FAREY SECTION IN $k(i)$ AND $k(\rho)$

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CONTENTS

An important tool in the study of the approximation to real numbers by rational numbers is provided by the theory of Farey sections. We develop in this paper an analogous theory for Farey fractions whose numerators and denominators are integers in the quadratic field $k(i)$ or in the quadratic field $k(\rho)$, where $\rho^2+\rho+1=0$. This leads us to results on approximation to a complex number by numbers of $k(i)$ or $k(\rho)$. In particular, we obtain new proofs of two theorems of Minkowski (theorems XIV and XV), which it is hoped are more transparent than those given by Minkowski himself and by Hlawka.

1. Introduction

Throughout this paper R, k, Ω will denote respectively the ring of rational integers, the field of rational numbers and the field of complex numbers. We put

$$i=\sqrt{(-1)}, \quad \rho=\frac{-1+i\sqrt{3}}{2}=\exp\frac{2\pi i}{3}.$$

In the usual way $R(i)$, $R(\rho)$, $k(i)$, $k(\rho)$ will denote the extensions of R and k by i and ρ. It is well known that $R(i)$ and $R(\rho)$ are the rings of integers in $k(i)$ and $k(\rho)$ respectively, and that all ideals in $R(i)$ and $R(\rho)$ are principal ideals.†

In part I of this paper we extend the notion of Farey section to $k(i)$ and $k(\rho)$ and study its properties; in part II we use the theory of part I to discuss the approximation of complex numbers by numbers of $k(i)$ and $k(\rho)$ and the related problem of the minimum of

$$\max\{|\alpha\xi+\beta\eta|, |\gamma\xi+\delta\eta|\}$$

for $\xi, \eta \in R(i)$ or $\in R(\rho)$, where $\alpha, \beta, \gamma, \delta \in \Omega$ are given. Finally, in part III we discuss further

† Indeed, the Euclidean algorithm holds in both.

certain regions of the Gauss plane related to Farey section which are introduced in part I. Apart from the definitions, part III is practically independent of parts I and II.

This paper may be regarded as carrying out in detail a programme sketched by Hurwitz (1891, § 8) but apparently not carried out either by himself or others. The thesis of Made (1903) claims to generalize Farey section to $k(i)$, but the generalization is on quite different lines from that adopted here and is rather artificial.

2. Preliminary discussion

If $N \geqslant 1$ is a positive number we denote by $\mathfrak{H}_N^*$ the set of all rational fractions which, when expressed in their lowest terms, have denominators not exceeding N. For example, $\mathfrak{H}_3^*$ consists of the infinite sequence of fractions

$$\ldots, -\tfrac{4}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1, \ldots.$$

We also denote by $\mathfrak{H}_N$ the set of all rational fractions which, when expressed in their lowest terms, have both numerators and denominators not exceeding N. Thus $\mathfrak{H}_3$ consists of the fifteen fractions

$$-3, -2, -\tfrac{3}{2}, -1, -\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{3}{2}, 2, 3,$$

together with the improper fraction $\infty = \frac{1}{0}$. Since $\mathfrak{H}_N^*$ has period 1, we may speak of $\mathfrak{H}_N^*$ and $\mathfrak{H}_N$ as the periodic and non-periodic Farey sections respectively. We may consider $\mathfrak{H}_N^*$ and $\mathfrak{H}_N$ as represented by points on the x-axis.

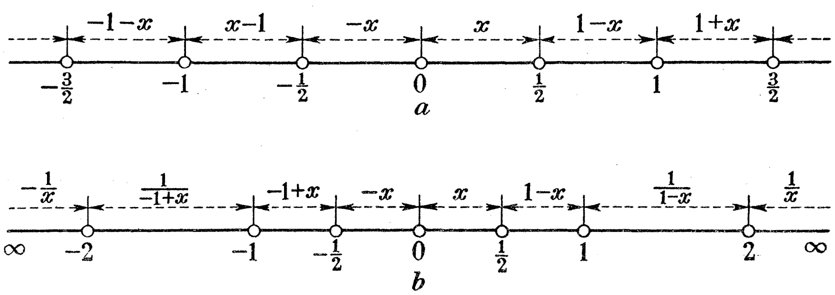

Figure 1

Clearly, $\mathfrak{H}_N^*$ is mapped into itself by the two sets of symmetries

$$x' = x+n, \quad x' = -x+n,$$

where n is any integer. Hence all of $\mathfrak{H}_N^*$ can be derived from the stretch between 0 and $\frac{1}{2}$ by translation and/or taking mirror images as in figure 1 *a*.

The only symmetries of $\mathfrak{H}_N$ are the two following:

$$x' = -x, \quad x' = \pm 1/x.$$

Further, clearly $\mathfrak{H}_N$ and $\mathfrak{H}_N^*$ coincide in the interval $|x| \leqslant 1$. Hence again $\mathfrak{H}_N$ can be built up from the stretch between 0 and $\frac{1}{2}$ as in figure 1 *b*.

The Farey sections $\mathfrak{H}_N^*$ were introduced to study the approximation of irrationals by rationals. It is convenient to define for each irreducible fraction a/b of $\mathfrak{H}_N^*$ the set $\mathfrak{R}^*(a, b)$ of x for which a/b is the best approximation in the sense that $|bx-a| \leqslant |b'x-a'|$ for any other irreducible fraction a'/b' in $\mathfrak{H}_N^*$. We have the following known theorems:

THEOREM A*.† *Let a'/b', a/b, a''/b'' be three consecutive reduced fractions of $\mathfrak{H}_N^*$, where $b, b', b'' \geqslant 0$. Then $\mathfrak{R}^*(a, b)$ is the line segment*

$$x \in \mathfrak{R}^*(a,b) \quad .\equiv. \quad \frac{a'+a}{b'+b} \leqslant x \leqslant \frac{a+a''}{b+b''}.$$

The fractions of the type $\frac{a+a'}{b+b'}$, where a'/b', a/b are consecutive, play an important part in the theory and are called the 'medians'.

THEOREM B*. *The necessary and sufficient condition that the reduced fractions a/b, a'/b' of $\mathfrak{H}_N^*$ be consecutive $(b, b' \geqslant 0)$, is that simultaneously*

(i) $|ab'-a'b| = 1.$

(ii) *the median* $\frac{a'+a}{b'+b}$ *is not in* $\mathfrak{H}_N^*$.

The condition (ii) is clearly necessary since $\frac{a'+a}{b'+b}$ lies between a'/b' and a/b.

THEOREM C*. *All terms of $\mathfrak{H}_{N+1}^*$ which are not already in $\mathfrak{H}_N^*$, are medians $\frac{a'+a}{b'+b}$ of consecutive terms a'/b', a/b of $\mathfrak{H}_N$.*

For example, the successive medians of $\mathfrak{H}_3^*$ are

$$\frac{0+1}{1+3} = \frac{1}{4} \quad \text{between} \quad 0 \text{ and } \frac{1}{3},$$

$$\frac{1+1}{3+2} = \frac{2}{5} \quad \text{between} \quad \frac{1}{3} \text{ and } \frac{1}{2},$$

$$\frac{1+2}{2+3} = \frac{3}{5} \quad \text{between} \quad \frac{1}{2} \text{ and } \frac{2}{3},$$

etc. Of these only $\frac{1}{4}$ belongs to $\mathfrak{H}_4^*$. Hence $\mathfrak{H}_4^*$ is

$$\ldots, -\tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \ldots.$$

Finally, we have a theorem on approximation:

THEOREM D*. *If $x \in \mathfrak{R}^*(a, b)$, then $|bx-a| \leqslant 1/N$.*

This may be proved by observing that the worst x in the interval $\mathfrak{R}^*(a, b)$ to approximate are those at the ends, i.e. the medians. It may then be verified that the medians do in fact satisfy the required inequality.

From Theorem D* we may derive a more general theorem on linear forms:

THEOREM E. *Let a, b, c, d be real numbers, $ad-bc \neq 0$. Then there are integers $(x, y) \neq (0, 0)$ such that simultaneously*

$$|ax+by| \leqslant \surd|ad-bc|,$$
$$|cx+dy| \leqslant \surd|ad-bc|.$$

Of course, theorem E is a direct consequence of Minkowski's theorem on convex regions; but the generalizations to $k(i)$ and $k(\rho)$, which we shall discuss later, are not.

In a precisely similar way, if a/b is a reduced fraction of $\mathfrak{H}_N$, we may define the region $\mathfrak{R}(a, b)$ to be the set of x for which $|bx-a| \leqslant |b'x-a'|$ for any other a'/b' in $\mathfrak{H}_N$. The analogues, theorems A, B, and C, of theorems A*, B*, and C* hold:

THEOREMS A, B and C. *Theorems A*, B* and C* still hold if $\mathfrak{H}_N$ and $\mathfrak{R}(a, b)$ are read for $\mathfrak{H}_N^*$ and $\mathfrak{R}^*(a, b)$.*

† In this preliminary discussion theorems and lemmas are denoted by letters. In the main work the theorems will be reformulated more precisely and be denoted by numbers.

Now let us generalize $\mathfrak{H}_N$ and $\mathfrak{H}_N^*$ to $R(i)$. We define $\mathfrak{H}_N^*(i)$ to be the set of all reduced fractions α/β with numerators and denominators in $R(i)$ and $|\beta|^2 \leqslant N$. Similarly, $\mathfrak{H}_N(i)$ is defined to be the set of α/β with $|\alpha|^2 \leqslant N$, $|\beta|^2 \leqslant N$. We regard $\mathfrak{H}_N^*(i)$, $\mathfrak{H}_N(i)$ as points on the complex plane closed with a single point at infinity.

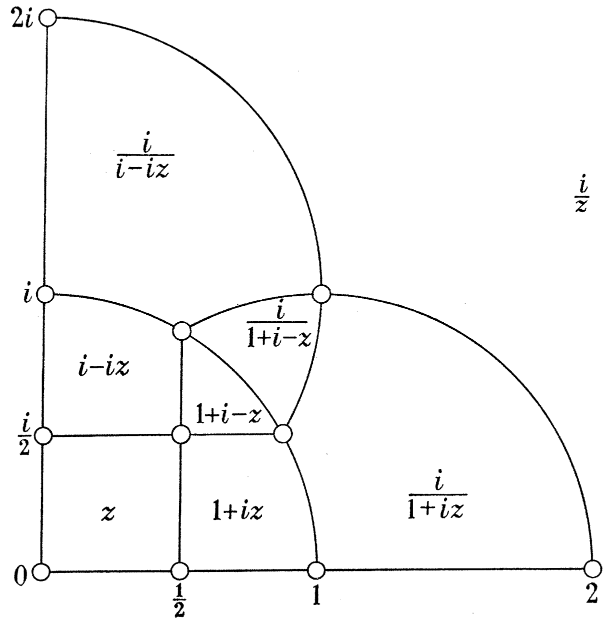

FIGURE 2†

Clearly, $\mathfrak{H}_N^*(i)$ is invariant under the mappings

$$z \to \bar{z}, \tag{2·1}$$

$$z \to i^k z + \nu \quad (k = 0, 1, 2, 3;\ \nu \in R(i)). \tag{2·2}$$

Hence all of $\mathfrak{H}_N^*(i)$ is obtainable from the portion in $0 \leqslant \mathscr{R}z \leqslant \frac{1}{2}$, $0 \leqslant \mathscr{I}z \leqslant \frac{1}{2}$ by translations and rotations.‡ A diagram of this portion of $\mathfrak{H}_N^*(i)$ is given at the end of the paper for $N = 25$.

Similarly, $\mathfrak{H}_N(i)$ is invariant under the mappings (2·1) and

$$z \to i^k z \quad (k = 0, 1, 2, 3), \tag{2·3}$$

and

$$z \to i^k/z \quad (k = 0, 1, 2, 3). \tag{2·4}$$

Further, $\mathfrak{H}_N(i)$ and $\mathfrak{H}_N^*(i)$ coincide in the circle $|z| \leqslant 1$. Hence all of $\mathfrak{H}_N(i)$ is obtainable from the portion in $0 \leqslant \mathscr{R}z \leqslant \frac{1}{2}$, $0 \leqslant \mathscr{I}z \leqslant \frac{1}{2}$ as shown in figure 2 (we give only one quadrant). Similarly $\mathfrak{H}_N(i)$ may be obtained from the portion in the first quadrant outside the circles $|z-1| = 1$, $|z-i| = 1$. Diagrams of $\mathfrak{H}_N(i)$ are given for $N = 5$, 10 at the end of the paper.

We now define $\mathfrak{R}^*(\alpha, \beta)$ for $\alpha/\beta \in \mathfrak{H}_N^*(i)$ to be the set of $z \in \Omega$ such that $|\beta z - \alpha| \leqslant |\beta' z - \alpha'|$ for any α'/β' in $\mathfrak{H}_N^*(i)$. Clearly, the boundary of $\mathfrak{R}^*(\alpha, \beta)$ consists of arcs of circles

$$|\beta z - \alpha| = |\beta' z - \alpha'|$$

for different α'/β' in $\mathfrak{H}_N^*(i)$. We now *define* α/β, α'/β' to be adjacent if $\mathfrak{R}^*(\alpha, \beta)$ and $\mathfrak{R}^*(\alpha', \beta')$

† An entry (e.g. $1+iz$) in a region means that all points of $\mathfrak{H}_N(i)$ in it can be obtained from those in the fundamental region $0 \leqslant \mathscr{R}z \leqslant \frac{1}{2}$, $0 \leqslant \mathscr{I}z \leqslant \frac{1}{2}$ by the appropriate transformation $z' = 1 + iz$.

‡ Indeed, because of (2·1) we need only the portion with say $0 \leqslant \mathscr{I}z \leqslant \mathscr{R}z \leqslant \frac{1}{2}$.

have a common point. Clearly, the set of regions $\mathfrak{R}^*(\alpha,\beta)$ have the same symmetries (2·1), (2·2) as $\mathfrak{H}_N^*(i)$.

Similarly, we may define $\mathfrak{R}(\alpha,\beta)$ for fractions $\alpha/\beta \in \mathfrak{H}_N(i)$. The set of regions $\mathfrak{R}(\alpha,\beta)$ has the same symmetries (2·3), (2·4) as $\mathfrak{H}_N(i)$. This is obvious for (2·3). As for (2·4), if $z \in \mathfrak{R}(\alpha,\beta)$ we have $\left|\frac{\alpha}{z}-\beta\right| \leqslant \left|\frac{\alpha'}{z}-\beta'\right|$ for any α'/β' in $\mathfrak{H}_N(i)$. Since β/α, β'/α' are in $\mathfrak{H}_N(i)$ if α/β, α'/β' are, it follows that $1/z \in \mathfrak{R}(\beta,\alpha)$, i.e. $\mathfrak{R}(\alpha,\beta)$ is obtained from $\mathfrak{R}(\beta,\alpha)$ by inversion. We have the further result:

Lemma A (i). *If $|\alpha/\beta|<1$, then $\mathfrak{R}(\alpha,\beta)$ lies entirely in the circle $|z|\leqslant 1$.*

We give the simple proof. If $\alpha/\beta \in \mathfrak{H}_N(i)$, then clearly also $\bar\beta/\bar\alpha \in \mathfrak{H}_N(i)$. If $z \in \mathfrak{R}(\alpha,\beta)$, then

$$|\beta z-\alpha| = \min_{\alpha'/\beta' \in \mathfrak{H}_N^{(i)}} |\beta' z-\alpha'| \leqslant |\bar\alpha z-\bar\beta|,$$

i.e. $|z|\leqslant 1$ if $|\alpha/\beta|<1$, as may easily be verified.

We should remark that the set of regions $\mathfrak{R}(\alpha,\beta)$ does not necessarily have the extra partial symmetries given by figure 1. For example, $\frac{2+i}{5}=\frac{1}{2-i}=\frac{\alpha}{\beta}$ (say) has the seven neighbours in $\mathfrak{H}_9(i)$ (taken clockwise)

$$\frac{1}{2},\quad \frac{3+i}{5}=\frac{1+i}{2+i},\quad \frac{1+i}{2}=\frac{1}{1-i},\quad \frac{1+i}{3},\quad \frac{1+i}{4}=\frac{1}{2-2i},\quad \frac{0}{1},\quad \frac{1}{3}, \tag{2·5}$$

but $1+\frac{i\alpha}{\beta}=\frac{4+2i}{5}=\frac{2}{2-i}$ has only the six neighbours

$$\frac{2+i}{2},\quad \frac{4+3i}{5}=\frac{1+2i}{2+i},\quad \frac{1+i}{2}=\frac{1}{1-i},\quad \frac{2+i}{3},\quad \frac{3+i}{4}=\frac{1+2i}{2+2i},\quad \frac{1}{1}.$$

There is no neighbour $1+\frac{1}{3}i=\frac{3+i}{3}$ corresponding to $\frac{1}{3}$ in (2·5), since $|3+i|^2=10>9$ and so $\frac{3+i}{3}\notin\mathfrak{H}_9(i)$.

In this paper, we extend to $k(i)$ the theorems given above for k. More precisely, we shall prove:

Theorem A* (i). *$\mathfrak{R}^*(\alpha,\beta)$ is a star domain about α/β.*

Theorem A (i). *$\mathfrak{R}(\alpha,\beta)$ is a star domain about α/β for $\beta\neq 0$ and sufficiently large N.*

There is little doubt that theorem A (i) holds for all N, but we have not constructed a proof.

The analogues of the medians in the rational case are the four numbers $\frac{\alpha+\epsilon\alpha'}{\beta+\epsilon\beta'}$ where α/β, $\alpha'/\beta' \in \mathfrak{H}_N^*(i)$ and ϵ is a unit, i.e. $\epsilon=\pm 1$ or $\pm i$. Clearly, all four medians lie on the circle $|\beta z-\alpha|=|\beta' z-\alpha'|$.

Theorem B* (i). *The necessary and sufficient condition that the reduced fractions α/β, $\alpha'/\beta' \in \mathfrak{H}_N^*(i)$ be adjacent, is that simultaneously*

(i) $|\alpha\beta'-\alpha'\beta|=1$ or $2^{\frac{1}{2}}$.

(ii) *At least one of the medians $\frac{\alpha+\epsilon\alpha'}{\beta+\epsilon\beta'}$ is not in $\mathfrak{H}_N^*(i)$.*

Theorem C* (i). *All terms of $\mathfrak{H}_{N+1}^*(i)$ which are not already in $\mathfrak{H}_N^*(i)$, are medians $\frac{\alpha+\epsilon\alpha'}{\beta+\epsilon\beta'}$ of adjacent terms α/β, α'/β' of $\mathfrak{H}_N^*(i)$.*

THEOREMS B (i) and C (i). *Theorems* B* (i) *and* C* (i) *hold if* $\mathfrak{H}_N(i)$ *is read for* $\mathfrak{H}_N^*(i)$.

THEOREM D* (i). *If* $z \in \mathfrak{R}^*(\alpha, \beta)$, *then* $|\beta z - \alpha| \leqslant \kappa/N^{\frac{1}{2}}$, *where* $\kappa = \sqrt{2}/(3-\sqrt{3})$.

From theorem D* (i) we derive

THEOREM E (i). *Let* $\alpha, \beta, \gamma, \delta \in \Omega$, $\alpha\delta - \beta\gamma \neq 0$. *Then there are* $\xi, \eta \in R(i)$, $(\xi, \eta) \neq (0,0)$ *such that simultaneously*

$$|\alpha\xi + \beta\eta| \leqslant \kappa^{\frac{1}{2}} \sqrt{|\alpha\delta - \beta\gamma|},$$
$$|\gamma\xi + \delta\eta| \leqslant \kappa^{\frac{1}{2}} \sqrt{|\alpha\delta - \beta\gamma|},$$

where κ *is defined in theorem* D* (i).

This theorem and the corresponding theorem for $k(\rho)$ are due to Minkowski (1907, chap. 6). A further proof has recently been given by Hlawka (1941). Both these proofs are somewhat complicated. The present investigation originated in an attempt to find a more natural approach.

We define a *node* of $\mathfrak{H}_N^*(i)$ or $\mathfrak{H}_N(i)$ to be a point where three or more regions $\mathfrak{R}^*(\alpha, \beta)$ or $\mathfrak{R}(\alpha, \beta)$ respectively meet. It is obvious from the diagram that there are only a few kinds of nodes. Indeed we have

THEOREMS F* (i) and F (i). *At most four regions* $\mathfrak{R}^*(\alpha, \beta)$ *or* $\mathfrak{R}(\alpha, \beta)$ *respectively meet at a node. If four such regions meet, they subtend equal angles* $\frac{1}{2}\pi$ *at the node. If only three regions meet at a node, they subtend either* $\frac{2}{3}\pi$, $\frac{2}{3}\pi$, $\frac{2}{3}\pi$ *or* $\frac{1}{2}\pi$, $\frac{3}{4}\pi$, $\frac{3}{4}\pi$.

We also generalize Farey section to $k(\rho)$. We define $\mathfrak{H}_N^*(\rho)$, $\mathfrak{H}_N(\rho)$ analogously to $\mathfrak{H}_N^*(i)$, $\mathfrak{H}_N(i)$ and then define $\mathfrak{R}^*(\alpha, \beta)$, $\mathfrak{R}(\alpha, \beta)$ for $\alpha/\beta \in \mathfrak{H}_N^*(\rho)$, $\mathfrak{H}_N(\rho)$ respectively.

Clearly, $\mathfrak{H}_N^*(\rho)$ is invariant under the mappings

$$z \to \bar{z},$$
$$z \to \rho^k z + \nu, \quad z \to -\rho^k z + \nu \quad (k = 0, 1, 2,\ \nu \in R(\rho)).$$

Hence $\mathfrak{H}_N^*(\rho)$ has a 'hexagonal' symmetry and it is enough to know $\mathfrak{H}_N^*(\rho)$ for

$$0 \leqslant \mathscr{R} z \leqslant \tfrac{1}{2}, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{6}\pi.$$

Similarly, $\mathfrak{H}_N(\rho)$ has a 'hexagonal' symmetry analogous to that of $\mathfrak{H}_N(i)$.

As the proof of theorems A (i) and A* (i) is laborious, we do not attempt to prove the analogues for $k(\rho)$, but there is no doubt they hold. We have the following analogues of the remaining theorems. The 'medians' of α/β, α'/β' in $k(\rho)$ are, of course, the six numbers $\dfrac{\alpha+\epsilon\alpha'}{\beta+\epsilon\beta'}$, where ϵ is a unit, i.e. $\epsilon = \pm 1$ or $\pm\rho$ or $\pm\rho^2$.

THEOREM B* (ρ). *The necessary and sufficient condition that the reduced fractions* α/β, α'/β' *of* $\mathfrak{H}_N^*(\rho)$ *be adjacent, is that simultaneously*

(i) $|\alpha\beta' - \alpha'\beta| = 1$ *or* $3^{\frac{1}{2}}$,

(ii) *at least one of the medians* $\dfrac{\alpha+\epsilon\alpha'}{\beta+\epsilon\beta'}$ *is not in* $\mathfrak{H}_N^*(\rho)$, *where if* $|\alpha\beta' - \alpha'\beta| = 3^{\frac{1}{2}}$, *we consider only* ϵ *such that* $\alpha + \epsilon\alpha' \equiv \beta + \epsilon\beta' \equiv 0\ (1-\rho)$.

THEOREM B (ρ). *Theorem* B* (ρ) *continues to hold if* $\mathfrak{H}_N(\rho)$ *is read for* $\mathfrak{H}_N^*(\rho)$.

THEOREMS C* (ρ), C (ρ), D* (ρ) and E (ρ). *Theorems* C* (i), C (i), D* (i) *and* E (i) *hold in* $k(\rho)$ *when* $\kappa = 1$ *is read in theorems* D* (ρ), E (ρ).

THEOREMS F* (ρ) and F (ρ). *At most four regions* $\mathfrak{R}^*(\alpha, \beta)$, $\mathfrak{R}(\alpha, \beta)$ *respectively meet at a node. If four regions meet at a node, they subtend angles* $\frac{1}{3}\pi$, $\frac{2}{3}\pi$, $\frac{1}{3}\pi$, $\frac{2}{3}\pi$. *If three regions meet at a node, they subtend either* $\frac{2}{3}\pi$, $\frac{2}{3}\pi$, $\frac{2}{3}\pi$ *or* $\frac{1}{3}\pi$, $\frac{5}{6}\pi$, $\frac{5}{6}\pi$.

In part I we prove and discuss more precise versions of theorems B (i), C (i) and F (i) and outline proofs of theorems B (ρ), C (ρ) and F (ρ). Since the proofs of the corresponding * theorems are parallel but easier we do not give them. In part II we prove theorems D* (i), D* (ρ), E (i) and E (ρ). Finally, in part III we prove theorems A* (i) and A (i). The treatment throughout is independent of this preliminary discussion.

PART I

3. Introduction

In § 4 we give an account of the properties of the ordinary Farey section in k which we intend to generalize. As the standard proofs of these properties do not generalize, we will give new proofs depending on the ideas which will be useful later. In this way, the main lines† of the proof will appear without the detail required in $k(i)$ and $k(\rho)$.

In § 5 we shall discuss the problem in $k(i)$ in general terms. §§ 6 to 8 carry out the programme in detail. In § 9 we discuss more briefly the analogous results for $k(\rho)$. The theory of § 8 is practically independent of the rest of the paper.

4. Farey section in k

If $N \geqslant 1$ is a positive number,‡ we denote by $\mathfrak{H}_N$ the set of all fractions $r = a/b$, where a, b are integers not exceeding N in absolute value:§

$$r \in \mathfrak{H}_N \quad .\equiv. \quad r = a/b; \quad a, b \in R; \quad |a|, |b| \leqslant N.$$

The improper fraction $\infty = \frac{1}{0}$ is included in $\mathfrak{H}_N$. For many purposes it is more convenient to consider the set $\mathfrak{F}_N$ of pairs (a, b) of numerators and denominators of fractions of $\mathfrak{H}_N$ in their lowest terms:||

$$(a, b) \in \mathfrak{F}_N \quad .\equiv. \quad a, b \in R; \quad |a|, |b| \leqslant N; \quad \gcd(a, b) = 1.$$

Finally, we shall denote by $\mathfrak{G}_N$ the set of all pairs (a, b) of numerators and denominators of fractions of $\mathfrak{H}_N$, not necessarily in their lowest terms:

$$(a, b) \in \mathfrak{G}_N \quad .\equiv. \quad a, b \in R; \quad a/b \in \mathfrak{H}_N.$$

We shall say that two elements of $\mathfrak{F}_N$ or of $\mathfrak{G}_N$ are equivalent, if they correspond to the same fraction of $\mathfrak{H}_N$. For most purposes equivalent elements will be regarded as the same.

For $(a, b) \in \mathfrak{F}_N$, we denote by $\mathfrak{R}(a, b)$ the set of all real numbers x for which a/b gives the best approximation in the sense that

$$x \in \mathfrak{R}(a, b) \quad .\equiv. \quad |bx - a| = \min_{(c, d) \in \mathfrak{F}_N} |dx - c|. \tag{4·1}$$

If $(c, d) \in \mathfrak{G}_N$, say $c = ef$, $d = eg$, $(f, g) \in \mathfrak{F}_N$, then $|cx - d| \geqslant |gx - f|$, so we may also write this as

$$x \in \mathfrak{R}(a, b) \quad .\equiv. \quad |bx - a| = \min_{(c, d) \in \mathfrak{G}_N} |dx - c|. \tag{4·2}$$

Clearly, $\mathfrak{R}(a, b) = \mathfrak{R}(a', b')$ if (a, b) and (a', b') are equivalent elements of $\mathfrak{F}_N$.

† The $k(i)$ and $k(\rho)$ proofs are of course complete without reference to the proofs in k.

‡ Note that we do not insist that N is an integer.

§ As explained in the introduction, there is another type of Farey section in which the condition $|a| \leqslant N$ is omitted. However, the Farey section defined above is the one whose properties are more difficult to prove, so we discuss it in part I.

|| As the symbol (a, b) is required to denote the ordered number pair, we use gcd (a, b) to denote the greatest common divisor of a and b.

We say that (a, b) and (c, d) are adjacent if $\mathfrak{R}(a, b)$ and $\mathfrak{R}(c, d)$ abut, i.e. if there is an x' such that

$$|bx'-a| = |dx'-c| = \min_{(e,f)\in\mathfrak{G}_N} |fx'-e|. \tag{4·3}$$

We first prove

THEOREM I. *The necessary and sufficient condition that* (a, b) *and* $(c, d) \in \mathfrak{F}_N$ *be adjacent, is that simultaneously*

(i) $|ad-bc| = 1$.

(ii) $(a \pm c, b \pm d) \notin \mathfrak{G}_N$ *for at least one of the two signs* $\pm$.

The proof is in two parts.

If (i) *and* (ii) *are true,* (a, b) *and* (c, d) *are adjacent.* If $(a\pm c, b\pm d) \notin \mathfrak{G}_N$ we put $x' = (a\pm c)/(b\pm d)$. Then $|bx'-a| = |dx'-c| = |ad-bc|/|b\pm d| = \dfrac{1}{|b\pm d|}$ and, if (e,f) is any element of $\mathfrak{G}_N$,

$$|fx'-e| = \frac{|f(a\pm c)-e(b\pm d)|}{|b\pm d|} \geqslant \frac{1}{|b\pm d|},$$

since $|f(a\pm c)-e(b\pm d)| = 0$ would imply $\dfrac{a\pm c}{b\pm d} = \dfrac{e}{f} \in \mathfrak{H}_N$, contrary to (ii). Hence

$$|bx'-a| = |dx'-c| = \min_{(e,f)\in\mathfrak{G}_N} |fx'-e|,$$

as required.

If (a, b) *and* (c, d) *are adjacent then* (i) *and* (ii) *hold.* We suppose that (4·3) holds for some x'. Then

$$bx'-a = \pm(dx'-c)$$

for some choice of sign and so $(b\mp d)x' - (a\mp c) = 0$. If $(a\mp c, b\mp d) \in \mathfrak{G}_N$, this is a contradiction with (4·3). Hence (ii) holds. To prove (i) we require the following lemma:

LEMMA 1. *Suppose* $a, b, c, d \in R$ *and* $\Delta = |ad-bc| > 1$. *Then there are* $p, q \in R$ *such that*

$$\left.\begin{aligned} ap+cq &\equiv 0\ (\Delta),\\ bp+dq &\equiv 0\ (\Delta),\end{aligned}\right\} \tag{4·4}$$

and

$$0 < |p|+|q| \leqslant \Delta. \tag{4·5}$$

The last sign of equality is required only when $\Delta = 2$, $p = q = 1$, *and then* $p = 1$, $q = -1$ *also satisfies* (4·4) *and* (4·5).

We note first that the points (p, q) for which (4·4) holds form a *lattice* Λ. On putting $ap+cq = r\Delta$, $bp+dq = s\Delta$, we have $\pm p = dr-cs$, $\mp q = br-as$, where r, s are any integers, and so the determinant $d(\Lambda)$ of Λ is $|ad-bc| = \Delta$. The convex region defined by

$$|p|+|q| \leqslant \Delta$$

has area $2\Delta^2$ and the existence of p, q follows now at once from Minkowski's convex body theorem if $\Delta > 2$. If $\Delta = 2$ and $a \equiv b \equiv 0\ (2)$ take $p = 1$, $q = 0$, and if $c \equiv d \equiv 0\ (2)$ take $p = 0$, $q = 1$. Otherwise, since $|ad-bc| = 2$, we have $a \equiv c$, $b \equiv d\ (2)$ and $p = q = 1$ and $p = -q = 1$ do what is required.

Suppose now that (4·3) holds and that $|ad-bc| = \Delta > 1$. Let p, q be the p, q of the lemma and put $ap+cq = r\Delta$, $bp+dq = s\Delta$, as before. Then

$$|r| = \left|\frac{ap+cq}{\Delta}\right| \leqslant \frac{|p|+|q|}{\Delta} N \leqslant N,$$

$$|s| \leqslant N,$$

so that
$$(r,s)\in\mathfrak{G}_N.$$

Further,
$$|rx'-s| = \left|\frac{p(bx'-a)+q(dx'-c)}{\Delta}\right|$$
$$\leqslant \frac{|p|+|q|}{\Delta}\max\{|bx'-a|,|dx'-c|\},$$

and hence, by lemma 1, if $\Delta>2$,
$$|rx'-s|<\max\{|bx'-a|,|dx'-c|\},$$
contrary to (4·3). Similarly, we reach a contradiction if $|p|+|q|<\Delta=2$. If
$$|p|+|q|=\Delta=2$$
then both $p=q=1$ and $p=-q=1$ satisfy (4·4) and (4·5) and then
$$|rx'-s| = \left|\frac{(bx'-a)\pm(dx'-c)}{2}\right| < |bx'-a| = |dx'-c|$$
for at least one choice of sign, again a contradiction. This concludes the proof of the theorem.

In order not to make the point $\infty=\frac{1}{0}$ exceptional, we complete the infinite line by a single point at infinity and so make it topologically equivalent to a circle. With this convention we have the

THEOREM II. *The* $\mathfrak{R}(a,b)$ *are intervals.*

It is enough to prove that the $\mathfrak{R}(a,b)$ are simply-connected. This is a topologically invariant property and so invariant under the transformation
$$y=\frac{dx-c}{bx-a},\quad x=\frac{ay-c}{by-d},$$
where c, d are any integers such that $\quad ad-bc=1$

[they exist since $\gcd(a,b)=1$]. The inequality (4·1) with this transformation and after multiplication by $|by-d|$ becomes
$$1=\min_{(e,f)\in\mathfrak{F}_N}|(af-be)y-(cf-de)|.$$

By the preceding theorem, the only neighbours (e,f) to (a,b) have $|af-be|=1$, and hence, by writing $(-e,-f)$ for e,f, if necessary, it will be enough to prove that the region
$$1\leqslant\min_{\substack{(e,f)\in\mathfrak{F}_N\\ af-be=1}}|y-(cf-de)| \tag{4·6}$$
is simply-connected. Put $h=cf-de$, and (4·6) becomes
$$1\leqslant\min|y-h| \tag{4·7}$$
taken over all h such that simultaneously
$$|e|=|c-ah|\leqslant N,\quad |f|=|d-bh|\leqslant N. \tag{4·8}$$
Clearly, (4·8) will be satisfied only for some consecutive set $h_1\leqslant h\leqslant h_2$ of values of h. Hence (4·7) is true if, and only if,
$$y\leqslant h_1-1\quad\text{or}\quad y\geqslant h_2+1\quad\text{or}\quad y=\infty.$$
This is a simply-connected region, and so the theorem is proved.

The foregoing account is, of course, unduly complicated as an account of Farey sections in k, since then the numbers $a/b\in\mathfrak{H}_N$ may be arranged in order of magnitude. Theorems I and II show that $\mathfrak{R}(a, b)$ is an interval about the fraction a/b. Hence the two pairs (a, b) and (c, d) are adjacent in our sense if, and only if, a/b and c/d are consecutive elements of $\mathfrak{H}_N$ in magnitude. However, in $k(i)$ and $k(\rho)$ there will be no analogue of the linear order of $\mathfrak{H}_N$, and we are driven to the definition of adjacency in terms of the regions $\mathfrak{R}(a, b)$, $\mathfrak{R}(c, d)$.

If N is an integer and $\mathfrak{F}_N$ is known, there is a simple rule by which $\mathfrak{F}_{N+1}$ may be generated, which runs as follows:

THEOREM III. *If $(e,f) \in \mathfrak{F}_{N+1}$ but $(e,f) \notin \mathfrak{F}_N$, then there are two adjacent pairs (a, b), $(c, d) \in \mathfrak{F}_N$ such that $(e,f) = (a+c, b+d)$.*

This means that the Farey sections may be constructed for increasing N by a simple algorithm. We prove the theorem for $N>3$; it can be verified directly for $N \leqslant 3$. The proof depends on the following lemma:

LEMMA 2. *Let $e, f \in R$, $\gcd(e,f) = 1$, and $\max(|e|, |f|) > 2$. Then there are $a', b' \in R$ such that*

$$a'f - b'e = 1 \tag{4·9}$$

and

$$|a'| \leqslant \tfrac{1}{2}|e|, \quad |b'| \leqslant \tfrac{1}{2}|f|. \tag{4·10}$$

We first note that $|ef| \neq 0$ and $|e| \neq |f|$; so, by symmetry, we may assume that

$$|e| > |f| \geqslant 1, \quad |e| > 2.$$

Since e, f are coprime, there are certainly $a'', b'' \in R$ such that

$$a''f - b''e = 1.$$

Then

$$a' = a'' + he, \quad b' = b'' + hf,$$

where $h \in R$, is also a solution of (4·9). We now choose h such that

$$|a'| \leqslant \tfrac{1}{2}|e|.$$

Then, by (4·9),

$$|2b'| = \left|\left(\frac{2a'}{e}\right)f + \frac{2}{e}\right| \leqslant |f| + \frac{2}{|e|} < |f| + 1.$$

Hence $|2b'| \leqslant |f|$, since $|2b'|$ and $|f|$ are integers. This proves the lemma.

COROLLARY. *We may choose $\epsilon = \pm 1$ such that*

$$|e - \epsilon a'| \leqslant |e|, \quad |f - \epsilon b'| \leqslant |f|,$$

where the first sign of equality is required only if $|e| = 1$ and the second only if $|f| = 1$.

Suppose first that $a'b' \neq 0$. Then, by (4·9)

$$\operatorname{sgn}(a'f) = \operatorname{sgn}(b'e)$$

and so

$$\operatorname{sgn}(a'e) = \operatorname{sgn}(b'f) = \epsilon \quad \text{(say)}.$$

Then ϵ does what is required. Further, $a' = 0$ is possible only if $|e| = 1$, and then we may take $\epsilon = \operatorname{sgn}(b'f)$. Similarly, for $b' = 0$. This concludes the proof.

The proof of theorem III is now immediate. We take for (a, b), the $(\epsilon a', \epsilon b')$ of lemma 2 and put

$$c = e - a, \quad d = f - b,$$

so that

$$(e,f) = (a+c, b+d). \tag{4·11}$$

Then

$$|af - be| = |ad - bc| = |cf - de| = 1 \tag{4·12}$$

and

$$\max\{|a|, |b|\} < \max\{|e|, |f|\} = N+1,$$
$$\max\{|c|, |d|\} < \max\{|e|, |f|\} = N+1.$$

Since $N \in R$ and since, by (4·12),

$$\gcd(a,b) = \gcd(c,d) = 1,$$

we have

$$(a,b) \in \mathfrak{F}_N, \quad (c,d) \in \mathfrak{F}_N.$$

It remains only to prove that (a,b) and (c,d) are neighbours. This, however, follows at once from theorem I. Condition (i) of the theorem is true by (4·12) and condition (ii) is true by (4·11), since $(e,f) \notin \mathfrak{G}_N$ by hypothesis.

5. The generalization to $k(i)$

For simplicity, we shall deal in this and the succeeding paragraphs primarily with $k(i)$. The situation in $k(\rho)$ is very similar and an account of it will be deferred to the end of this part I (§9).

We denote by $\mathfrak{H}_N(i)$, or by $\mathfrak{H}_N$ if the omission of the (i) causes no ambiguity, the set of all fractions α/β:

$$\alpha/\beta \in \mathfrak{H}_N \quad .\equiv. \quad \alpha, \beta \in R(i), \quad \mathcal{N}(\alpha), \mathcal{N}(\beta) \leqslant N,$$

where $\mathcal{N}(\xi) = |\xi|^2$ is the norm of ξ. The improper fraction $\infty = \frac{1}{0}$ is included in $\mathfrak{H}_N$. Similarly, $\mathfrak{F}_N$ and $\mathfrak{G}_N$ will be the set of all pairs (α,β), α, $\beta \in R(i)$ such that $\alpha/\beta \in \mathfrak{H}_N$, and the $(\alpha,\beta) \in \mathfrak{F}_N$ will have the further property that the fraction α/β is in its lowest terms,† i.e. $\gcd(\alpha,\beta) = 1$. Two elements (α,β) and (α',β') of $\mathfrak{F}_N$ or $\mathfrak{G}_N$ will be called equivalent if $\alpha/\beta = \alpha'/\beta'$. For most purposes equivalent elements will be regarded as the same.

For $(\alpha,\beta) \in \mathfrak{F}_N$, we define $\mathfrak{R}(\alpha,\beta)$ to be the set of $z \in \Omega$ for which α/β gives the best approximation in the sense that

$$z \in \mathfrak{R}(\alpha,\beta) \quad .\equiv. \quad |\beta z - \alpha| = \min_{(\gamma,\delta) \in \mathfrak{F}_N} |\delta z - \gamma|, \tag{5·1}$$

and, as in the real case, this is equivalent to

$$z \in \mathfrak{R}(\alpha,\beta) \quad .\equiv. \quad |\beta z - \alpha| = \min_{(\gamma,\delta) \in \mathfrak{G}_N} |\delta z - \gamma|. \tag{5·2}$$

Two pairs (α,β), $(\gamma,\delta) \in \mathfrak{F}_N$ will be called adjacent if $\mathfrak{R}(\alpha,\beta)$, $\mathfrak{R}(\gamma,\delta)$ have a point z' in common, i.e. if

$$|\beta z' - \alpha| = |\delta z' - \gamma| = \min_{(\eta,\theta) \in \mathfrak{F}_N} |\theta z' - \eta|, \tag{5·3}$$

or, what is the same thing,

$$|\beta z' - \alpha| = |\delta z' - \gamma| = \min_{(\eta,\theta) \in \mathfrak{G}_N} |\theta z' - \eta|. \tag{5·4}$$

The condition that (α,β) and (γ,δ) be adjacent is later shown to be a natural generalization of theorem I (theorem IV).

The regions $\mathfrak{R}(\alpha,\beta)$ can be considered as regions on the Gauss plane completed with a single point at infinity. Some diagrams of the set of regions belonging to $\mathfrak{F}_N$ for various N are given at the end of the paper. Clearly each $\mathfrak{R}(\alpha,\beta)$ is bounded by a finite set of arcs of the circles

$$|\beta z - \alpha| = |\delta z - \gamma|,$$

where (γ,δ) runs through all the neighbours of (α,β). A rather long, though elementary, argument shows that all the $\mathfrak{R}(\alpha,\beta)$ are star domains (part III), but otherwise they appear to have no simple properties as regions.

The points which belong to three or more regions are called nodes. It is remarkable that all nodes are of only three distinct types (theorem V), and that no more than four regions

† All ideals in both $R(i)$ and $R(\rho)$ are principal ideals, i.e. every fraction in $k(i)$ or $k(\rho)$ does have a representation in lowest terms.

meet at a node. Since all regions meeting at a node are adjacent, the properties of nodes may be investigated using the properties of adjacent regions.

Finally, it is shown that the $\mathfrak{F}_N$ may be built up successively by a process analogous to that of theorem III (theorem VI).

6. Adjacent regions

In this section we shall prove the following theorem:

THEOREM IV (cf. theorem I). *The necessary and sufficient condition that* (α,β), $(\gamma,\delta)\in\mathfrak{F}_N$ *be adjacent, is that simultaneously*

(i) $|\alpha\delta-\beta\gamma| = 1$ or $2^{\frac{1}{2}}$,

(ii) $(\alpha+\epsilon\gamma,\beta+\epsilon\delta)\notin\mathfrak{G}_N$ for some choice of $\epsilon=\pm 1$ or $\pm i$.

The proof of this theorem depends on two lemmas:

LEMMA 3. *Suppose* α, β, γ, $\delta\in R(i)$, gcd $(\alpha,\beta)=$ gcd $(\gamma,\delta)=1$, and $|\alpha\delta-\beta\gamma|=2^{\frac{1}{2}}$. *Then*

$$\alpha\equiv\gamma\ (1+i),\quad \beta\equiv\delta\ (1+i).$$

Clearly $|\alpha\delta-\beta\gamma|=2^{\frac{1}{2}}$ implies

$$\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}\equiv 0\quad (1+i).$$

The lemma now follows at once by enumeration of cases, since each of α, β, γ, δ is congruent either to 0 or to 1 modulo $1+i$, and $\alpha\equiv\beta\equiv 0$ or $\gamma\equiv\delta\equiv 0$ is excluded by the proviso that α, β and γ, δ are coprime respectively.

LEMMA 4 (cf. lemma 1). *Suppose* α, β, γ, $\delta\in R(i)$ *and* $\Delta=\alpha\delta-\beta\gamma$, $\mathcal{N}(\Delta)>2$; *then there are* ξ, $\eta\in R(i)$ *such that*

$$\alpha\xi+\gamma\eta\equiv\beta\xi+\delta\eta\equiv 0\ (\Delta), \tag{6·1}$$

$$0<|\xi|+|\eta|\leqslant|\Delta|. \tag{6·2}$$

The last sign of equality is required only when $|\Delta|=2$ *or* $|\Delta|=2^{\frac{1}{2}}$, *but then* $|\xi|<|\Delta|$, $|\eta|<|\Delta|$ *and* ξ, η *may be so chosen that* ξ, $\pm\eta$ *is a solution for both signs.*

As in the proof of lemma 1, the values of ξ, η which satisfy (6·1) form a two-dimensional complex lattice of determinant Δ. Hence if we put $\xi=x+iy$, $\eta=u+iv$ where x, y, u, v are real, the values of x, y, u, v form a four-dimensional lattice Λ of determinant

$$d(\Lambda)=\mathcal{N}(\Delta)=|\Delta|^2.$$

We have to show the existence of a lattice-point other than $(0,0,0,0)$ in the convex region $\mathfrak{S}$:

$$(x,y,u,v)\in\mathfrak{S}\quad .\!\equiv\!.\quad (x^2+y^2)^{\frac{1}{2}}+(u^2+v^2)^{\frac{1}{2}}\leqslant|\Delta|,$$

defined by (6·2). An elementary calculation shows that $\mathfrak{S}$ has volume

$$\mathscr{V}(\mathfrak{S})=\int_{r=0}^{|\Delta|}\pi(|\Delta|-r)^2\,d(\pi r^2)=\frac{\pi^2|\Delta|^4}{6}.$$

Hence, by Minkowski's convex body theorem, there is certainly a point of Λ other than $(0,0,0,0)$ in the interior of $\mathfrak{S}$ provided

$$\mathscr{V}(\mathfrak{S})>2^4 d(\Lambda),$$

i.e. provided $$\tfrac{1}{6}\pi^2|\Delta|^4>2^4|\Delta|^2,\quad |\Delta|^2>\frac{96}{\pi^2}.$$

This proves the lemma for $|\Delta|^2\geqslant 10$, since $96/\pi^2<10$.

To complete the proof of the lemma we verify it for the remaining values of Δ individually. Since $|\Delta|^2 > 2$ by hypothesis, the remaining values of Δ are

$$\Delta = \epsilon(1+i)^2, \quad \epsilon(1+i)^3, \quad \epsilon(2\pm i), \quad 3\epsilon \quad (\epsilon = \pm 1 \text{ or } \pm i). \tag{6·3}$$

By the general theory of determinants there is certainly a solution ξ_0, η_0 of (6·1) with $\gcd(\xi_0, \eta_0, \Delta) = 1$; and so, since (6·3) are all prime powers, either $\gcd(\xi_0, \Delta) = 1$ or $\gcd(\eta_0, \Delta) = 1$ (or both). By symmetry, we may assume that $\gcd(\xi_0, \Delta) = 1$. If we now choose η_1 such that $\eta_0 \equiv \xi_0 \eta_1\ (\Delta)$, there is a solution $\xi = 1$, $\eta = \eta_1$ of (6·1). Further, $\xi = 1$, $\eta = \eta_2$ is a solution if $\eta_2 \equiv \eta_1\ (\Delta)$. We prove the lemma for $\Delta = \epsilon(2\pm i)$, 3ϵ by showing that we may choose a complete set of residues η_2 modulo Δ such that $1 + |\eta_2| < |\Delta|$. Indeed, we have

$$\Delta = 3\epsilon: \qquad \eta_2 = 0, \ \pm 1, \ \pm i, \ \pm 1 \pm i; \quad 1 + |\eta_2| \leqslant 1 + 2^{\frac{1}{2}} < 3 = |\Delta|,$$

$$\Delta = \epsilon(2\pm i): \qquad \eta_2 = 0, \ \pm 1, \ \pm i; \quad 1 + |\eta_2| \leqslant 2 < 5^{\frac{1}{2}} = |\Delta| \quad \text{(independent signs)}.$$

There remain only $\Delta = \epsilon(1+i)^2 = 2i\epsilon$, $\Delta = \epsilon(1+i)^3$. If $\eta_1 \equiv 0\ (\Delta)$, there is nothing to prove. If $\eta_1 \equiv 0\ (1+i)$ but $\eta_1 \not\equiv 0\ (\Delta)$, we may choose θ such that $0 < |\theta| < |\Delta|$ and $\eta_1 \theta \equiv 0\ (\Delta)$. Then $\xi = \theta$, $\eta = 0$ is a solution of (6·1) and of (6·2) with $<$. Hence we may assume $\gcd(\eta_1, \Delta) = 1$. If $\Delta = \epsilon(1+i)^2 = 2i\epsilon$ we have now

$$\eta_1 \equiv 1 \equiv -1 \ (2) \quad \text{or} \quad \eta_1 \equiv i \equiv -i \ (2);$$

so either $\qquad (\xi, \eta) = (1, \pm 1) \quad \text{or,} \quad (\xi, \eta) = (1, \pm i)$

satisfy both (6·1) and (6·2). If $\Delta = \epsilon(1+i)^3 = 2i\epsilon(1+i)$, we have again

$$\eta_1 \equiv 1 \equiv -1 \ (2) \quad \text{or} \quad \eta_1 \equiv i \equiv -i \ (2),$$

and $\qquad (\xi, \eta) = (1+i, \pm(1+i)) \quad \text{or} \quad (1+i, \pm i(1+i))$

satisfy (6·1) and (6·2).

This concludes the proof of the lemma.†

We now complete the proof of the theorem. There are parts corresponding to the necessity and sufficiency of the criteria.

If (i) *and* (ii) *are true*, (α, β), (γ, δ) *are adjacent.* By (ii) we may choose ϵ such that

$$(\alpha + \epsilon\gamma, \beta + \epsilon\delta) \notin \mathfrak{G}_N,$$

and we put $z' = (\alpha + \epsilon\gamma)/(\beta + \epsilon\delta)$. Then

$$|\beta z' - \alpha| = |\delta z' - \gamma| = \frac{|\Delta|}{|\beta + \epsilon\delta|}.$$

Further, if $(\eta, \theta) \in \mathfrak{F}_N$, we have

$$|\theta z' - \eta| = \frac{|\theta(\alpha + \epsilon\gamma) - \eta(\beta + \epsilon\delta)|}{|\beta + \epsilon\delta|}.$$

Now $|\theta(\alpha + \epsilon\gamma) - \eta(\beta + \epsilon\delta)| \neq 0$, since $\eta/\theta \in \mathfrak{H}_N$ but $\dfrac{\alpha + \epsilon\gamma}{\beta + \epsilon\delta} \notin \mathfrak{H}_N$ by hypothesis. Hence

$$|\theta(\alpha + \epsilon\gamma) - \eta(\beta + \epsilon\delta)| \geqslant 1.$$

† Alternatively, if ξ, η run through all values satisfying (6·1) it may be shown that $|\xi|^2 + |\eta|^2$ runs through all the values taken by a certain binary Hermitian form for variables in $R(i)$. We may then make use of the known estimate for the minimum of such forms to show that $|\xi|^2 + |\eta|^2 < \frac{1}{2}|\Delta|^2$ for some permissible ξ, η if $|\Delta|^2 > 8$. Hence, by Cauchy's inequality, (6·2) with $<$ is satisfied if $|\Delta|^2 > 8$. The cases $|\Delta|^2 \leqslant 8$ have then to be treated separately.

Again, we might have used the minimum of a quaternary quadratic form. The method in the text seems, however, more elementary.

This proves the result if $|\Delta| = 1$. If $|\Delta| = 2^{\frac{1}{2}}$, we have $\alpha+\epsilon\gamma\equiv\beta+\epsilon\delta\equiv 0\ (1+i)$ by lemma 3 and hence $|\theta(\alpha+\epsilon\gamma)-\eta(\beta+\epsilon\delta)|\geqslant 2^{\frac{1}{2}}$ as required.

If (α,β), (γ,δ) *are adjacent,* (ii) *is true.* If (α,β), (γ,δ) are adjacent, (5·4) is true for some z'. We may now choose $\epsilon=\pm 1, \pm i$ such that

$$|(\beta+\epsilon\delta)z'-(\alpha+\epsilon\gamma)| = |(\beta z'-\alpha)+\epsilon(\delta z'-\gamma)| < |\beta z'-\alpha| = |\delta z'-\gamma|.$$

Hence $(\alpha+\epsilon\gamma, \beta+\epsilon\delta)\notin\mathfrak{G}_N$ by (5·4). This proves (ii).

If (α,β), (γ,δ) *are adjacent,* (i) *is true.* Suppose *per absurdum* that $|\Delta|>2^{\frac{1}{2}}$. We may choose ξ, η to satisfy the conclusions of lemma 4. We put

$$\alpha\xi+\gamma\eta = \Delta\phi, \quad \beta\xi+\delta\eta = \Delta\psi,$$

where ϕ, $\psi\in R(i)$. Then, by lemma 4,

$$|\phi| = \left|\frac{\alpha\xi+\gamma\eta}{\Delta}\right| \leqslant \frac{|\xi|+|\eta|}{|\Delta|}\max\{|\alpha|, |\gamma|\}\leqslant N,$$

$$|\psi|\leqslant N,$$

so $(\phi,\psi)\in\mathfrak{G}_N$. Further, if there is inequality in (6·2), we have

$$|\psi z'-\phi| = \frac{|\xi(\beta z'-\alpha)+\eta(\delta z'-\gamma)|}{|\Delta|} < \max\{|\beta z'-\alpha|, |\delta z'-\gamma|\}$$

in contradiction with (5·4). If there is equality in (6·2), so that $\Delta=\epsilon(1+i)^2$ or $\Delta=\epsilon(1+i)^3$, then we have

$$\alpha\xi\pm\gamma\eta = \Delta\phi_\pm, \quad \beta\xi\pm\delta\eta = \Delta\psi_\pm,$$

where ϕ_+, ϕ_-, ψ_+, $\psi_-\in R(i)$. Hence

$$|\psi_\pm z'-\phi_\pm| = \left|\frac{\xi(\beta z'-\alpha)\pm\eta(\delta z'-\gamma)}{\Delta}\right| \leqslant |\beta z'-\alpha| = |\delta z'-\gamma|,$$

with strict inequality for one sign $\pm$. This, again, is in contradiction with (5·4).

This concludes the proof of the theorem.

7. Nodes

In this section we show that there are only three kinds of nodes. We remember that the boundary between $\mathfrak{R}(\alpha,\beta)$ and $\mathfrak{R}(\gamma,\delta)$ is part of the circle $|\beta z-\alpha| = |\delta z-\gamma|$. We summarize what is to be proved in the following theorem:

THEOREM V. *At most four regions* $\mathfrak{R}(\alpha,\beta)$ *meet at a node* z'. *There is always at least one pair of these,* $\mathfrak{R}(\alpha,\beta)$ *and* $\mathfrak{R}(\gamma,\delta)$ *such that* $|\alpha\delta-\beta\gamma|=1$. *On replacing* (α,β) *and* (γ,δ) *by equivalent elements and interchanging them if necessary, the regions* $\mathfrak{R}$ *meeting in* z' *and the angles subtended at* z' *are represented*† *in one of the following three diagrams* (figure 3). *If* $z' = \dfrac{\alpha+\gamma t'}{\beta+\delta t'}$ *the value of* t' *is as shown. Further*

$$\min_{(\phi,\psi)\in\mathfrak{G}_N}|\psi z'-\phi| = \frac{1}{|\beta+\delta t'|}.$$

We first prove a lemma:

LEMMA 5. *If* $\mathfrak{R}(\alpha,\beta)$, $\mathfrak{R}(\gamma,\delta)$ *and* $\mathfrak{R}(\eta,\theta)$ *all meet at a node, then at least one of* $|\alpha\delta-\beta\gamma|$, $|\alpha\theta-\beta\eta|$, $|\gamma\theta-\delta\eta|$ *is* 1.

† Of course, the diagrams are purely schematic. In general the regions are bounded by arcs of circles and the orientation is quite arbitrary.

For otherwise, since $\mathfrak{R}(\alpha,\beta)$, $\mathfrak{R}(\gamma,\delta)$, $\mathfrak{R}(\eta,\theta)$ are neighbours, we should have, by theorem IV,

$$|\alpha\delta-\beta\gamma| = |\alpha\theta-\beta\eta| = |\gamma\theta-\delta\eta| = 2^{\frac{1}{2}},$$

or, say,

$$\left.\begin{aligned}\alpha\delta-\beta\gamma &= \epsilon_1(1+i),\\ \alpha\theta-\beta\eta &= \epsilon_2(1+i),\\ \gamma\theta-\delta\eta &= \epsilon_3(1+i),\end{aligned}\right\} \tag{7·1}$$

a: $t' = -\rho$ or $-\rho^2$.

b: $t' = 1$ or i.

c: $t' = 1$ or i.

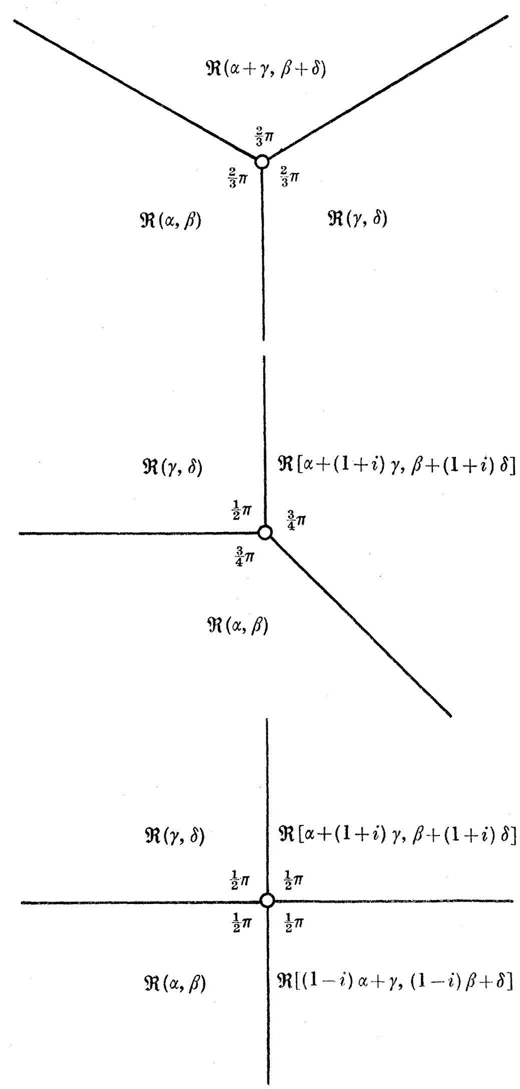

FIGURE 3*a* to *c*

where ϵ_1, ϵ_2, ϵ_3 are units ± 1, $\pm i$. Since $|\alpha\delta-\beta\gamma| \neq 0$, we may solve the equations

$$\left.\begin{aligned}\eta &= \lambda\alpha+\mu\gamma,\\ \theta &= \lambda\beta+\mu\delta,\end{aligned}\right\} \tag{7·2}$$

for λ and $\mu \in k(i)$. Indeed, by (7·1) we have $\lambda = -\epsilon_1^{-1}\epsilon_3$, $\mu = \epsilon_1^{-1}\epsilon_2 \in R(i)$, and so λ, μ are units. Hence, by lemma 3, and since $\lambda \equiv \mu \equiv 1 \ (1+i)$, we have

$$\eta \equiv \alpha+\gamma \equiv 2\alpha \equiv 0 \quad (1+i),$$
$$\theta \equiv \beta+\delta \equiv 2\beta \equiv 0 \quad (1+i),$$

contrary to the condition that $\gcd(\eta,\theta) = 1$.

COROLLARY. *If* (α,β), (γ,δ) *are such that* $|\alpha\delta-\beta\gamma|=1$ *then, by replacing* (η,θ) *by an equivalent pair if necessary, we have*

$$(\eta,\theta)=\lambda(\alpha,\beta)+\mu(\gamma,\delta), \tag{7·3}$$

where λ, μ *is one of the three pairs*

$$\left.\begin{aligned}\lambda&=1, & \mu&=\epsilon,\\ \lambda&=1, & \mu&=\epsilon(1+i),\\ \lambda&=1-i, & \mu&=\epsilon,\end{aligned}\right\} \tag{7·4}$$

and ϵ *is a unit.*

Since $|\alpha\delta-\beta\gamma|=1$ we can solve (7·3) for $\lambda,\mu\in R(i)$. Then

$$2^{\frac{1}{2}}\geqslant|\alpha\theta-\beta\eta|=|\mu|\,|\alpha\delta-\beta\gamma|=|\mu|,$$
$$2^{\frac{1}{2}}\geqslant|\lambda|.$$

Hence† $\lambda=\epsilon'$ or $\epsilon'(1-i)$ and $\mu=\epsilon''$ or $\epsilon''(1+i)$, where ϵ', ϵ'' are units. The corollary follows with $\epsilon=\epsilon'^{-1}\epsilon''$ on replacing (η,θ) by $(\eta\epsilon',\theta\epsilon')$. Since $\gcd(\lambda,\mu)=\gcd(\eta,\theta)=1$, the case $\lambda=\epsilon'(1-i)$, $\mu=\epsilon''(1+i)$ does not occur.

We now proceed to the proof of theorem V. The substitution

$$z=\frac{\alpha+\gamma t}{\beta+\delta t}$$

is conformal. Further,

$$\beta z-\alpha=\frac{-(\alpha\delta-\beta\gamma)\,t}{\beta+\delta t},\quad\therefore\quad|\beta z-\alpha|=\frac{|t|}{|\beta+\delta t|},$$

$$\delta z-\gamma=\frac{\alpha\delta-\beta\gamma}{\beta+\delta t},\quad\therefore\quad|\delta z-\gamma|=\frac{1}{|\beta+\delta t|},$$

and, by (7·3), $$\theta z-\eta=\frac{(\alpha\delta-\beta\gamma)(-\lambda t+\mu)}{\beta+\delta t},\quad\therefore\quad|\theta z-\eta|=\frac{|\lambda t-\mu|}{|\beta+\delta t|}.$$

Hence the equations of the boundaries become:

$$\left.\begin{array}{llll}\text{between} & \mathfrak{R}(\alpha,\beta) \text{ and } \mathfrak{R}(\gamma,\delta): & |t|=1,\\ \text{between} & \mathfrak{R}(\alpha,\beta) \text{ and } \mathfrak{R}(\eta,\theta): & |t|=|\lambda t-\mu|,\\ \text{between} & \mathfrak{R}(\gamma,\delta) \text{ and } \mathfrak{R}(\eta,\theta): & 1=|\lambda t-\mu|,\end{array}\right\} \tag{7·5}$$

and the node corresponds to a value t' of t such that

$$1=|t'|=|\lambda t'-\mu|.$$

We now examine the various cases in (7·4) in turn.

First case: $\lambda=1$, $\mu=\epsilon$. Then the three circles (7·5) become

$$|t|=1,\quad|t|=|t-\epsilon|,\quad 1=|t-\epsilon|.$$

These three circles (see figure 4) meet at $t=-\epsilon\rho$ and $t=-\epsilon\rho^2$, and in both cases make an angle of $\frac{2}{3}\pi$ with each other.

On putting $(\epsilon\gamma,\epsilon\delta)$ for (γ,δ) this gives case a of the theorem.

Second case: $\lambda=1$, $\mu=\epsilon(1+i)$. The three circles (7·5) become

$$|t|=1,\quad|t|=|t-\epsilon(1+i)|,\quad|t-\epsilon(1+i)|=1.$$

These meet at $t=\epsilon$ and $t=\epsilon i$ and make the angles $\frac{1}{2}\pi$, $\frac{3}{4}\pi$, $\frac{3}{4}\pi$ with each other (see figure 4). On putting $(\epsilon\gamma,\epsilon\delta)$ for (γ,δ) this gives case b of the theorem.

† By using $1-i$ in the expression for λ and $1+i$ in that for μ a more elegant formulation is obtained.

Third case: $\lambda = (1-i)$, $\mu = \epsilon$. The three circles (7·5) become

$$|t| = 1, \quad |t| = |(1-i)t-\epsilon|, \quad |(1-i)t-\epsilon| = 1.$$

These again meet at $t = \epsilon$ and $t = \epsilon i$ and make the angles $\frac{1}{2}\pi$, $\frac{3}{4}\pi$, $\frac{3}{4}\pi$ with each other. On putting $(\epsilon\alpha, \epsilon\beta)$ for (γ, δ) and $(-i\gamma, -i\delta)$ for (α, β), this again gives case b of the theorem.

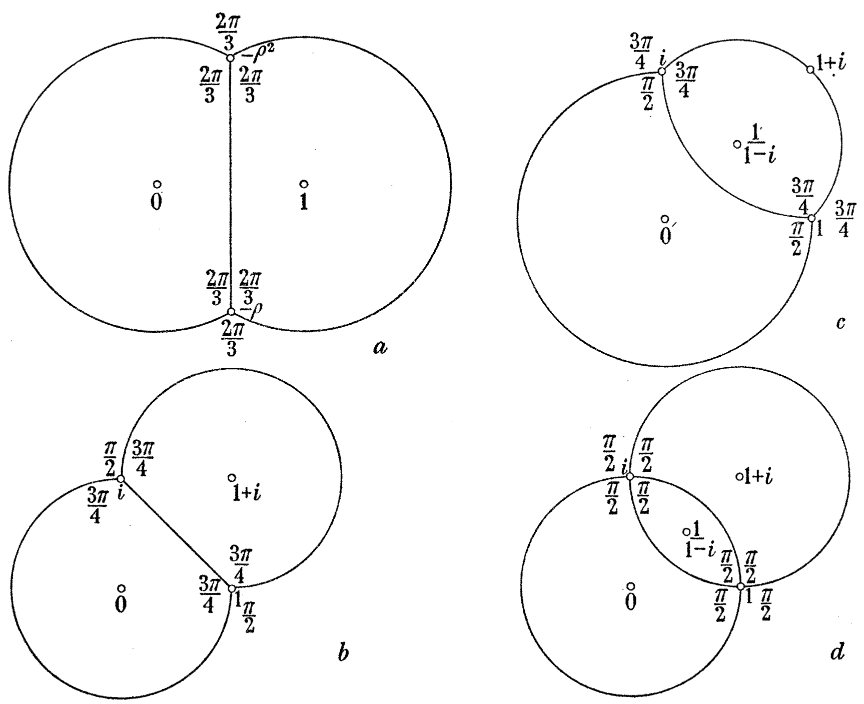

FIGURE 4. Diagrams of $(\epsilon = 1)$. a, first case. b, second case. c, third case (this is just the second case inverted in $|t| = 1$). d, fourth case.

It follows that all nodes where just three regions meet are of types a or b of the theorem. Suppose now four or more regions meet at z'. Then, by lemma 5 and its corollary, they can, in a suitable notation, be denoted by

$$\mathfrak{R}(\alpha,\beta), \quad \mathfrak{R}(\gamma,\delta), \quad \mathfrak{R}(\eta_1,\theta_1), \quad \mathfrak{R}(\eta_2,\theta_2), \quad \ldots,$$

where $$|\alpha\delta-\beta\gamma| = 1, \quad (\eta_j,\theta_j) = \lambda_j(\alpha,\beta)+\mu_j(\gamma,\delta) \quad (j = 1,2,\ldots)$$

$$\left.\begin{array}{llll} \text{and} & \lambda_j = 1, & \mu_j = \epsilon_j, \\ \text{or} & \lambda_j = 1, & \mu_j = \epsilon_j(1+i), \\ \text{or} & \lambda_j = 1-i, & \mu_j = \epsilon_j. \end{array}\right\} \tag{7·6}$$

Further, since $\mathfrak{R}(\eta_1,\theta_1)$ and $\mathfrak{R}(\eta_j,\theta_j)$ are adjacent, we have

$$2^{\frac{1}{2}} \geqslant |\eta_1\theta_j-\eta_j\theta_1| = |\lambda_1\mu_j-\mu_1\lambda_j|. \tag{7·7}$$

The truth of the theorem will now follow from a sequence of remarks.

(a) *If* $\lambda_1 = 1$, $\mu_1 = \epsilon_1$, *there can be no* $(\eta_2,\theta_2), \ldots$. Suppose, *per absurdum*, that (η_2,θ_2) exists. Then the values of t' corresponding to $\mathfrak{R}(\alpha,\beta)$, $\mathfrak{R}(\gamma,\delta)$, $\mathfrak{R}(\eta_1,\theta_1)$ and to $\mathfrak{R}(\alpha,\beta)$, $\mathfrak{R}(\gamma,\delta)$, $\mathfrak{R}(\eta_2,\theta_2)$ must be the same. Hence, since $(7·4_2)$ and $(7·4_3)$ give $t' \in k(i)$, but $(7·4_1)$ does not, we must have $\lambda_2 = 1$, $\mu_2 = \epsilon_2$. Hence

$$t' = (-\epsilon_1\rho \text{ or } -\epsilon_1\rho^2) = (-\epsilon_2\rho \text{ or } -\epsilon_2\rho^2).$$

Since $\epsilon_1, \epsilon_2 \in R(i)$, this is possible only when $\epsilon_1 = \epsilon_2$, i.e. when $(\eta_1, \theta_1) = (\eta_2, \theta_2)$. This proves the remark.

(*b*) *It is impossible that* $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \epsilon_1(1+i)$, $\mu_2 = \epsilon_2(1+i)$. If we substitute these values in (7·7), we get

$$|\epsilon_1 - \epsilon_2| \leqslant 1,$$

which is impossible for units $\epsilon_1, \epsilon_2 \in R(i)$ except when $\epsilon_1 = \epsilon_2$.

(*c*) *It is impossible that* $\lambda_1 = \lambda_2 = 1-i$, $\mu_1 = \epsilon_1$, $\mu_2 = \epsilon_2$. This follows as in (*b*).

(*d*) *At most four regions meet at a node. By interchanging* (η_1, θ_1) *with* (η_2, θ_2), *if necessary, we can have only*

$$\lambda_1 = 1, \quad \mu_1 = \epsilon_1(1+i), \quad \lambda_2 = 1-i, \quad \mu_2 = \epsilon_1.$$

Except for the fact that $\epsilon_1 = \epsilon_2$, this follows at once from (*a*), (*b*) and (*c*). That $\epsilon_1 = \epsilon_2$ follows at once from (7·7).

By putting $(\epsilon_1\gamma, \epsilon_1\delta)$ for (γ, δ) this gives us case *c* of the theorem. That the angles are as shown follows from figure 4 in the *t*-plane.

This concludes the proof of the theorem.

8. Construction of $\mathfrak{H}_{N+1}$ from $\mathfrak{H}_N$

In this section we prove the following theorem for integral N:

THEOREM VI (cf. theorem III of § 3). *If* $(\xi, \eta) \in \mathfrak{F}_{N+1}$ *but* $(\xi, \eta) \notin \mathfrak{F}_N$, *then there are adjacent elements* (α, β), $(\gamma, \delta) \in \mathfrak{F}_N$ *such that*

$$|\alpha\delta - \beta\gamma| = 1, \quad (\xi, \eta) = (\alpha+\gamma, \beta+\delta). \tag{8·1}$$

The proof of this result will follow immediately from theorem VIII. Theorems VII and VIII have no explicit reference to Farey section either in the enunciation or the proofs.

If $z \in \Omega$, we shall use $\mathscr{R}z$ and $\mathscr{I}z$ to denote the real and imaginary part of z respectively. We first quote a well-known lemma without proof.

LEMMA 6. *For any* $z \in \Omega$ *there is a* $\phi \in R(i)$ *such that*

$$|\mathscr{R}(z-\phi)| \leqslant \tfrac{1}{2}, \quad |\mathscr{I}(z-\phi)| \leqslant \tfrac{1}{2}, \tag{8·2}$$

and hence

$$|z-\phi| \leqslant 2^{-\frac{1}{2}}. \tag{8·3}$$

We use this to prove the following theorem:

THEOREM VII (cf. Lemma 2 of § 3). *Let* $\xi, \eta \in R(i)$ *and let*

$$\gcd(\xi, \eta) = 1, \quad |\xi\eta| > 1. \tag{8·4}$$

Then there exist $\alpha, \beta \in R(i)$ *such that*

$$\alpha\eta - \beta\xi = 1 \tag{8·5}$$

and

$$\left|\frac{\alpha}{\xi}\right| \leqslant 2^{-\frac{1}{2}}, \quad \left|\frac{\beta}{\eta}\right| \leqslant 2^{-\frac{1}{2}}. \tag{8·6}$$

Since $\gcd(\xi, \eta) = 1$, there certainly exists one solution $\alpha = \alpha_0$, $\beta = \beta_0$ of (8·5). Further,

$$\alpha = \alpha_0 - \phi\xi, \quad \beta = \beta_0 - \phi\eta, \tag{8·7}$$

for any $\phi \in R(i)$, is another solution of (8·5). We show that ϕ can be chosen so as to satisfy (8·6) as follows:

We first give a rough sketch of the proof. We have

$$\frac{\alpha}{\xi} = \frac{\alpha_0}{\xi} - \phi, \quad \frac{\beta}{\eta} = \frac{\beta_0}{\eta} - \phi, \tag{8·8}$$

and hence, by lemma 6, we can certainly satisfy each half of (8·6) separately, by taking $z = \alpha_0/\xi$ and $z = \beta_0/\eta$ respectively. However, by (8·5),

$$\frac{\alpha}{\xi} - \frac{\beta}{\eta} = \frac{1}{\xi\eta}, \tag{8·9}$$

and $1/\xi\eta$ is comparatively small; so we hope that we can satisfy both halves of (8·6) simultaneously.

The detailed proof requires some modifications in the above simple programme. We write

$$X = |\xi|^2, \quad Y = |\eta|^2.$$

The theorem is trivial if $X = 1$ or $Y = 1$; and so, by writing $-\eta$, $-\xi$ for ξ, η respectively if necessary, we may assume that

$$Y \geqslant X > 1. \tag{8·10}$$

We choose an arbitrary real $e > 0$, so small that

$$\left|\frac{e}{\xi\eta}\right| < \frac{1}{2Y}, \tag{8·11}$$

and put

$$z = \frac{\beta_0}{\eta} + \frac{e}{\xi\eta}. \tag{8·12}$$

Finally, we choose for ϕ the ϕ of lemma 6 appropriate to this z.

In the first place, by (8·2), (8·11) and (8·12),

$$\left|\mathscr{R}\left(\frac{\beta}{\eta}\right)\right| = \left|\mathscr{R}\left(z - \phi - \frac{e}{\xi\eta}\right)\right| \leqslant \frac{1}{2} + \left|\frac{e}{\xi\eta}\right| < \frac{1}{2} + \frac{1}{2Y}, \tag{8·13}$$

and, similarly,

$$\left|\mathscr{I}\left(\frac{\beta}{\eta}\right)\right| < \frac{1}{2} + \frac{1}{2Y}. \tag{8·14}$$

But

$$\frac{\beta}{\eta} = \frac{\beta\bar{\eta}}{\eta\bar{\eta}} = \frac{p + iq}{Y},$$

where $p, q \in R$. Hence, by (8·13) and (8·14),

$$|2p| < Y + 1, \quad |2q| < Y + 1,$$

i.e.

$$|2p| < Y, \quad |2q| \leqslant Y.$$

Thus finally

$$\left|\frac{\beta}{\eta}\right|^2 = \left(\frac{p}{Y}\right)^2 + \left(\frac{q}{Y}\right)^2 \leqslant \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which proves the second half of (8·6).

Further, by (8·9),

$$\frac{\alpha}{\xi} = \frac{\beta}{\eta} + \frac{1}{\xi\eta} = z - \phi + \frac{(1-e)}{\xi\eta};$$

and so, by (8·10) and since $e > 0$,

$$\left|\mathscr{R}\left(\frac{\alpha}{\xi}\right)\right| \leqslant \frac{1}{2} + \frac{1-e}{|\xi\eta|} < \frac{1}{2} + \frac{1}{X}, \tag{8·15}$$

$$\left|\mathscr{I}\left(\frac{\alpha}{\xi}\right)\right| < \frac{1}{2} + \frac{1}{X}. \tag{8·16}$$

As before, we have

$$\frac{\alpha}{\xi} = \frac{\alpha\bar{\xi}}{\xi\bar{\xi}} = \frac{l + im}{X},$$

where l, $m \in R$ and

$$l^2 + m^2 = \mathscr{N}(\alpha\bar{\xi}) \equiv 0 \ \ (X). \tag{8·17}$$

All that remains to be proved now is that $|\alpha/\xi|^2 \leqslant \frac{1}{2}$, i.e. that

$$l^2 + m^2 \leqslant \tfrac{1}{2}X^2. \tag{8·18}$$

By (8·15), (8·16) we have $$|2l|<X+2, \quad |2m|<X+2. \tag{8·19}$$
If X is even, (8·19) implies that $|2l|\leqslant X$, $|2m|\leqslant X$, and so that (8·18) holds. Hence for the rest of the proof we may assume that X is odd and so, by (8·10) and (8·19), that

$$X\geqslant 3, \quad |l|\leqslant\tfrac{1}{2}(X+1), \quad |m|\leqslant\tfrac{1}{2}(X+1). \tag{8·20}$$

We subdivide the proof into a number of cases

(i) $|l|\leqslant\frac{1}{2}(X-1), \quad |m|\leqslant\frac{1}{2}(X-1)$,

(ii) $|l|=\frac{1}{2}(X+1), \quad |m|\leqslant\frac{1}{2}(X-3)$,

(iii) $|l|\leqslant\frac{1}{2}(X-3), \quad |m|=\frac{1}{2}(X+1)$,

(iv) $|l|=|m|=\frac{1}{2}(X+1)$,

(v) $|l|=\frac{1}{2}(X+1), \quad |m|=\frac{1}{2}(X-1)$ or $|l|=\frac{1}{2}(X-1), \quad |m|=\frac{1}{2}(X+1)$.

In case (i) the inequality (8·18) follows at once. In cases (ii) and (iii) we have, by (8·20),

$$|l|^2+|m|^2\leqslant\tfrac{1}{4}\{2X^2-4X+10\}<\tfrac{1}{2}X^2,$$

and so again (8·20) follows. Finally, cases (iv) and (v) cannot really occur since they contradict (8·17). Indeed, (iv) and (v) would give

$$0\equiv l^2+m^2=2\left(\frac{X+1}{2}\right)^2\equiv\frac{X+1}{2} \quad (X),$$

and

$$0\equiv l^2+m^2=\left(\frac{X+1}{2}\right)^2+\left(\frac{X-1}{2}\right)^2\equiv\frac{-X+1}{2} \quad (X).$$

This is impossible since $$0<\left|\frac{X+1}{2}\right|<X, \quad 0<\left|\frac{-X+1}{2}\right|<X$$
by (8·20).

This concludes the proof of the theorem. We note that we have also proved a corollary:

COROLLARY. $$\left|\mathscr{R}\left(\frac{\beta}{\eta}\right)\right|\leqslant\tfrac{1}{2}, \quad \left|\mathscr{I}\left(\frac{\beta}{\eta}\right)\right|\leqslant\tfrac{1}{2}.$$

We now prove the existence of a pair γ, $\delta\in R(i)$ with certain properties.

THEOREM VIII. *Let the conditions of theorem* VII *hold. Then there are* γ, δ, $\epsilon\in R(i)$ *such that* $|\epsilon|=1$ *and*

$$|\gamma\eta-\delta\xi|=|\alpha\delta-\beta\gamma|=|\delta\xi-\beta\eta|=1, \tag{8·21}$$

$$(\xi,\eta)=\epsilon(\alpha,\beta)+(\gamma,\delta) \tag{8·22}$$

and $$|\gamma|\leqslant|\xi|, \quad |\delta|\leqslant|\eta|. \tag{8·23}$$

The signs of equality in (8·23) *are required only when* $|\xi|=1$ *or* $|\eta|=1$ *respectively.*

The proof when $|\xi|=1$ or $|\eta|=1$ is immediate; so we may assume that (8·10) holds.

The proof depends on the following lemma:

LEMMA 7. *Let* $z\in\Omega$ *and* $$|\mathscr{R}z|\leqslant\tfrac{1}{2}, \quad |\mathscr{I}z|\leqslant\tfrac{1}{2};$$
then there is an $\epsilon\in R(i)$, $|\epsilon|=1$ *such that*

$$|\mathscr{R}(1-\epsilon z)|+|\mathscr{I}(1-\epsilon z)|\leqslant 1. \tag{8·24}$$

We may choose ϵ so that $$\tfrac{1}{2}\geqslant\mathscr{R}(\epsilon z)\geqslant|\mathscr{I}(\epsilon z)|.$$
The truth of the lemma now follows.

We now apply this lemma, as we may by the corollary to theorem VII, to

$$z=\frac{\beta}{\eta}.$$

With this choice of ϵ we now put

$$\gamma = \xi - \epsilon\alpha, \quad \delta = \eta - \epsilon\beta.$$

Then (8·21) and (8·22) hold. (8·23) remains to be proved.

We note first that

$$\left|\mathscr{R}\left(1-\frac{\epsilon\beta}{\eta}\right)\right| = \left|\mathscr{R}\left(\frac{\delta}{\eta}\right)\right| \neq 1, \tag{8·25}$$

since otherwise we should, by lemma 7, have $\left|\mathscr{I}\left(1-\frac{\epsilon\beta}{\eta}\right)\right| = \left|\mathscr{I}\left(\frac{\delta}{\eta}\right)\right| = 0$ and so $\delta = \pm\eta$, contrary to (8·21). Similarly

$$\left|\mathscr{I}\left(1-\frac{\epsilon\beta}{\eta}\right)\right| = \left|\mathscr{I}\left(\frac{\delta}{\eta}\right)\right| \neq 1. \tag{8·26}$$

Hence by (8·24), (8·25) and (8·26)

$$\left|\frac{\delta}{\eta}\right|^2 = \left|\mathscr{R}\left(\frac{\delta}{\eta}\right)\right|^2 + \left|\mathscr{I}\left(\frac{\delta}{\eta}\right)\right|^2 < 1$$

as required.

Further

$$\frac{\gamma}{\xi} - \frac{\delta}{\eta} = -\epsilon\left(\frac{\alpha}{\xi} - \frac{\beta}{\eta}\right) = \frac{-\epsilon}{\xi\eta},$$

and so

$$\left|\mathscr{R}\left(\frac{\gamma}{\xi}\right)\right| + \left|\mathscr{I}\left(\frac{\gamma}{\xi}\right)\right| \leqslant \left|\mathscr{R}\left(\frac{\delta}{\eta}\right)\right| + \left|\mathscr{I}\left(\frac{\delta}{\eta}\right)\right| + \left|\mathscr{R}\left(\frac{\epsilon}{\xi\eta}\right)\right| + \left|\mathscr{I}\left(\frac{\epsilon}{\xi\eta}\right)\right|$$

$$\leqslant 1 + \frac{\surd 2}{|\xi\eta|} \leqslant 1 + \frac{\surd 2}{X} \tag{8·27}$$

by (8·24) and (8·10).

But

$$\frac{\gamma}{\xi} = \frac{\gamma\bar{\xi}}{\xi\bar{\xi}} = \frac{u+iv}{X},$$

where $u, v \in R$ and

$$u^2 + v^2 = |\gamma\bar{\xi}|^2 \equiv 0 \ (X). \tag{8·28}$$

and, by (8·27), $|u| + |v| \leqslant X + \surd 2$, i.e.

$$|u| + |v| \leqslant X + 1.$$

We consider the following cases:

(i) $\max\{|u|, |v|\} < X$,

(ii) $|u| = 0, \quad |v| = X+1$ or $|u| = X+1, \quad |v| = 0$,

(iii) $|u| = 1, \quad |v| = X$ or $|u| = X, \quad |v| = 1$,

(iv) $|u| = 0, \quad |v| = X$ or $|u| = X, \quad |v| = 0$.

In case (i) we have

$$\left|\frac{\gamma}{\xi}\right|^2 = \left(\frac{u}{X}\right)^2 + \left(\frac{v}{X}\right)^2 \leqslant \frac{(X-1)^2 + 2^2}{X^2} < 1$$

if $X \geqslant 3$, and if $X = 2$ we have $|u| \leqslant 1$, $|v| \leqslant 1$ and again $(u/X)^2 + (v/X)^2 < 1$. Hence (8·23) with $<$ always holds in case (i). We show that the remaining cases cannot in fact occur. Cases (ii) and (iii) contradict the criterion (8·28). Case (iv) would imply that $\gamma = \epsilon'\xi$ with $|\epsilon'| = 1$, and so we should have $\gamma\eta - \delta\xi \equiv 0 \ (\xi)$ in contradiction with (8·10) and (8·21).

The proof of theorem VI is now immediate. On writing α, β for $\epsilon\alpha$, $\epsilon\beta$ we see that (8·1) holds. Further, by theorems VII and VIII

$$\max\{\mathscr{N}(\alpha), \mathscr{N}(\beta), \mathscr{N}(\gamma), \mathscr{N}(\delta)\} < \max\{\mathscr{N}(\xi), \mathscr{N}(\eta)\} \leqslant N+1,$$

and hence

$$\max\{\mathscr{N}(\alpha), \mathscr{N}(\beta), \mathscr{N}(\gamma), \mathscr{N}(\delta)\} \leqslant N.$$

Also

$$\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1 \quad \text{by (8·1)}.$$

Hence $$(\alpha,\beta)\in\mathfrak{F}_N, \quad (\gamma,\delta)\in\mathfrak{F}_N.$$

Finally, (α,β) and (γ,δ) are adjacent by (8·1) and theorem IV (§ 6).

9. The corresponding results for $k(\rho)$

We may define $\mathfrak{F}_N$, $\mathfrak{G}_N$, $\mathfrak{H}_N$ and $\mathfrak{R}(\alpha,\beta)$ as at the beginning of § 5 except that $R(\rho)$ is read for $R(i)$. All of the preceding argument may be carried out in $k(\rho)$ more or less as in $k(i)$. We shall not carry out the details but summarize the results in the following theorems followed by a few brief comments.

Theorem IX (cf. theorem IV). *The necessary and sufficient condition that (α,β) and $(\gamma,\delta)\in\mathfrak{F}_N$ be adjacent is that simultaneously*

(i) $|\alpha\delta-\beta\gamma| = 1$ or $3^{\frac{1}{2}}$,

(ii) $(\alpha+\epsilon\gamma,\beta+\epsilon\delta)\notin\mathfrak{G}_N$ *for some choice of* $\epsilon=\pm 1, \pm\rho, \pm\rho^2$, *where ϵ must also satisfy*
$$\alpha+\epsilon\gamma\equiv\beta+\epsilon\delta\equiv 0 \quad (1-\rho) \quad \text{if} \quad |\alpha\delta-\beta\gamma| = 3^{\frac{1}{2}}.$$

The proof of this theorem depends on the following analogue of lemma 4, which is proved in the same way:

Lemma 8 (cf. lemma 4, § 6). *Suppose α, β, γ, $\delta\in R(\rho)$ and $\Delta=\alpha\delta-\beta\gamma$, $\mathcal{N}(\Delta)>3$. Then there are ξ, $\eta\in R(\rho)$ such that*
$$\alpha\xi+\gamma\eta\equiv\beta\xi+\delta\eta\equiv 0 \quad (\Delta)$$
and
$$0<|\xi|+|\eta|\leqslant|\Delta|.$$
The last $\leqslant$ can be replaced by $<$ except when $|\Delta|=2$ and then $|\xi|<|\Delta|$, $|\eta|<|\Delta|$.

From theorem IX we may deduce the following analogue of theorem V:

Theorem X (cf. theorem V, § 7). *At most four regions $\mathfrak{R}(\alpha,\beta)$ meet at a node z'. There is always at least one pair of these, $\mathfrak{R}(\alpha,\beta)$ and $\mathfrak{R}(\gamma,\delta)$, such that $|\alpha\delta-\beta\gamma|=1$. On replacing (α,β) and (γ,δ) by equivalent elements and interchanging them if necessary, the regions $\mathfrak{R}$ meeting at z' and the angles subtended at z' are as represented in one of the three figures 5a to c. If*
$$z'=\frac{\alpha+\gamma t'}{\beta+\delta t'},$$
the value of t' is as shown. Further
$$\min_{(\phi,\psi)\in\mathfrak{F}_N}|\psi z'-\phi| = \frac{1}{|\beta+\delta t'|}.$$

We note that there is a corollary, the analogue of which is not true in $k(i)$.

Corollary. *All nodes $z'\in k(\rho)$.*

We shall see that this means that the problems of part II are more easily solved in $k(\rho)$ than in $k(i)$.

Finally the analogue of theorem VI is

Theorem XI (cf. theorem VI, § 8). *If $N\in R$ and $(\xi,\eta)\in\mathfrak{F}_{N+1}$ but $(\xi,\eta)\notin\mathfrak{F}_N$, there are adjacent elements (α,β), $(\gamma,\delta)\in\mathfrak{F}_N$ such that*
$$|\alpha\delta-\beta\gamma|=1, \quad (\xi,\eta)=(\alpha,\beta)+(\gamma,\delta).$$

The proof, as before, depends on two theorems with no reference to Farey section.

Theorem XII (cf. theorem VII, § 8). *Let $\xi,\eta\in R(\rho)$ and let*
$$\gcd(\xi,\eta)=1, \quad |\xi\eta|>1.$$
Then there exist $\alpha,\beta\in R(\rho)$ such that
$$\alpha\eta-\beta\xi=1$$
and
$$\left|\frac{\alpha}{\xi}\right|\leqslant 3^{-\frac{1}{2}}, \quad \left|\frac{\beta}{\eta}\right|\leqslant 3^{-\frac{1}{2}}.$$

a: $t' = -\rho$ or $-\rho^2$.

b: $t' = 1$ or $-\rho$.

c: $t' = -\rho^2$.

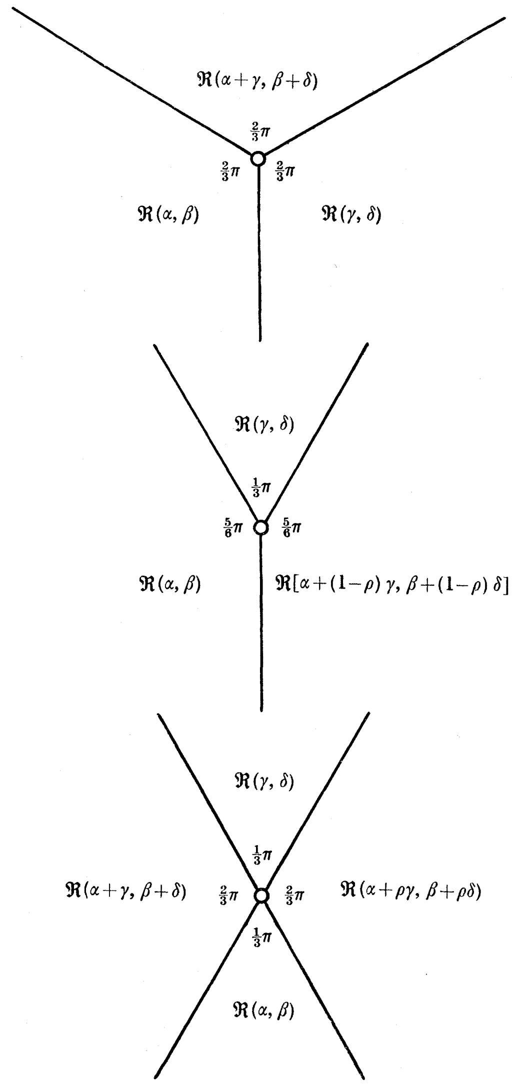

FIGURE 5*a* to *c*

THEOREM XIII (cf. theorem VIII, § 8). *There are further $\gamma, \delta, \epsilon \in R(\rho)$ with $|\epsilon| = 1$ such that*

$$|\gamma\eta - \delta\xi| = |\alpha\delta - \beta\eta| = 1,$$

$$(\xi, \eta) = \epsilon(\alpha, \beta) + (\gamma, \delta),$$

and
$$|\gamma| \leqslant |\xi|, \quad |\delta| \leqslant |\eta|.$$

The signs of equality are required only when $|\xi| = 1$ or $|\eta| = 1$ respectively.

Theorem XII is rather difficult to prove neatly, so we sketch the lemma on which the proof depends:

LEMMA 9 (cf. lemma 6, § 8). *For any $z \in \Omega$ there is a $\phi \in R(\rho)$ such that*

$$|\mathscr{R}(z-\phi)| \leqslant \tfrac{1}{2}, \quad |\mathscr{R}\{\rho(z-\phi)\}| \leqslant \tfrac{1}{2}, \quad |\mathscr{R}\{\rho^2(z-\phi)\}| \leqslant \tfrac{1}{2}.$$

The proof of this is immediate on dividing the Gauss plane of z by three sets of parallel lines

$$\left.\begin{aligned} \mathscr{R}(z-\phi) &= \pm\tfrac{1}{2}, \\ \mathscr{R}\{\rho(z-\phi)\} &= \pm\tfrac{1}{2}, \\ \mathscr{R}\{\rho^2(z-\phi)\} &= \pm\tfrac{1}{2}, \end{aligned}\right\} \quad \phi \in R(\rho)$$

into congruent hexagons.

COROLLARY. $$|z-\phi|\leqslant 3^{-\frac{1}{2}}.$$

On writing $$u=\mathscr{R}(z-\phi),\quad v=\mathscr{R}\{\rho(z-\phi)\},\quad w=\mathscr{R}\{\rho^2(z-\phi)\},$$
we have $$u+v+w=\mathscr{R}\{(1+\rho+\rho^2)\,(z-\phi)\}=0$$
and $$|u|\leqslant\tfrac{1}{2},\quad |v|\leqslant\tfrac{1}{2},\quad |w|\leqslant\tfrac{1}{2}. \tag{9·1}$$
Further $$\begin{aligned}|z-\phi|^2&=-\tfrac{4}{3}(uv+vw+wu)\\&=\tfrac{4}{3}(u^2-vw)=\tfrac{4}{3}(v^2-uw)=\tfrac{4}{3}(w^2-uv).\end{aligned} \tag{9·2}$$
Since two of u, v, w must have the same sign (0 counted as both $+$, $-$), the corollary follows from (9·1), (9·2).

The proof of theorems XII, XIII then follows on the same general lines as that of theorems VII, VIII.

PART II

10. INTRODUCTION

In this part II we shall use the results of part I to prove the following two theorems:

THEOREM XIV. *Let $\alpha, \beta, \gamma, \delta \in \Omega$ and $u>0$, $v>0$ such that*
$$uv\geqslant\frac{\sqrt{2}}{3-\sqrt{3}}|\alpha\delta-\beta\gamma|>0. \tag{10·1}$$
Then there are $\xi, \eta \in R(i)$, not both zero, such that
$$|\alpha\xi+\beta\eta|\leqslant u,\quad |\gamma\xi+\delta\eta|\leqslant v. \tag{10·2}$$
The theorem would be false if the $\leqslant$ signs in (10·2) were replaced by $<$.

THEOREM XV. *Let $\alpha, \beta, \gamma, \delta \in \Omega$ and $u>0$, $v>0$ such that*
$$uv\geqslant|\alpha\delta-\beta\gamma|>0. \tag{10·3}$$
Then there are $\xi, \eta \in R(\rho)$, not both zero, such that
$$|\alpha\xi+\beta\eta|\leqslant u,\quad |\gamma\xi+\delta\eta|\leqslant v. \tag{10·4}$$
The theorem would be false if the $\leqslant$ signs in (10·4) were replaced by $<$.

These theorems have been proved by Minkowski (1907), and another proof of theorem XIV has been given by Hlawka (1941); but it is hoped that the proofs given here are more transparent. The theorems are, of course, generalizations of Minkowski's fundamental theorem:

THEOREM XVI. *Let a, b, c, d be real numbers and $u>0$, $v>0$ such that*
$$uv\geqslant|ad-bc|>0. \tag{10·5}$$
Then there are $x, y \in R$, not both zero, such that
$$|ax+by|\leqslant u,\quad |cx+dy|\leqslant v. \tag{10·6}$$
The $\leqslant$ signs in (10·6) cannot be replaced by $<$.

Theorem XVI is a direct consequence of Minkowski's convex-body theorem, but theorems XIV and XV are not.

In § 11 we show that theorems XIV and XV can be deduced from what are apparently special cases. In § 12 we show the relevance of part I to these special cases and in §§ 13 and 14 respectively we conclude the arguments for $k(\rho)$ and $k(i)$.

In the rest of this section we show by giving actual examples that the last sentences of theorems XIV and XV are true.

For theorem XV we just put $\alpha = \delta = 1$, $\beta = \gamma = 0$ and $u = v = 1$ so that (10·3) holds. If we could have $<$ in (10·4) we should have $|\xi|<1$, $|\eta|<1$, which is impossible for $\xi, \eta \in R(\rho)$ except in the excluded case $\xi = \eta = 0$.

For theorem XIV we put $\alpha = \gamma = 1$, $\beta = -\rho$, $\delta = -\rho^2$, $u = 1$, $v = \dfrac{\sqrt{3}+1}{\sqrt{2}} = |i-\rho^2|$. Then (10·1) is true (with $=$). We shall show that

$$|\xi - \rho\eta| < 1, \tag{10·7}$$

$$|\xi - \rho^2\eta| < \frac{\sqrt{3}+1}{\sqrt{2}}, \tag{10·8}$$

together imply $\xi = \eta = 0$. Suppose not. First, it follows at once from (10·7) that

$$\xi \neq 0, \quad \eta \neq 0.$$

Next,
$$3^{\frac{1}{2}}|\eta| = |(\xi-\rho\eta)-(\xi-\rho^2\eta)| < 1 + \frac{\sqrt{3}+1}{\sqrt{2}},$$
and so, since $\eta \in R(i)$,
$$|\eta| = 1 \quad \text{or} \quad |\eta| = 2^{\frac{1}{2}}.$$
Similarly
$$|\xi| = 1 \quad \text{or} \quad |\xi| = 2^{\frac{1}{2}}.$$
If $|\xi| = |\eta| = 2^{\frac{1}{2}}$, then gcd (ξ, η) would be $1+i$ and $\xi' = \xi/(1+i)$, $\eta' = \eta/(1+i)$ would be another solution of (10·7) and (10·8). Hence we may assume

$$\min\{|\xi|, |\eta|\} = 1.$$

If $|\eta| = 1$, put $\theta = \xi/\eta$, and if $|\xi| = 1$, put $\theta = \bar{\eta}/\bar{\xi}$. Then, in any case, (10·7) and (10·8) become

$$\left.\begin{aligned} &|\theta - \rho| < 1, \\ &|\theta - \rho^2| < \frac{\sqrt{3}+1}{\sqrt{2}}, \end{aligned}\right\} \tag{10·9}$$

where $|\theta| = 1$ or $2^{\frac{1}{2}}$, i.e. $\theta = \pm 1$, $\pm i$ or $\pm 1 \pm i$ (independent signs). As is easily verified, none of these θ satisfy (10·9). Hence our initial assumption that we could read $<$ in (10·2) is false.

Our method of proof has the disadvantage compared with that of Minkowski that it does not show that these are essentially the only cases where equality is required in (10·2) and (10·4).

11. A simplification of the problem

A special case ($a = 0$, $b = c = 1$, $d = -\theta$, $u = N$, $v = 1/N$) of theorem XVI is the following result:

Corollary. *Given any real numbers $N > 0$ and θ, there are $x, y \in R$, not both zero, such that*

$$|y| \leqslant N, \quad |y\theta - x| \leqslant 1/N.$$

There is a well-known argument due to Hilbert (cf. Minkowski 1907, chap. 1, §§ 6 to 10) which conversely deduces theorem XVI from this corollary. In this paragraph we show that similarly theorems XIV and XV may be deduced from special cases. The proof is a straightforward generalization of Hilbert's argument.

We shall denote by $k(\sqrt{(-m)})$ a quadratic-imaginary extension of k and by $R(\sqrt{(-m)})$ its ring of integers, and prove the following theorem:

Theorem XVII. *Let all ideals in $R(\sqrt{(-m)})$ be principal (e.g. $R(\sqrt{(-m)})$ is $R(i)$ or $R(\rho)$). Suppose there is a $c > 0$ depending only on m with the property that for every $z \in \Omega$ and $N > 0$ there are $\xi, \eta \in R(\sqrt{(-m)})$, not both zero, such that*†

$$\mathscr{N}(\eta) \leqslant N, \quad |\eta z - \xi| \leqslant cN^{-\frac{1}{2}}.$$

† By considering $0 < N < 1$ it is easily seen to be necessary that $c \geqslant 1$.

Then for every $\alpha, \beta, \gamma, \delta \in \Omega$ *and every* $u, v > 0$ *such that*

$$uv \geqslant c\,|\,\alpha\delta - \beta\gamma\,| > 0, \tag{11·1}$$

there are $\xi, \eta \in R(\surd(-m))$, *not both zero, such that*

$$|\,\alpha\xi + \beta\eta\,| \leqslant u, \quad |\,\gamma\xi + \delta\eta\,| \leqslant v. \tag{11·2}$$

We first prove a lemma.

Lemma 10. *The conclusion of theorem XVII holds if* $\alpha, \beta, \gamma, \delta \in R(\surd(-m))$.

Since $R(\surd(-m))$ *is a principal ideal ring we may put*

$$\gamma = \psi\gamma_1, \quad \delta = \psi\delta_1,$$

where $\quad \gamma_1, \delta_1, \psi \in R(\surd(-m)), \quad \gcd(\gamma_1, \delta_1) = 1,$

and may choose $\alpha_1, \beta_1 \in R(\surd(-m))$ such that

$$\alpha_1\delta_1 - \beta_1\gamma_1 = 1.$$

The unimodular transformation

$$\left.\begin{aligned}\xi_1 &= \alpha_1\xi + \beta_1\eta,\\ \eta_1 &= \gamma_1\xi + \delta_1\eta,\end{aligned}\right\} \tag{11·3}$$

then gives $\quad \alpha\xi + \beta\eta = \phi\xi_1 + \chi\eta_1, \quad \gamma\xi + \delta\eta = \psi\eta_1,$

where $\phi, \chi \in R(\surd(-m))$ and

$$|\,\alpha\delta - \beta\gamma\,| = |\,\phi\psi\,| > 0. \tag{11·4}$$

We now put $\quad N = \dfrac{v^2}{|\,\psi\,|^2}, \quad z = -\chi/\phi$

in the definition of c. Then there are $\xi_1, \eta_1 \in R(\surd(-m))$, not both zero, such that

$$\mathcal{N}(\eta_1) \leqslant \frac{v^2}{|\,\psi\,|^2}, \quad \left|\frac{\chi}{\phi}\eta_1 + \xi_1\right| \leqslant \frac{c\,|\,\psi\,|}{v},$$

i.e. such that $\quad |\,\psi\eta_1\,| \leqslant v, \quad |\,\phi\xi_1 + \chi\eta_1\,| \leqslant \dfrac{c\,|\,\phi\psi\,|}{v} \leqslant u$

by (11·1) and (11·4). Since $\xi, \eta \in R(\surd(-m))$ may now be determined from (11·3), this proves the lemma.

Corollary. *The conclusions of theorem XVII hold if* $\alpha, \beta, \gamma, \delta \in k((\surd -m))$.

We may choose $\omega \in R(\surd(-m))$ such that $\omega\alpha, \omega\beta, \omega\gamma, \omega\delta \in R(\surd(-m))$ and apply the previous lemma with $\omega\alpha$, $\omega\beta$, $\omega\gamma$, $\omega\delta$, $|\,\omega\,|\,u$, $|\,\omega\,|\,v$ for α, β, γ, δ, u, v.

We may now prove theorem XVII. Given α, β, γ, $\delta \in \Omega$ and $u, v > 0$ satisfying (11·1) we may choose an infinite sequence of sets

$$\alpha^{(j)}, \beta^{(j)}, \gamma^{(j)}, \delta^{(j)} \in k(\surd(-m)), \quad u^{(j)} > 0, \quad v^{(j)} > 0 \quad (j = 1, 2, \ldots),$$

such that $\quad u^{(j)}v^{(j)} \geqslant c\,|\,\alpha^{(j)}\delta^{(j)} - \beta^{(j)}\gamma^{(j)}\,| > 0,$

and $\quad \alpha^{(j)} \to \alpha, \quad \beta^{(j)} \to \beta, \quad \gamma^{(j)} \to \gamma, \quad \delta^{(j)} \to \delta, \quad u^{(j)} \to u, \quad v^{(j)} \to v.$

Then, by the last corollary we may find $\xi^{(j)}, \eta^{(j)} \in R(\surd(-m))$, not both zero, such that

$$|\,\alpha^{(j)}\xi^{(j)} + \beta^{(j)}\eta^{(j)}\,| \leqslant u^{(j)}, \quad |\,\gamma^{(j)}\xi^{(j)} + \delta^{(j)}\eta^{(j)}\,| \leqslant v^{(j)}.$$

Since, as is easily verified, all the pairs $\xi^{(j)}, \eta^{(j)}$ lie in a finite part of the ξ, η plane, some pair ξ', η' must occur for an infinite sequence $j = j_l$ $(l = 1, 2, \ldots)$ of values of j. Then, by letting l tend to infinity, we have $\quad |\,\alpha\xi' + \beta\eta'\,| \leqslant u, \quad |\,\gamma\xi' + \delta\eta'\,| \leqslant v,$

as required.

12. The Farey argument. Introduction of $\mathfrak{H}_N^*$, etc.

By theorem XVII the proofs of theorems XIV and XV are reduced to a consideration of

$$\min_{\alpha,\beta} |\beta z - \alpha| \tag{12·1}$$

over all $\alpha, \beta \in R(i)$, $R(\rho)$ respectively, such that

$$\mathcal{N}(\beta) \leqslant N,$$

where $z \in \Omega$ and $N < 0$ are given. We note that no bound is put on $\mathcal{N}(\alpha)$. We are therefore led to consider $\mathfrak{H}_N^*$, the set of all reduced fractions α/β:

$$\alpha/\beta \in \mathfrak{H}_N^* \quad .\equiv. \quad \mathcal{N}(\beta) \leqslant N,$$

and $\alpha, \beta \in R(i)$, $R(\rho)$, respectively. We now define $\mathfrak{F}_N^*$, $\mathfrak{G}_N^*$ in terms of $\mathfrak{H}_N^*$ precisely as $\mathfrak{F}_N$, $\mathfrak{G}_N$ were defined in terms of $\mathfrak{H}_N$.

The theory of $\mathfrak{H}_N^*$ runs entirely parallel to that of $\mathfrak{H}_N$. It is slightly simpler because of the lack of restriction on $\mathcal{N}(\alpha)$. In particular, we may define the region $\mathfrak{R}^*(\alpha,\beta)$ belonging to $(\alpha,\beta) \in \mathfrak{F}_N^*$. As may easily be verified theorems IV, V, VI, X and XII hold if $\mathfrak{F}^*$, $\mathfrak{G}^*$, $\mathfrak{H}^*$, $\mathfrak{R}^*$, ... are read for $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{R}$, ..., etc.

With this notation we shall write $\Delta_N(z)$ for (12·1), i.e.

$$\Delta_N(z) = \min_{(\alpha,\beta) \in \mathfrak{F}_N^*} |\beta z - \alpha|. \tag{12·2}$$

Lemma 11. *We may assume $\beta \neq 0$ in (12·2).*

If $\beta = 0$, we must have $\alpha \neq 0$, and so $|\beta z - \alpha| \geqslant 1$. Since† $N \geqslant 1$ we may put $\beta = 1$ in (12·2) and then choose α so that $|z - \alpha| \leqslant 2^{-\frac{1}{2}} < 1$ (for $R(i)$), $|z-\alpha| \leqslant 3^{-\frac{1}{2}} < 1$ (for $R(\rho)$). Thus the minimum is not attained for $\beta = 0$.

Corollary. *$\mathfrak{R}^*(\alpha, 0)$ is null for all α.*

We now prove the following theorem:

Theorem XVIII. *In both $R(i)$ and $R(\rho)$ the maximum of $\Delta_N(z)$ is attained at a node of $\mathfrak{F}_N^*$.*

We give the proof for $R(i)$. That for $R(\rho)$ is similar.

We first note that $\Delta_N(z)$ cannot attain its maximum at an inner point of an $\mathfrak{R}^*(\alpha,\beta)$, since there

$$\Delta_N(z) = \min_{(\gamma,\delta) \in \mathfrak{F}_N^*} |\delta z - \gamma| = |\beta z - \alpha|$$

and $\beta \neq 0$ by lemma 11, corollary. Hence if theorem XVIII were false, $\Delta_N(z)$ would attain its maximum at an inner point of the boundary between two regions, $\mathfrak{R}^*(\alpha,\beta)$, $\mathfrak{R}^*(\gamma,\delta)$ say. The boundary is an arc or arcs of the circle

$$|\beta z - \alpha| = |\delta z - \gamma|. \tag{12·3}$$

The truth of the theorem is then a consequence of the following lemma:

Lemma 12. *Let (α,β), $(\gamma,\delta) \in \mathfrak{F}_N^*$, and let z_0 be a value of z on the circle*

$$|\beta z - \alpha| = |\delta z - \gamma|,$$

for which $|\beta z - \alpha| = |\delta z - \gamma|$ is a (local) maximum. Then there is a $(\lambda,\mu) \in \mathfrak{F}_N^*$ such that

$$|\mu z_0 - \lambda| < |\beta z_0 - \alpha| = |\gamma z_0 - \delta|.$$

The z satisfying (12·3) are of the form

$$z = z(\theta) = \frac{\alpha + \gamma e^{i\theta}}{\beta + \delta e^{i\theta}},$$

† We defined $\mathfrak{F}_N$ and $\mathfrak{F}_N^*$ only for $N \geqslant 1$.

where θ is real. $z_0 = z(\theta_0)$, where θ_0 is a θ for which

$$|\beta z-\alpha| = |\delta z-\gamma| = \frac{|\alpha\delta-\beta\gamma|}{|\beta+\delta e^{i\theta}|}$$

is a (local) maximum, i.e. $|\beta+\delta e^{i\theta}|$ is a (local) minimum. This can only be the θ_0 defined by

$$e^{i\theta_0} = \operatorname{sgn}\left(\frac{-\beta}{\delta}\right).$$

We may now choose $l \in R$ such that $\quad |\theta_0-\tfrac{1}{2}l\pi| \leqslant \tfrac{1}{4}\pi.$

Put $\epsilon = e^{\frac{1}{2}l\pi i} = i^l$. Then

$$\begin{aligned}|\beta+\epsilon\delta|^2 &= |\beta|^2+|\delta|^2-2|\beta\delta|\cos(\theta_0-\tfrac{1}{2}l\pi)\\ &\leqslant |\beta|^2+|\delta|^2-\sqrt{2}\,|\beta\delta|\\ &\leqslant \max\{|\beta|^2, |\delta|^2\}.\end{aligned}$$

Hence $\mathcal{N}(\beta+\epsilon\delta) \leqslant N$. On writing

$$\alpha+\epsilon\beta = \nu\lambda, \quad \gamma+\epsilon\delta = \nu\mu, \quad \gcd(\lambda,\mu) = 1,$$

we have then $\qquad (\lambda,\mu) \in \mathfrak{F}_N^*$

and $\qquad |\mu z_0-\lambda| = \dfrac{1}{|\nu|}\dfrac{|e^{i\theta_0}-e^{\frac{1}{2}l\pi i}|\,|\alpha\delta-\beta\gamma|}{|\beta+\delta e^{i\theta_0}|} < \dfrac{|\alpha\delta-\beta\gamma|}{|\beta+\delta e^{i\theta_0}|} = |\beta z_0-\alpha| = |\delta z_0-\gamma|,$

as required.

This concludes the proof of lemma 12, and so of theorem XVIII.

13. Proof of theorem XV

In this section we deal only with $k(\rho)$. Our object is to prove the following theorem, of which theorem XV is a consequence:

Theorem XIX. *Let $N>0$ and let $\nu \in R(\rho)$ be such that $N_1 = \mathcal{N}(\nu) > N$ is the least norm greater than N. Then for all $z \in \Omega$ there are $\alpha, \beta \in R(\rho)$, not both zero, such that*

$$\mathcal{N}(\beta) \leqslant N, \quad |\beta z-\alpha| \leqslant N_1^{-\frac{1}{2}}.$$

The last $\leqslant$ cannot be replaced by $<$.

If $N<1$, we have $N_1 = 1$ and the theorem is trivial ($\beta = 0, \alpha = 1$). Hence we may assume that $N \geqslant 1$ and so apply the results of § 12. In that language theorem XIX asserts that

$$\max_{z\in\Omega} \Delta_N(z) = N_1^{-\frac{1}{2}}.$$

By theorem XVIII we need consider only nodes z'. By theorem X and its corollary (§ 9), $z' = \lambda/\mu \in k(\rho)$. If $\lambda, \mu \in R(\rho)$, $\gcd(\lambda,\mu) = 1$, we have

$$\Delta_N\left(\frac{\lambda}{\mu}\right) = \frac{1}{|\mu|} \leqslant N_1^{-\frac{1}{2}},$$

since $\lambda/\mu \notin \mathfrak{H}_N^*$ and so $\mathcal{N}(\mu) > N$, $\mathcal{N}(\mu) \geqslant N_1$. Further, clearly, for the ν of the theorem,

$$\Delta_N\left(\frac{1}{\nu}\right) = \frac{1}{|\nu|} = N_1^{-\frac{1}{2}},$$

which proves the last sentence.

Since $N_1 > N$, theorem XV follows from theorems XIX and XVII (with $c = 1$).

14. Proof of theorem XIV

In this section we deal only with $k(i)$. Our object is to prove the following theorem of which theorem XIV is a consequence:

Theorem XX. *For every $N>0$ and for all $z \in \Omega$ there are $\alpha, \beta \in R(i)$, not both zero, such that*

$$\mathcal{N}(\beta) \leqslant N, \quad |\beta z-\alpha| < \frac{\sqrt{2}}{3-\sqrt{3}} N^{-\frac{1}{2}}.$$

The constant $\sqrt{2}/(3-\sqrt{3})$ cannot be replaced by a smaller one.

If $N<1$, the theorem is trivial so we may assume $N \geqslant 1$ and apply the results of § 12. In that language theorem XX asserts that

$$\max_{z \in \Omega} \Delta_N(z) < \frac{\sqrt{2}}{3-\sqrt{3}} N^{-\frac{1}{2}}.$$

By theorem XVIII we need consider only nodes z'. By theorem IV (§ 6) either $z' \in k(i)$ or $z' \notin k(i)$, $z' \in k(i,\rho)$.

Lemma 13. *If $z' \in k(i)$ then* $\qquad \Delta_N(z') < N^{-\frac{1}{2}}.$

The proof is exactly similar to that of the previous theorem XIX (§ 13), and will be omitted.

Since $\sqrt{2}/(3-\sqrt{3}) > 1$ we need therefore consider only nodes $z' \notin k(i)$.

Lemma 14. *If $z' \notin k(i)$ is a node, then*

$$z' = \frac{\alpha-\rho^2\gamma}{\beta-\rho^2\delta}; \quad \alpha,\beta,\gamma,\delta \in R(i); \quad |\alpha\delta-\beta\gamma| = 1, \tag{14·1}$$

where†

$$\max\{|\beta|, |\delta|, |\beta+\delta|\} \leqslant N^{\frac{1}{2}}, \tag{14·2}$$

$$\min\{|\beta+i\delta|, |\delta-i(\beta+\delta)|, |\beta+\delta-i\beta|\} > N^{\frac{1}{2}}. \tag{14·3}$$

From theorem V (§ 7)‡ it follows that (14·1) and (14·2) hold, since (14·2) just expresses the fact that (α,β), (γ,δ), $(\alpha+\gamma, \beta+\delta) \in \mathfrak{F}_N^*$. It remains to verify (14·3). Suppose, *per absurdum*, that $|\beta+i\delta| \leqslant N^{\frac{1}{2}}$ and so $(\alpha+i\gamma, \beta+i\delta) \in \mathfrak{G}_N^*$. Then we would have

$$\begin{aligned}\min_{(\eta,\theta)\in\mathfrak{F}_N^*} |\theta z'-\eta| &= \min_{(\eta,\theta)\in\mathfrak{G}_N^*} |\theta z'-\eta| \\ &\leqslant |(\beta+i\delta)z'-(\alpha+i\gamma)| = \frac{|i+\rho^2|}{|\beta-\rho^2\delta|} \\ &< \frac{1}{|\beta-\rho^2\delta|} = |\beta z'-\alpha|.\end{aligned}$$

Hence $z' \notin \mathfrak{R}^*(\alpha,\beta)$. This is a contradiction, since z' can only have the form given if $z' \in \mathfrak{R}^*(\alpha,\beta)$. The rest of (14·3) is proved similarly.

The converse that if (14·1), (14·2) and (14·3) hold, then z' is a node, is true but we do not prove it as we do not require it. The proof is straightforward but a little tedious.

Lemma 15. *If $\beta,\delta \in \Omega$ and* (14·2) *and* (14·3) *hold, then*

$$|\beta-\rho^2\delta| > \frac{3-\sqrt{3}}{\sqrt{2}} N^{\frac{1}{2}}. \tag{14·4}$$

Put $\qquad \lambda = \beta-\rho^2\delta, \quad \mu = \beta-\rho\delta,$

so that $\quad (\rho-\rho^2)\beta = \rho\lambda-\rho^2\mu, \quad (\rho-\rho^2)\delta = \lambda-\mu, \quad (\rho-\rho^2)(\beta+\delta) = -(\rho^2\lambda-\rho\mu).$

Then, by (14·2), $\qquad \max_{l=0,1,2} |\lambda-\rho^l\mu| \leqslant |\rho-\rho^2| N^{\frac{1}{2}} = (3N)^{\frac{1}{2}},$

and hence

$$\max\{|\lambda|, |\mu|\} \leqslant (3N)^{\frac{1}{2}}. \tag{14·5}$$

† Note the cyclical symmetry in (α,β), (γ,δ), $(-\alpha-\gamma, -\beta-\delta)$.

‡ Case a. If $t' = -\rho$ in the theorem interchange (α,β) and (γ,δ).

Further, by (14·3), $$\min_{l=0,1,2} |(\rho+i)\lambda-\rho^l(\rho^2+i)\mu|>(3N)^{\frac{1}{2}},$$
and so $$\max\{|(\rho+i)\lambda|, |(\rho^2+i)\mu|\}>(3N)^{\frac{1}{2}}. \tag{14·6}$$
Since $$|\rho+i| = \frac{1+\sqrt{3}}{\sqrt{2}}>1>\frac{\sqrt{3}-1}{\sqrt{2}} = |\rho^2+i|,$$
we have $|(\rho^2+i)\mu|<(3N)^{\frac{1}{2}}$ by (14·5). Hence $|(\rho+i)\lambda|>(3N)^{\frac{1}{2}}$ by (14·6), which is the required inequality.

COROLLARY. *If* $z'\notin k(i)$, $$\Delta_N(z')<\frac{\sqrt{2}}{3-\sqrt{3}}N^{-\frac{1}{2}}.$$

By theorem V (§ 7), we have $\Delta_N(z') = |\beta-\rho^2\delta|^{-1}$, so the corollary follows immediately.

Finally, theorem XX follows at once from theorem XVIII and lemmas 14 and 15; and theorem XIV follows from theorems XX and XVII. That the constant in theorem XX is best possible follows from the already proved fact (§ 10) that theorem XIV is best possible. If theorem XX could be improved, then so, by theorem XVII, could theorem XIV. Indeed, by reversing the argument of this paragraph and using the converse of lemma 14, it is not difficult to prove that
$$\lim_{N\to\infty} N^{\frac{1}{2}} \max_{z\in\Omega} \Delta_N(z) = \frac{\sqrt{2}}{3-\sqrt{3}},$$
but we shall not go into details of the proof.

PART III

15. INTRODUCTION

In this part III we show that both $\mathfrak{R}^*(\alpha,\beta)$ and $\mathfrak{R}(\alpha,\beta)$ for $\beta\neq 0$ are star domains about α/β in the sense of Minkowski. The proof that $\mathfrak{R}^*(\alpha,\beta)$ is a star domain is rather long but quite elementary. The proof that $\mathfrak{R}(\alpha,\beta)$ is a star domain is more complicated, and we only carry the proof through for 'large enough' N (i.e. for $N\geqslant N_0$, where N_0 is an absolute constant which we have not troubled to determine). There is little doubt that $\mathfrak{R}(\alpha,\beta)$ is a star domain for *all* N, but it does not seem worth while pursuing the subject further as $\mathfrak{R}^*(\alpha,\beta)$ is perhaps the more natural object of investigation. We discuss throughout only $k(i)$. Doubtless analogous results hold for $k(\rho)$.

In what follows we denote the interior, closure, frontier and complement of a point set $\mathfrak{M}$ by $\mathscr{I}\mathfrak{M}$, $\overline{\mathfrak{M}}$, $\mathscr{F}\mathfrak{M}$ and $\mathscr{C}\mathfrak{M}$ respectively.† All point sets will be in the two-dimensional plane with co-ordinates x, y or, what is the same, in the Gauss plane of $z = x+iy$.

In § 16 we prove two theorems of a general nature; in §§ 17 and 18 we apply them to $\mathfrak{R}^*(\alpha,\beta)$ and $\mathfrak{R}(\alpha,\beta)$ respectively.

16. GENERAL PREPARATION

It is easily seen that Minkowski's definition of a star domain is equivalent to the following one:

DEFINITION. *A non-null bounded point set* $\mathfrak{D}$ *in the plane is said to be a star domain*‡ *about a point* O *if* $O\in\mathscr{I}\mathfrak{D}$ *and whenever* $P\in\overline{\mathfrak{D}}$, *all the points of the line segment* OP *except possibly* P *are in* $\mathscr{I}\mathfrak{D}$.

We note for later use the following two lemmas:

† There should be no confusion with the previous use of $\mathscr{I}z$ to denote the imaginary part of the complex number z.

‡ We note that a star domain need be neither open nor closed, so $\mathfrak{D}$ is not necessarily a domain in the topological sense. One can show, however, that $\mathscr{I}\mathfrak{D}$ is a topological domain.

LEMMA 16. *If* $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_r$ *are star domains about* O, *then so are also*

$$\mathfrak{D}_1 \cup \mathfrak{D}_2 \ldots \cup \mathfrak{D}_r$$

and

$$\mathfrak{D}_1 \cap \mathfrak{D}_2 \ldots \cap \mathfrak{D}_r.$$

LEMMA 17. *Let* $\mathfrak{D}$ *be a star domain about* $O = (0,0)$. *Then the set* $\mathfrak{E}$ *obtained from* $\mathscr{C}\mathfrak{D}$ *by the transformation*

$$x' + iy' = \frac{1}{x+iy}$$

is also a star domain. (*Here* $O = \frac{1}{\infty}$ *is considered to belong to* $\mathfrak{E}$.)

We suppress the proofs.

In the rest of § 16 'star domain' will mean 'star domain about O', until the contrary is stated. If a point set $\mathfrak{D}$ is given, a point $P \neq O$ on $\mathscr{F}\mathfrak{D}$ is called *black* if there is a $\delta > 0$ (depending possibly on P) such that

$$tP \in \mathscr{I}\mathfrak{D} \quad (1-\delta < t < 1),$$

$$tP \in \mathscr{C}\overline{\mathfrak{D}} \quad (1 < t < 1+\delta),$$

where tP, for positive t, is the point on the radius vector OP at a distance $t\,|OP|$ from O.

LEMMA 18. *If all frontier points of a bounded point-set* $\mathfrak{D}$ *are black, then* $\mathfrak{D}$ *is a star domain.*

Let $P \in \mathfrak{D}$. Suppose that $tP \notin \mathscr{I}\mathfrak{D}$ for some t in $0 \leqslant t < 1$. Denote by τ the l.u.b. of such t. Then $0 \leqslant \tau < 1$, since either $P \in \mathscr{I}\mathfrak{D}$ and $\mathscr{I}\mathfrak{D}$ is open or $P \in \mathscr{F}\mathfrak{D}$ and then P is black by hypothesis. Further,

$$tP \in \mathscr{I}\mathfrak{D} \quad (\tau < t < 1), \tag{16·1}$$

since τ is an upper bound. But then clearly $\tau P \in \mathscr{F}\mathfrak{D}$. Hence τP is black. This, however, is in contradiction with (16·1). Hence $tP \in \mathscr{I}\mathfrak{D}$ $(0 \leqslant t < 1)$, i.e. $\mathfrak{D}$ is a star domain.

LEMMA 19. *Let* $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{D}_2 \ldots \cup \mathfrak{D}_r$, *where* $\mathfrak{D}_1, \ldots, \mathfrak{D}_r$ *are bounded point-sets. Let* $P \in \mathscr{F}\mathfrak{D}$, *say* $P \in \mathscr{F}\mathfrak{D}_j$ *for* $j = j_1, j_2, \ldots, j_s$ *and* $P \in \mathscr{C}\overline{\mathfrak{D}}_j$ *for* $j \neq j_1, \ldots, j_s$. *Then* P *is a black point of* $\mathfrak{D}$ *if it is a black point of* $\mathfrak{D}_{j_1}, \ldots, \mathfrak{D}_{j_s}$.

The proof is obvious.

LEMMA 20. *Let* $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_r$ *be bounded point-sets and* $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \ldots \cup \mathfrak{D}_r$. *Suppose that, for every* j, *every point* $P \in \mathscr{F}\mathfrak{D}_j$, $P \notin \mathscr{I}\mathfrak{D}$ *is a black point of* $\mathfrak{D}_j$. *Then* $\mathfrak{D}$ *is a star domain.*

For by lemma 19 every point of $\mathscr{F}\mathfrak{D}$ is a black point of $\mathfrak{D}$. Hence by lemma 18, $\mathfrak{D}$ is a star domain.

Let u_0, v_0 be fixed real numbers. We denote by Λ the set of all numbers pairs (u, v);

$$u \equiv u_0 \ (1), \quad v \equiv v_0 \ (1).$$

For any point $P = (x', y')$ we denote by

$$\mathfrak{C}(P) = \mathfrak{C}(x', y')$$

the open circular domain

$$(x, y) \in \mathfrak{C}(x', y') \quad .\!\equiv\!. \quad (x-x')^2 + (y-y')^2 < 1$$

of centre P and unit radius.

Finally, for $r > 0$ we denote by $\mathfrak{S}$ the region

$$(x, y) \in \mathfrak{S} \quad .\!\equiv\!. \quad x^2 + y^2 \leqslant r^2.$$

THEOREM XXI.

$$\mathfrak{R} = \bigcup_{P \in \Lambda \cap \mathfrak{S}} \mathfrak{C}(P)$$

is either the null set or a star domain with respect to $O = (0, 0)$.

If $P=(u,v)\in\Lambda\cap\mathfrak{S}$, the points of $\mathscr{F}\mathfrak{C}(u,v)$ are of the form

$$P(\theta)=(u+\cos\theta, v+\sin\theta) \quad (-\pi<\theta\leqslant\pi).$$

We require a condition for $P(\theta)$ to be black.

LEMMA 21. (i) *If $u^2+v^2<1$, then all the points $P(\theta)$ are black points of $\mathfrak{C}(u,v)$.*

(ii) *If $u^2+v^2=1$, then all the points $P(\theta)$ except $O=(0,0)$ are black points of $\mathfrak{C}(u,v)$.*

(iii) *If $u^2+v^2>1$, and* $$p(\theta)=u\cos\theta+v\sin\theta>-1, \tag{16·2}$$ *then $P(\theta)$ is a black point of $\mathfrak{C}(u,v)$.*

We put $$\phi(t)=(u-tu-t\cos\theta)^2+(v-tv-t\sin\theta)^2,$$

so that $\phi(t)<1, =1, >1$, according as

$$tP(\theta)\in\mathscr{I}\mathfrak{C}(u,v), \quad \mathscr{F}\mathfrak{C}(u,v), \quad \mathscr{C}\overline{\mathfrak{C}}(u,v).$$

Further $$\phi(1)=1; \quad \frac{1}{2}\left[\frac{d}{dt}\phi(t)\right]_{t=1}=u\cos\theta+v\sin\theta+1.$$

Hence if (16·2) holds, $P(\theta)$ is black by definition. This proves (iii). If $u^2+v^2\leqslant 1$, then

$$u\cos\theta+v\sin\theta\geqslant-\surd(u^2+v^2)\geqslant-1,$$

with equality only when $u=-\cos\theta$, $v=-\sin\theta$; i.e. with this exception (16·2) holds. This proves (i) and (ii).

LEMMA 22. *Let $(u,v)\in\Lambda$.*

(i) *If $(u-1,v)\in\mathfrak{S}$, then $P(\theta)\in\mathfrak{R}$ for $\frac{2}{3}\pi<|\theta|\leqslant\pi$.*

(ii) *If $(u,v-1)\in\mathfrak{S}$, then $P(\theta)\in\mathfrak{R}$ for $-\frac{5}{6}\pi<\theta<-\frac{1}{6}\pi$.*

(iii) *If $(u-1,v-1)\in\mathfrak{S}$, then $P(\theta)\in\mathfrak{R}$ for $-\pi<\theta<-\frac{1}{2}\pi$.*

(iv) *If $(u-1,v+1)\in\mathfrak{S}$, then $P(\theta)\in\mathfrak{R}$ for $\frac{1}{2}\pi<\theta<\pi$.*

We prove (i). We suppose $\pi\geqslant|\theta|>\frac{2}{3}\pi$ and put

$$P(\theta)=(u+\cos\theta, v+\sin\theta)=(x',y').$$

Then $$\begin{aligned}(u-1-x')^2+(v-y')^2&=(1+\cos\theta)^2+\sin^2\theta\\&=2+2\cos\theta,\\&<1\end{aligned}$$

since $\cos\theta<-\frac{1}{2}$. Hence $$P(\theta)\in\mathfrak{C}(u-1,v).$$

Further, $(u-1,v)\in\Lambda$ since $(u,v)\in\Lambda$. Hence

$$P(\theta)\in\mathfrak{C}(u-1,v)\subset\bigcup_{P\in\mathfrak{S}\cap\Lambda}\mathfrak{C}(P)=\mathfrak{R}$$

as asserted.

The proofs of (ii), (iii) and (iv) are similar.

LEMMA 23. *Let $(u,v)\in\mathfrak{S}\cap\Lambda$ and let† $P(\theta)\notin\mathfrak{R}$. Then $P(\theta)$ is a black point of $\mathfrak{C}(u,v)$ with respect to $O=(0,0)$.*

We may suppose, by symmetry, that $$u\geqslant v\geqslant 0. \tag{16·3}$$

We are given that $(u,v)\in\mathfrak{S}$, i.e. $$u^2+v^2\leqslant r^2. \tag{16·4}$$

We distinguish a large number of cases, O_1, O_2, A_1, A_2, B_1, B_2, C_1, C_2 (say).

Case O. $u^2+v^2\leqslant 1$.

Subcase O_1. $u^2+v^2<1$. Then all $P(\theta)$ are black by lemma 21 (i).

Subcase O_2. $u^2+v^2=1$. Then $u\geqslant\dfrac{1}{\surd 2}$ since $u\geqslant v\geqslant 0$, and $(u-1)^2+v^2<1=u^2+v^2\leqslant r^2$.

† I.e. $P(\theta)\in\mathscr{F}\mathfrak{R}\cap\mathscr{F}\mathfrak{C}(u,v)$ since $\mathfrak{R}$ and $\mathfrak{C}(u,v)\subset\mathfrak{R}$ are open.

Hence $(u-1, v) \in \mathfrak{S} \cap \Lambda$ and $O = (0,0) \in \mathfrak{C}(u-1, v) \subset \bigcup_{P \in \mathfrak{S} \cap \Lambda} \mathfrak{C}(P) = \mathfrak{R}$. But by lemma 21 (ii) all points $P(\theta)$ are black except O. The lemma is thus proved in this case.

We may therefore assume from now on that

$$u^2+v^2>1. \tag{16·5}$$

Hence by lemma 21 (iii), to show $P(\theta)$ is black we need only prove

$$p(\theta) = u\cos\theta + v\sin\theta > -1.$$

Case A. $u \leqslant \frac{11}{10}$.

Subcase A_1. $v \leqslant \frac{11}{20}$, so $u > \frac{1}{2}$ by (16·5). Then, by (16·4),

$$(u-1)^2+v^2 = u^2+v^2+(1-2u) < u^2+v^2 < r^2,$$

and so

$$(u-1, v) \in \mathfrak{S}.$$

Hence, since $P(\theta) \notin \mathfrak{R}$, we have, by lemma 22 (i),

$$|\theta| \leqslant \tfrac{2}{3}\pi.$$

If $|\theta| \leqslant \frac{1}{2}\pi$, then $\qquad p(\theta) = u\cos\theta+v\sin\theta \geqslant -v > -1.$

If $\frac{1}{2}\pi \leqslant |\theta| \leqslant \frac{2}{3}\pi$, say $|\theta| = \frac{1}{2}\pi+\delta$, where $0 \leqslant \delta \leqslant \frac{1}{6}\pi$, then

$$\begin{aligned} p(\theta) &\geqslant -u\,|\cos\theta| - v\,|\sin\theta| \\ &= -u\sin\delta - v\cos\delta \\ &\geqslant -\tfrac{11}{10}\sin\delta - \tfrac{11}{20}\cos\delta \\ &\geqslant -\frac{11}{10}\cdot\frac{1}{2} - \frac{11}{20}\cdot\frac{\sqrt{3}}{2} > -1. \end{aligned}$$

Hence always $p(\theta) > -1$ and so $P(\theta)$ is black.

Subcase A_2. $v \geqslant \frac{11}{20}$, so $\frac{11}{10} \geqslant u \geqslant v \geqslant \frac{11}{20}$. Then

$$\begin{aligned} (u-1)^2+(v-1)^2 &< (\tfrac{1}{2})^2+(\tfrac{1}{2})^2 < 1 < u^2+v^2 \leqslant r^2, \\ (u-1)^2+v^2 &= u^2+v^2+(1-2u) < u^2+v^2 \leqslant r^2, \\ u^2+(v-1)^2 &= u^2+v^2+(1-2v) < u^2+v^2 \leqslant r^2, \end{aligned}$$

and hence $\qquad (u-1, v-1) \in \mathfrak{S}, \quad (u-1, v) \in \mathfrak{S}, \quad (u, v-1) \in \mathfrak{S}.$

Hence, by lemma 22 ((i), (ii) and (iii))

$$-\tfrac{1}{6}\pi \leqslant \theta \leqslant \tfrac{2}{3}\pi.$$

Then

$$\begin{aligned} p(\theta) &= u\cos\theta+v\sin\theta \geqslant u\cos\theta \geqslant -\tfrac{1}{2}u > -1 \quad (0 \leqslant \theta \leqslant \tfrac{2}{3}\pi), \\ p(\theta) &\geqslant v\sin\theta \geqslant -\tfrac{1}{2}v > -1 \quad (-\tfrac{1}{6}\pi \leqslant \theta \leqslant 0), \end{aligned}$$

and so always $p(\theta) > -1$, as asserted.

Case B. $u \geqslant \frac{11}{10}$, $v \leqslant \frac{3}{5}$.

Subcase B_1. $u \leqslant \frac{9}{5}$. Then $\qquad (u-1)^2+v^2 < u^2+v^2 \leqslant r^2,$

and

$$\begin{aligned} (u-1)^2+(v-1)^2 &= u^2+(1+(v-1)^2-2u) \\ &\leqslant u^2+(1+1-\tfrac{11}{5}) \\ &< u^2 \leqslant u^2+v^2 \leqslant r^2, \end{aligned}$$

and so $\qquad (u-1, v) \in \mathfrak{S}, \quad (u-1, v-1) \in \mathfrak{S},$

Hence, by lemma 22 ((i) and (iii)), we have

$$-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{2}{3}\pi,$$

and so
$$p(\theta) = u\cos\theta + v\sin\theta \geqslant -v > -1 \qquad (|\theta| \leqslant \tfrac{1}{2}\pi),$$
$$p(\theta) \geqslant u\cos\theta \geqslant -\tfrac{1}{2}u > -1 \qquad (\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{2}{3}\pi).$$

Hence always $p(\theta) > -1$, as asserted.

Subcase B_2. $u \geqslant \frac{9}{5}$. Then

$$(u-1)^2 + (v+1)^2 = u^2 + v^2 + (2 - 2u + 2v) < u^2 + v^2 \leqslant r^2,$$

so
$$(u-1, v+1) \in \mathfrak{S}.$$

Also, as in case B_1,
$$(u-1, v) \in \mathfrak{S}, \quad (u-1, v+1) \in \mathfrak{S}.$$

Hence by lemma 22 ((i), (iii) and (iv)), since $P(\theta) \notin \mathfrak{K}$

$$|\theta| \leqslant \tfrac{1}{2}\pi.$$

Further, we have then
$$p(\theta) = u\cos\theta + v\sin\theta \geqslant -v > -1,$$
as asserted.

Case C. $u \geqslant \frac{11}{10}$, $v \geqslant \frac{3}{5}$.

Subcase C_1. $u \leqslant 2 + \frac{3}{2}v$. Clearly

$$(u-1)^2 + v^2 < u^2 + v^2 \leqslant r^2,$$
$$u^2 + (v-1)^2 < u^2 + v^2 \leqslant r^2,$$

and so
$$(u-1, v) \in \mathfrak{S}, \quad (u, v-1) \in \mathfrak{S}.$$

Then by lemma 22 ((i) and (ii)), as before

$$-\tfrac{1}{6}\pi \leqslant \theta \leqslant \tfrac{2}{3}\pi.$$

Further
$$p(\theta) = u\cos\theta + v\sin\theta \geqslant 0 \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi),$$
$$\begin{aligned} p(\theta) &\geqslant u\cos\theta - v|\sin\theta| \\ &\geqslant u(\cos\theta - |\sin\theta|) \\ &\geqslant 0 \quad (-\tfrac{1}{6}\pi \leqslant \theta \leqslant 0) \end{aligned}$$

and, if
$$\tfrac{1}{2}\pi \leqslant \theta = \tfrac{1}{2}\pi + \delta \leqslant \tfrac{2}{3}\pi,$$
$$\begin{aligned} p(\theta) &= v\cos\delta - u\sin\delta \\ &\geqslant -2\sin\delta + v(\cos\delta - \tfrac{3}{2}\sin\delta) \\ &> -2\sin\delta \\ &\geqslant -1. \end{aligned}$$

Hence always $p(\theta) > -1$, as asserted.

Subcase C_2. $u \geqslant 2 + \frac{3}{2}v$. Then

$$\begin{aligned} (u-1)^2 + (v+1)^2 &= u^2 + v^2 + (2 + 2v - 2u) \\ &< u^2 + v^2 \leqslant r^2, \end{aligned}$$

and hence
$$(u-1, v+1) \in \mathfrak{S}.$$

Also, as in C_1,
$$(u-1, v) \in \mathfrak{S}, \quad (u, v-1) \in \mathfrak{S}.$$

Hence, by lemma 22 ((i), (ii) and (iv)) we have

$$-\tfrac{1}{6}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

Then
$$\begin{aligned} p(\theta) &= u\cos\theta + v\sin\theta \geqslant 0 \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi), \\ p(\theta) &\geqslant u\cos\theta - v\,|\sin\theta\,| \\ &> u(\cos\theta - |\sin\theta\,|) \\ &> 0, \qquad\qquad (-\tfrac{1}{6}\pi \leqslant \theta \leqslant 0), \end{aligned}$$
and so always $p(\theta) > -1$, as asserted.

Since all u, v satisfying $u \geqslant v \geqslant 0$ fall under one of O_1, O_2, A_1, A_2, B_1, B_2, C_1, C_2, the lemma is proved.

Theorem XXI now follows directly from lemmas 20 and 23, since $\mathfrak{K} = \mathscr{I}\mathfrak{K}$ is open.

The foregoing lemma will suffice for the $\mathfrak{R}^*(\alpha, \beta)$. To deal with the $\mathfrak{R}(\alpha, \beta)$ we require a more elaborate version.

Let Λ be as before, and let $s > 0$, p, q be real numbers. We define $\mathfrak{S}'$ by
$$(x, y) \in \mathfrak{S}' \quad .\equiv. \quad (x-p)^2 + (y-q)^2 \leqslant s^2.$$

THEOREM XXII. *There exist absolute constants m, M with the following property:*

Suppose that
$$\text{either} \quad p^2 + q^2 \leqslant m, \quad s \geqslant 1, \tag{16·6}$$
$$\text{or} \quad |p| \leqslant \tfrac{1}{2}, \; |q| \leqslant \tfrac{1}{2}, \quad s^2 \geqslant M. \tag{16·7}$$
Then
$$\mathfrak{K}' = \bigcup_{P \in \mathfrak{S}' \cap \Lambda} \mathfrak{C}(P)$$
is a star domain.

Suppose further that
$$\mathfrak{S}'' = \mathfrak{S}' \cap \mathfrak{S}$$
and
$$\mathfrak{K}'' = \bigcup_{P \in \mathfrak{S}'' \cap \Lambda} \mathfrak{C}(P).$$
Then if either (16·6) *or* (16·7) *holds, $\mathfrak{K}''$ is a star domain. It is also a star domain for $r^2 = s^2 \geqslant M$ and $|p| = 1$, $q = 0$ or $p = 0$, $|q| = 1$.*

The conditions (16·6) or (16·7) imply that $\mathfrak{S}'$ is very nearly a circle with centre O, i.e. very nearly an $\mathfrak{S}$, and hence so is $\mathfrak{S}''$. In the proof of theorem XXI all the inequalities were 'weak', i.e. we could have proved stronger inequalities than were actually needed. The proof of theorem XXII consists in going through that of theorem XXI and verifying that we may choose m and M so that the inequalities continue to hold. We omit the details.

17. $\mathfrak{R}^*(\alpha, \beta)$ IS A STAR DOMAIN

We prove the following theorem:

THEOREM XXIII. *$\mathfrak{R}^*(\alpha, \beta)$ is a star domain about α/β.*

$\mathfrak{R}^*(\alpha, \beta)$ was defined as the set of points z such that
$$|\beta z - \alpha| = \min_{\substack{\gcd(\eta, \theta) = 1 \\ |\theta| \leqslant N^{\frac{1}{2}}}} |\theta z - \eta|,$$
and so
$$|\beta z - \alpha| = \min_{0 < |\theta| \leqslant N^{\frac{1}{2}}} |\theta z - \eta|. \tag{17·1}$$

Let λ, μ be a solution in $R(i)$ of the equation
$$\lambda\beta - \mu\alpha = 1, \tag{17·2}$$
so that every pair of integers (η, θ) may be put in the form
$$(\eta, \theta) = A(\alpha, \beta) + B(\lambda, \mu), \tag{17·3}$$

where
$$\left.\begin{aligned} A &= \theta\lambda-\eta\mu \\ B &= \beta\mu-\alpha\theta \end{aligned}\right\} \in R(i). \tag{17·4}$$

We apply the conformal transformation
$$z = \frac{\lambda+\alpha t}{\mu+\beta t}, \quad t = \frac{\lambda-\mu z}{-\alpha+\beta z}. \tag{17·5}$$

Then equation (17·1) becomes $\quad 1 = \min |Bt-A|,$ $\quad$ (17·6)

subject to the condition $\quad |\theta| = |A\beta+B\mu| \leqslant N^{\frac{1}{2}},$

i.e.
$$\left|A+\frac{B\mu}{\beta}\right| \leqslant \frac{N^{\frac{1}{2}}}{|\beta|}. \tag{17·7}$$

For given $B \in R(i)$ we denote by $\mathfrak{D}_B$ the set of all t with
$$|Bt-A|<1 \tag{17·8}$$
for some A satisfying (17·7).

Lemma 24. *$\mathfrak{D}_B$ is a star domain about $-\mu/\beta$ or the null set.*

We apply the transformation $\quad Bt = t^*$ $\quad$ (17·9)

and denote by $\mathfrak{D}_B^*$ the set of $t^* = Bt$, $t \in \mathfrak{D}_B$. We write
$$A+\frac{B\mu}{\beta} = u+iv, \tag{17·10}$$
and put
$$t^*+\frac{B\mu}{\beta} = x+iy. \tag{17·11}$$

We use now theorem XXI and the corresponding notation. Thus (17·6) is equivalent to
$$(x,y) \in \mathfrak{C}(u,v)$$
and (17·7) to
$$u^2+v^2 \leqslant \frac{N}{|\beta^2|} = r^2 \quad \text{(say)}.$$
Finally, we write
$$\frac{B\mu}{\beta} = u_0+iv_0,$$
so that
$$u \equiv u_0\ (1), \quad v \equiv v_0\ (1).$$
Hence by theorem XXI with the appropriate definition of Λ, $\mathfrak{S}$ in terms of u_0, v_0 and r
$$\mathfrak{D}_B^* = \bigcup_{P \in \Lambda \cap \mathfrak{S}} \mathfrak{C}(P)$$
is a star domain about $t^* = -B\mu/\beta$ (corresponding to $x = y = 0$). Since $\mathfrak{D}_B$ is obtained from $\mathfrak{D}_B^*$ by a rotation and a magnification, this proves the lemma.

From lemma 16 we have the corollary:

Corollary. *$\mathfrak{D} = \bigcup_B \mathfrak{D}_B$ is a star domain.*

Finally, $\mathscr{C}\mathfrak{D}$ in the t-plane is the transform of $\mathfrak{R}^*(\alpha,\beta)$ in the z-plane by (17·5). The transformation (17·5) may be written in the form
$$(\beta t+\mu)(\beta z-\alpha) = \lambda\beta-\alpha\mu = 1.$$
This is a transformation from $x+iy = \beta z-\alpha$ to $x'+iy' = \beta t+\mu$ of the type discussed in lemma 17. Hence
$$\mathfrak{R}^*(\alpha,\beta)$$
is a star domain about α/β, as asserted.

18. $\mathfrak{R}(\alpha, \beta)$ IS A STAR DOMAIN FOR $\beta \neq 0$

We shall prove the following theorem:

THEOREM XXIV. *Let M, m be the constants of theorem XXII. There is a constant N_0 such that $\mathfrak{R}(\alpha,\beta)$ is a star domain about α/β whenever $\beta \neq 0$ and*

$$N \geqslant N_0. \tag{18·1}$$

There seems little doubt that the condition (18·1) is unnecessary. The $\mathfrak{R}(\alpha,\beta)$ are in fact star domains for all N for which diagrams have been drawn.

We require the following lemma:

LEMMA 25. *$\mathfrak{R}(\alpha,\beta)$ is the set of all $z \in \Omega$ for which*

$$|\beta z - \alpha| \leqslant \min |\theta z - \eta| \tag{18·2}$$

subject to $$|\eta| \leqslant N^{\frac{1}{2}}, \quad |\theta| \leqslant N^{\frac{1}{2}}, \quad 0 < |\alpha\theta - \beta\eta| \leqslant 2^{\frac{1}{2}}. \tag{18·3}$$

We recollect that $\mathfrak{R}(\alpha,\beta)$ is defined by

$$z \in \mathfrak{R}(\alpha,\beta) \quad .\equiv. \quad |\beta z - \alpha| = \min_{(\eta,\theta)\in\mathfrak{F}_N} |\theta z - \eta|. \tag{18·4}$$

Since $(\eta,\theta) \in \mathfrak{F}_N$ is equivalent to $|\eta| \leqslant N^{\frac{1}{2}}$, $|\theta| \leqslant N^{\frac{1}{2}}$, $\gcd(\eta,\theta) = 1$, we may replace (18·4) by

$$z \in \mathfrak{R}(\alpha,\beta) \quad .\equiv. \quad |\beta z - \alpha| = \min_{\substack{|\eta| \leqslant N^{\frac{1}{2}} \\ |\theta| \leqslant N^{\frac{1}{2}} \\ (\eta,\theta) \neq (0,0)}} |\theta z - \eta|. \tag{18·5}$$

Suppose, *per absurdum*, that there is some z for which (18·2) and (18·3) are true, but (18·5) is false, i.e. there is some (γ, δ) such that $|\gamma| \leqslant N^{\frac{1}{2}}$, $|\delta| \leqslant N^{\frac{1}{2}}$ and

$$|\delta z - \gamma| < |\beta z - \alpha|,$$

where necessarily $$|\Delta| > 2^{\frac{1}{2}}, \quad \Delta = \alpha\delta - \beta\gamma,$$

by (18·3). We suppose γ, δ are chosen so that Δ has the least value for which a contradiction occurs.

Let ξ, η be the ξ, η of lemma 4 (§ 6) and put

$$\gamma_1 = \frac{\alpha\xi + \gamma\eta}{\Delta} \in R(i), \quad \delta_1 = \frac{\beta\xi + \delta\eta}{\Delta} \in R(i).$$

Then much as in the proof of theorem IV (§ 6) we have

$$|\gamma_1| \leqslant N^{\frac{1}{2}}, \quad |\delta_1| \leqslant N^{\frac{1}{2}}$$

and $$|\delta_1 z - \gamma_1| \leqslant \left|\frac{\xi}{\Delta}\right| |\beta z - \alpha| + \left|\frac{\eta}{\Delta}\right| |\delta z - \alpha| < |\beta z - \alpha|.$$

Finally, $$|\Delta_1| = |\alpha\delta_1 - \beta\gamma_1| = \left|\frac{\eta}{\Delta}\right| |\alpha\delta - \beta\gamma| = |\eta| < |\Delta|.$$

Since $|\Delta|$ was assumed to be minimal this is a contradiction. Hence the lemma is true.

Suppose first that $$\alpha\beta \neq 0. \tag{18·6}$$

Then we choose λ, $\mu \in R(i)$ such that (17·2) holds, and make the transformation (17·4). Equation (18·2) becomes $$1 \leqslant \min |Bt - A|,$$

where A, $B \in R(i)$ are given in terms of η, θ by (17·3) and (17·4) and so are subject to the conditions of (18·3), i.e.

$$\left.\begin{aligned} |\eta| &= |A\alpha + B\lambda| \leqslant N^{\frac{1}{2}}, \\ |\theta| &= |A\beta + B\mu| \leqslant N^{\frac{1}{2}}, \end{aligned}\right\} \tag{18·7}$$

and
$$0<|\alpha\theta-\beta\eta|=|B|\leqslant 2^{\frac{1}{2}}. \qquad (18\cdot 8)$$
For given $B \in R(i)$ we denote by $\mathfrak{D}''_B$ the set of all t with
$$|Bt-A|<1$$
for some A satisfying (18·7).

LEMMA 26. *$\mathfrak{D}''_B$ is a star domain about $-\mu/\beta$ (or null) for $|B|\leqslant 2^{\frac{1}{2}}$ and $N\geqslant 2M/m$ (M, m of theorem XXII) except, possibly, when $|B|=2^{\frac{1}{2}}$, $|\alpha\beta|\leqslant 2^{\frac{3}{2}}$.*

The proof is similar to that of lemma 24 except that theorem XXII is invoked instead of theorem XXI. We make the substitutions (17·9), (17·10) and (18·11) and put
$$\frac{B\mu}{\beta}-\frac{B\lambda}{\alpha}=\frac{-B}{\alpha\beta}=p+iq.$$
Then (18·9) is the same as
$$(x,y)\in\mathfrak{C}(u,v),$$
and (18·7) becomes
$$\left.\begin{aligned}\left|A+\frac{B\lambda}{\alpha}\right|&\leqslant\frac{N^{\frac{1}{2}}}{|\alpha|},\\ \left|A+\frac{B\mu}{\beta}\right|&\leqslant\frac{N^{\frac{1}{2}}}{|\beta|},\end{aligned}\right\}$$
and so
$$\left.\begin{aligned}(u-p)^2+(v-q)^2&\leqslant s^2,\\ u^2+v^2&\leqslant r^2,\end{aligned}\right\}$$
where
$$s^2=\frac{N}{|\alpha|^2},\quad r^2=\frac{N}{|\beta|^2}.$$
Also
$$p^2+q^2=\left|\frac{B}{\alpha\beta}\right|^2\leqslant\frac{2}{|\alpha|^2|\beta|^2},$$
and so
$$\frac{s^2}{p^2+q^2}\geqslant\frac{N|\beta|^2}{2}\geqslant\frac{M}{m},$$
since $|\beta|\geqslant 1$ and $N\geqslant 2M/m$ by hypothesis. Thus
$$\textit{either}\quad s^2\geqslant M \quad \textit{or}\quad p^2+q^2\leqslant m \quad \text{(or both)}.$$
Finally,
$$s^2=\frac{N}{|\alpha|^2}\geqslant 1,$$
and p, q satisfy the conditions of theorem XXII. Lemma 26 now follows from theorem XXII just as lemma 24 follows from theorem XXI.

COROLLARY.
$$\mathfrak{D}''=\bigcup_{0<|B|\leqslant 2^{\frac{1}{2}}}\mathfrak{D}''_B$$
is a star domain for $N\geqslant N_0$.

This follows from the lemma since $\mathfrak{D}_{1+i}\subset\mathfrak{D}_1$ for $|\alpha\beta|\leqslant 2^{\frac{3}{2}}$ and large enough N.

Finally, by lemma 25, $\mathscr{C}\mathfrak{D}''$ in the t-plane is the transform by (17·5) of $\mathfrak{R}(\alpha,\beta)$ in the z-plane. Hence, as in the proof of theorem XXIII, we deduce that $\mathfrak{R}(\alpha,\beta)$ is a star domain about α/β.

There remains the case when (18·6) is false, i.e. $\alpha=0$ since $\beta\neq 0$ by hypothesis. Here we make the transformation $t=1/z$ and invoke theorem XXI. We omit the details which are similar to those in the proof of theorem XXIII.

19. CONCLUSION

It will be apparent that many of the methods of this paper can be applied to the corresponding problems in other quadratic imaginary fields $k(\sqrt{(-m)})$. We have not made any detailed investigation of any other special cases, but the following two general theorems are easily proved. We use $\mathfrak{F}_N$, $\mathfrak{G}_N$, $\mathfrak{H}_N$, $\mathfrak{R}(\alpha,\beta)$, etc., to refer to $k(\sqrt{(-m)})$.

THEOREM XXV (cf. theorems IV, IX). *There is a constant C_0 depending only on m such that if $\Re(\alpha,\beta)$ and $\Re(\gamma,\delta)$ are adjacent,* $|\alpha\delta-\beta\gamma| \leqslant C_0$.

THEOREM XXVI (cf. theorems V, X). *There is a constant C_1 depending only on m such that no more than C_1 regions meet at a node. The angles which they subtend at the node belong to one of a bounded set of schemes.*

It might be of interest if further values of m, e.g. $m = 2$, were investigated in detail.

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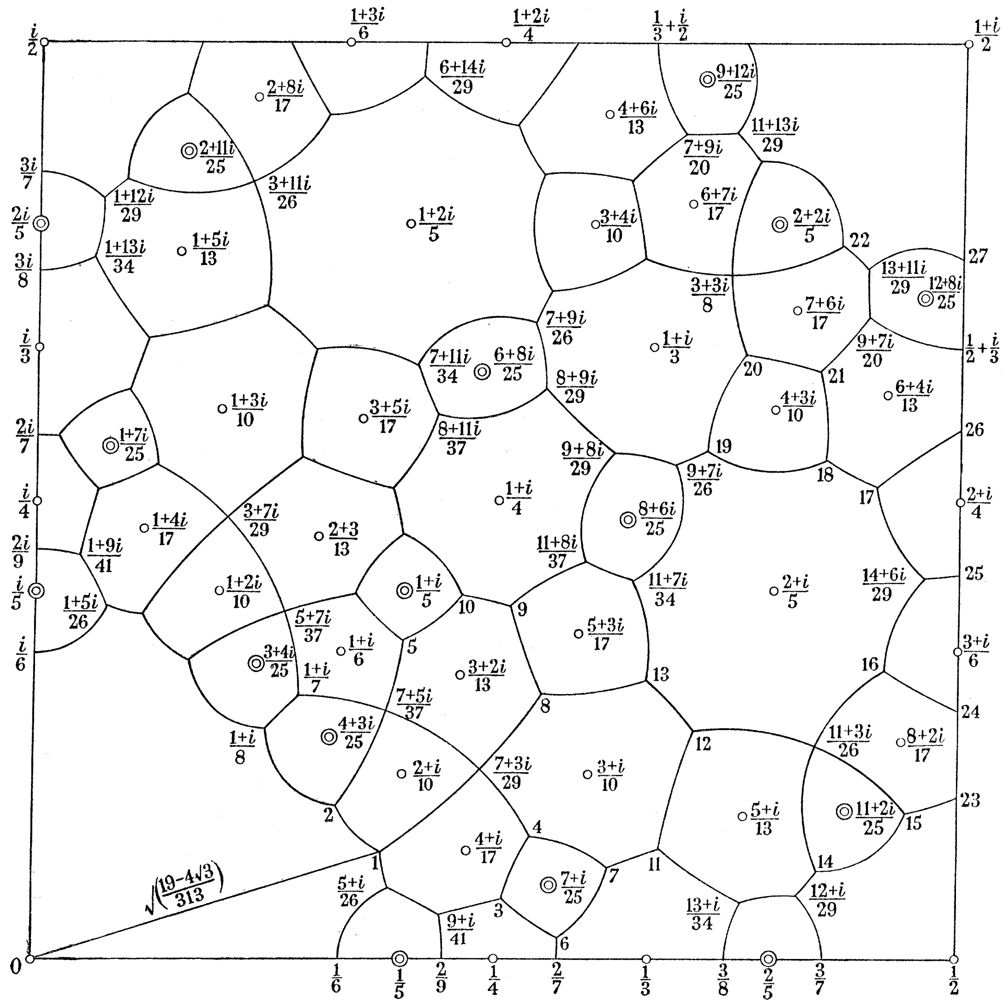
Examples of Farey Sections†

$\mathfrak{H}^*_N(i)$ for $N = 25$.



$N = 25$

1: $\left(\frac{140-13\sqrt{3}}{626}; \frac{57-12\sqrt{3}}{626}\right)$

2: $\left(\frac{172-9\sqrt{3}}{962}; \frac{115-20\sqrt{3}}{962}\right)$

3: $\left(\frac{51-8\sqrt{3}}{146}; \frac{10-3\sqrt{3}}{146}\right)$

4: $\frac{7+\sqrt{3}i}{26}$

5: $\left(\frac{38+3\sqrt{3}}{218}; \frac{55-10\sqrt{3}}{218}\right)$

6: $\left(\frac{98+7\sqrt{3}}{386}; \frac{25-12\sqrt{3}}{386}\right)$

7: $\left(\frac{124+15\sqrt{3}}{482}; \frac{17+4\sqrt{3}}{482}\right)$

8: $\left(\frac{22-\sqrt{3}}{74}; \frac{21-6\sqrt{3}}{74}\right)$

9: $\left(\frac{40-5\sqrt{3}}{122}; \frac{13+6\sqrt{3}}{122}\right)$

10: $\left(\frac{87+2\sqrt{3}}{458}; \frac{80+15\sqrt{3}}{458}\right)$

11: $\left(\frac{14-3\sqrt{3}}{26}; \frac{5-2\sqrt{3}}{26}\right)$

12: $\frac{5+\sqrt{3}i}{14}$

13: $\left(\frac{55+10\sqrt{3}}{218}; \frac{38-3\sqrt{3}}{218}\right)$

14: $\left(\frac{96-\sqrt{3}}{222}; \frac{7-2\sqrt{3}}{74}\right)$

15: $\left(\frac{291+16\sqrt{3}}{674}; \frac{68-9\sqrt{3}}{674}\right)$

16: $\left(\frac{164-11\sqrt{3}}{314}; \frac{39+6\sqrt{3}}{314}\right)$

17: $\left(\frac{104-9\sqrt{3}}{194}; \frac{57-4\sqrt{3}}{194}\right)$

18: $\left(\frac{136+5\sqrt{3}}{338}; \frac{113-12\sqrt{3}}{338}\right)$

19: $\left(\frac{149-10\sqrt{3}}{362}; \frac{116-9\sqrt{3}}{362}\right)$

20: $\left(\frac{90-9\sqrt{3}}{194}; \frac{57+4\sqrt{3}}{194}\right)$

21: $\left(\frac{21+6\sqrt{3}}{74}; \frac{22+\sqrt{3}}{74}\right)$

22: $\left(\frac{321+20\sqrt{3}}{818}; \frac{314+3\sqrt{3}}{818}\right)$

23: $\frac{1}{2}+\frac{4-\sqrt{3}}{26}i$

24: $\frac{1}{2}+\frac{2-\sqrt{3}}{2}i$

25: $\frac{1}{2}+\frac{\sqrt{2}-1}{2}i$ ‡

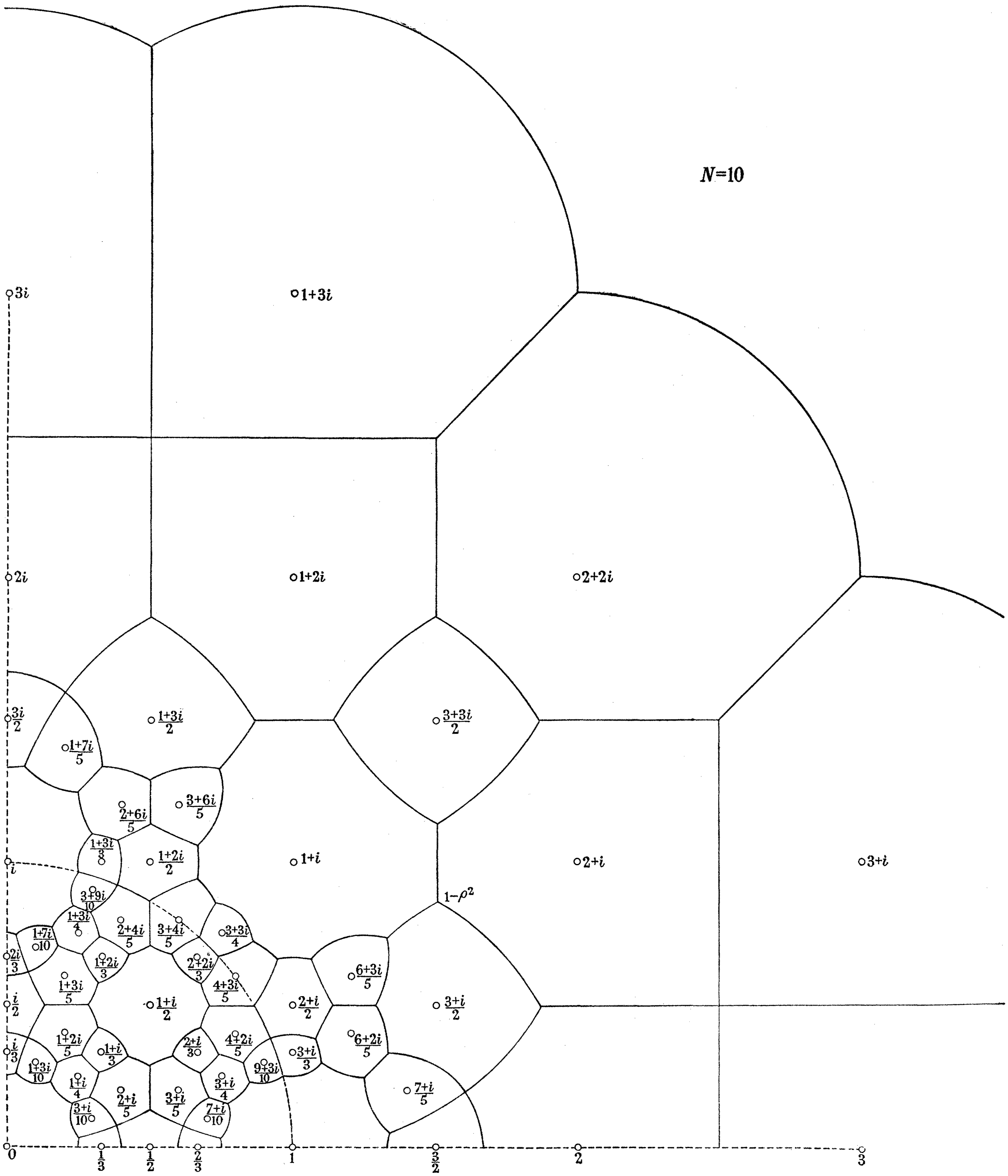
26: $\frac{1}{2}+\frac{\sqrt{3}}{6}i$

27: $\frac{1}{2}+\frac{16+\sqrt{3}}{46}i$

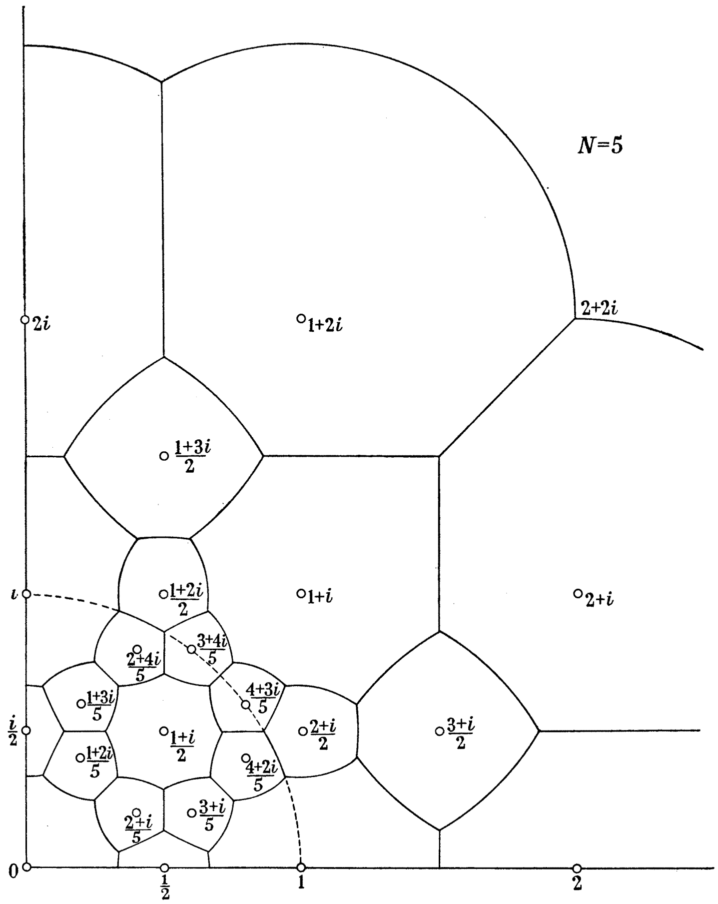
† One of us (K. Mahler) has drawn a complete set of diagrams of the regions $\mathfrak{R}(\alpha, \beta)$ belonging to $\mathfrak{F}_N$ in $k(i)$ and in $k(\rho)$ for values of N up to 25. These diagrams have been lodged with the Royal Society, where they may be inspected, as, for reasons of economy, it was impossible to reproduce them all in the present paper.

‡ Unnecessary vertex.

$\mathfrak{H}_N(i)$ for $N = 10$.



$\mathfrak{H}_N(i)$ for $N = 5$.



$\mathfrak{H}^*_N(\rho)$ and $\mathfrak{H}_N(\rho)$ for $N = 19$.