact seems to support the conjecture that Q_4' is actually the minimum of $Q(K)$ for all convex domains K.

[1] We are greatly indebted to Mr. D. F. Ferguson, M. A. for helping us with most of the numerical work of this paper.