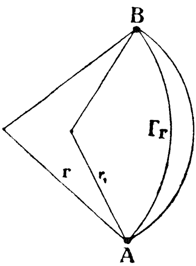
ON A QUESTION IN ELEMENTARY GEOMETRY

by K. Mahler (Manchester)

Through any two points A and B in the plane pass an infinity of different circles. Each such circle Γ_r, of radius r say, is divided by A and B into two separate arcs. Denote by γ_r the length of the smaller one of these two arcs; this length has a meaning as long as $2r \geqq \overline{AB}$; here $\overline{AB}$ denotes the distance of the two points A, B, *i.e.* the length of the line segment bounded by them.



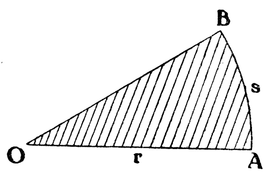
Γ, must be Γ_{r_1}.

It is nearly obvious from the figure that γ_r decreases when r increases, and this can also easily be proved by means of calculus or equivalent methods.

I give in this note an elementary proof of this property of γ_r, using only the ideas and methods of Euclid's "Elements". For clearness's sake, the proof is split into a number of lemmas. Everywhere, $\widehat{AB} = \gamma_r$ denotes the length of the smaller of the two arcs of a circle bounded by A and B; if several such arcs pass through A and B, then the context will make it clear which of them is being considered.

L e m m a 1: Let OAB be a circular sector of radius $r = \overline{OA}$ and bounded by the circular arc AB of length $s = \widehat{AB}$. Then OAB is of area $^1/_2\, rs = {}^1/_2\, \overline{OA} \times \widehat{AB}$.

This is a well-known theorem in Euclid's Elements, and is proved there by the exhaustion method, i.e. by means of a limiting process. This lemma enables us to carry through the proof of the monotonity of γ_r without using any further non-finite processes.



L e m m a 2: Let OAB be a circular sector as in Lemma 1 such that the angle $\sphericalangle\, AOB$ is acute. Let D lie on OB, and let CD be perpendicular to OC. Then $\overline{CD} < \widehat{AB}$.

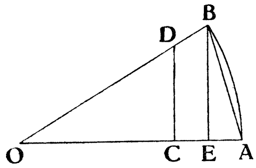
Proof: Denote by E the point on OA for which EB is perpendicular to OE. The triangle OCD is contained in, and similar to, the triangle OEB, and possibly coincides with it; therefore

$$(a) \qquad \overline{CD} \leqq \overline{EB}$$

Next the triangle OAB is a proper subset of the circular sector OAB, hence is of smaller area, so that by Lemma 1,

$$(b) \qquad {}^1/_2\,\overline{OA} \times \overline{EB} < {}^1/_2\,\overline{OA} \times \widehat{AB}, \text{ whence } \overline{EB} < \widehat{AB}$$

The assertion is immediate from (*a*) and (*b*).

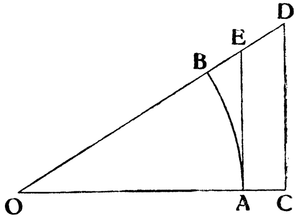


Lemma 3: Let OAB be a circular sector as in Lemma 1 such that the angle $\sphericalangle\ AOB$ is acute. Let C lie on the line through OA beyond A, and let D lie on the line OB beyond B and be such that CD is perpendicular to OC. Then

$$\overline{CD} > \widehat{AB}.$$

Proof: Denote by E the point on OB beyond B for which AE is perpendicular to OA. The triangle OCD contains, and is similar to the triangle OAE, and possibly coincides with it; therefore

$$(a) \qquad \overline{CD} \geqq \overline{AE}$$

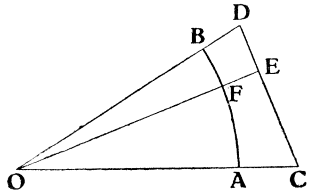


Next the circular sector OAB is a proper subset of the triangle OAE, hence is of smaller area. Therefore by Lemma 1,

$$(b) \qquad {}^1/_2\,\overline{OA} \times \widehat{AB} < {}^1/_2\,\overline{OA} \times \overline{AE}, \text{ whence } \widehat{AB} < \overline{AE}.$$

The assertion follows from (*a*) and (*b*).

Lemma 4: Let OAB be a circular sector as in Lemma 1, such that the angle $\sphericalangle\ AOB$ is less than 180°. Let C lie on OA beyond A, and D on OB beyond B, and let the line segment CD meet the circular arc AB in at most one point. Then $\overline{CD} > \widehat{AB}$.

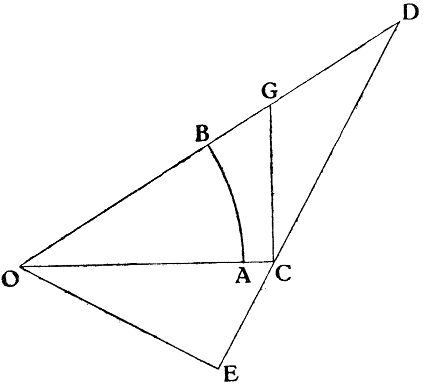


Proof: Denote by E the point on the line through CD for which OE is perpendicular to CD. If E coincides with either C or D, then the assertion has already been proved in Lemma 3.

Assume next that E lies between C and D, hence that the line OE lies in the angle $\sphericalangle AOB$. Then OE meets the arc AB in a unique point, F say. By applying Lemma 3 twice, we find that

$$\widehat{AF} < \overline{CE}, \widehat{FB} < \overline{ED},$$

and the assertion follows on adding these inequalities.



There remains the case that the line OE falls outside the angle $\sphericalangle AOB$. To fix the ideas, let us assume that E lies on CD beyond C. Then the angle $\sphericalangle OCD$ is greater than a right one, and so there is a point G on OD such that CG is perpendicular to OC; hence by Lemma 3,

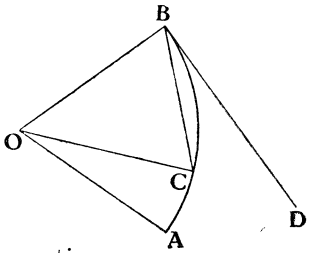
$$(a) \qquad \widehat{AB} < \overline{CG}.$$

The triangle OCG has, by construction, a right angle at C; hence $\sphericalangle CGO$ is acute, and so $\sphericalangle CGD$ is the largest angle of the triangle CDG. Then the side opposite this angle is the largest of the triangle:

$$(b) \qquad \overline{CG} < \overline{CD},$$

and the assertion follows from (a) and (b).

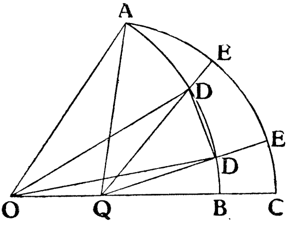
L e m m a 5: Let AB be a circular arc with centre at O. Let C be a point on this arc, and let BD be the tangent at B of this arc. Then the angle $\sphericalangle CBD$ is half the angle $\sphericalangle COB$.



Proof: Evidently

(a) $\sphericalangle CBO = 90° - {}^1/_2 \sphericalangle COB$, and
(b) $\sphericalangle CBO = 90° - \sphericalangle CBD$,

whence the assertion.



D and E at the top must be D′ and E′.

L e m m a 6: Let AB and AC be two circular arcs of centres O and Q, respectively, such that $OQBC$ lie on one line in just this order. Assume that $\sphericalangle AOB$ and $\sphericalangle AQB$ are at most 180°. Let E and E' be two points on the arc AC such that E is nearer to C, and E' is nearer to A; let further the radii EQ and $E'Q$ intersect the arc AB at D and D', respectively. Then

$$\overline{DE} > \overline{D'E'}.$$

Proof: Since

$$\overline{QE} = \overline{QE'} = \overline{QA},$$

it suffices to show that

(*a*) $$\overline{QD} < \overline{QD'}.$$

Draw the lines OD, OD' and DD'. The triangle ODD' has the two equal sides $\overline{OD} = \overline{OD'} = \overline{OA}$, so that

(*b*) $$\sphericalangle\, ODD' = \sphericalangle\, OD'D.$$

On the other hand,

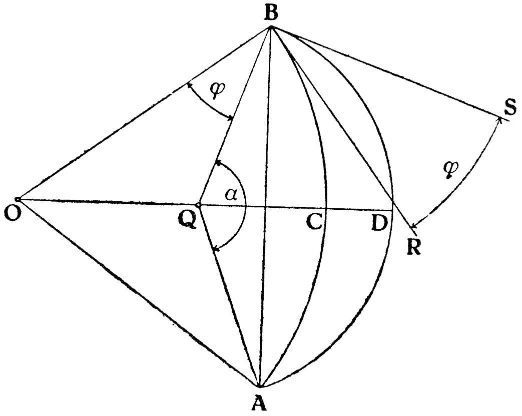
(*c*) $$\sphericalangle\, QDD' > \sphericalangle\, ODD' \text{ and } \sphericalangle\, QD'D < \sphericalangle\, OD'D.$$

From (*b*) and (*c*),

$$\sphericalangle\, QDD' > \sphericalangle\, QD'D,$$

and (*a*) follows immediately since in the triangle QDD' the side $\overline{QD'}$ lies opposite a larger angle than the side $\overline{QD}$.

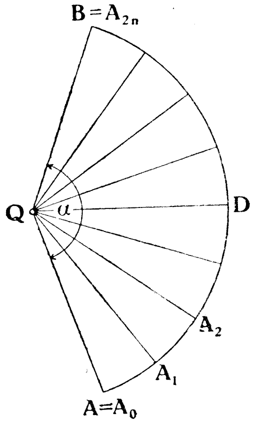
After these preparations, we can now start with the proof that γ_r decreases when r increases.



We consider two circular arcs ACB and ADB through A and B, of radius $\overline{OA} = \overline{OB}$, and $\overline{QA} = \overline{QB}$, respectively, where

$$\overline{OA} > \overline{QA}.$$

We assume that both arcs are at most semi-circles, and that both lie on the opposite side of the line through AB on which are O and Q. For shortness, denote by α the angle $\sphericalangle\ AQB$, and by φ the angle $\sphericalangle\ QBO$. Then φ is also the angle $\sphericalangle\ RBS$ of the tangents BR of the arc ACB and BS of the arc ADB at the point B common to both arcs; for the radius and the tangent through the same point of a circle are perpendicular.



A_{2n} must be A_{2^n}.

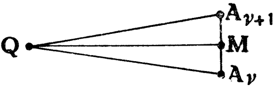
By Archimedes's Axiom (which is both stated and used in Euclid's Elements), there exists a positive integer n such that

$$(a) \qquad 2^{n+1}\varphi > \alpha.$$

Divide the angle $\alpha = \sphericalangle\ AQB$ into 2^n equal parts; this can be done by repeatedly halving the angle. The radii belonging to the so obtained fractions of α meet the arc ADB at the equidistant points $A_0 = A,\ A_1,\ A_2, \ldots,$ $A_{2^n} = B$.

Next form the line segments

$$AA_1 = A_0A_1,\ A_1A_2,\ A_2A_3,\ \ldots, A_{2^n-1}A_{2^n} = A_{2^n-1}B;$$



for each such line segment

$$A_\nu A_{\nu+1} \qquad (\nu = 0, 1, \ldots, 2^n-1)$$

denote by M_ν its centre: $\overline{A_\nu M_\nu} = \overline{M_\nu A_{\nu+1}}$.

The line QM_ν is then perpendicular to the line $A_\nu A_{\nu+1}$. Since further all 2^n triangles

$$QA_\nu A_{\nu+1} \qquad (\nu = 0, 1, \ldots, 2^n-1)$$

are congruent, the 2^n points

$$M_0, M_1, M_2, \ldots, M_{2^n-1}$$

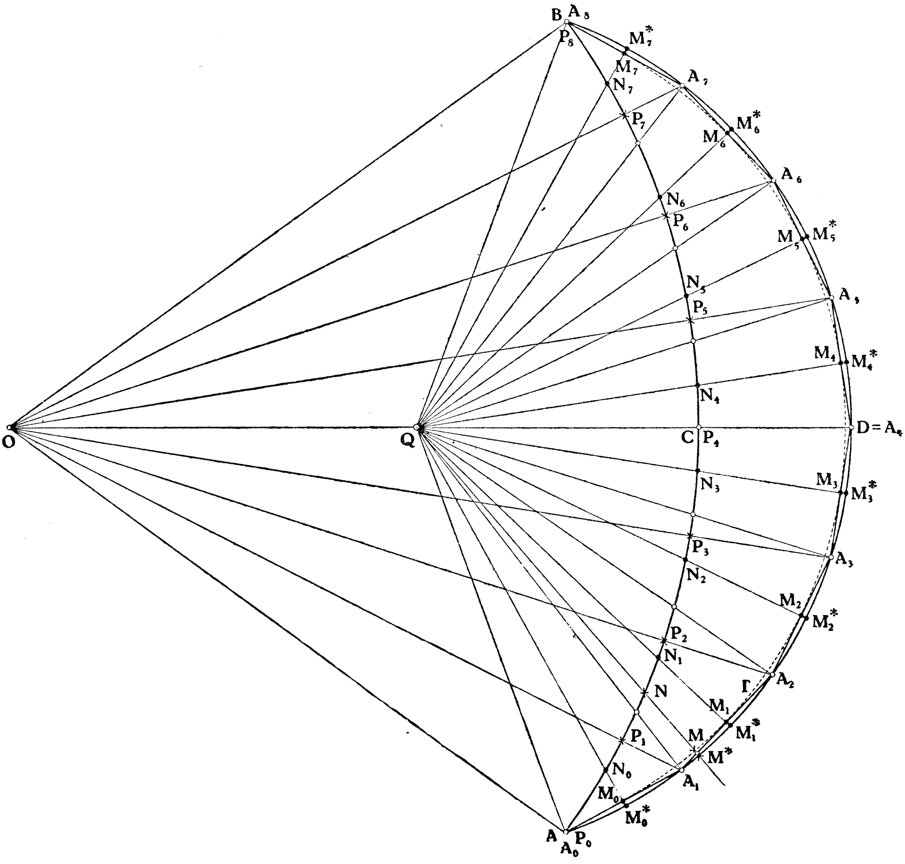
have all the same distance, ϱ say, from Q.

Denote now by Γ the circular arc through $M_0M_1 \ldots M_{2^n-1}$ of centre Q and radius ϱ; its endpoints are M_0 and M_{2^n-1}. Denote further by $\triangle$ the curve consisting of the two line segments AM_0 and $M_{2^n-1}B$ and of the arc Γ. Since for $\nu = 0, 1, \ldots, 2^n-1$, the midpoint M_ν is always that point on $A_\nu A_{\nu+1}$ which is nearest to Q, it is clear that the broken line

$$A_0A_1A_2 \ldots A_{2^n-1}A_{2^n}$$

lies everywhere between the curve $\triangle$ and the circular arc ADB. We show now that, on the other hand, $\triangle$ is separated from Q, and so also from O, by the circular arc ACB which has the same endpoints A and B.

Firstly, the angle $\sphericalangle\ A_0QA_1$ is, by construction and by (a), equal to $\frac{\alpha}{2^n} < 2\varphi$. By Lemma 5, this implies that the angle between the line AA_1 and the tangent to ADB at A is equal to $\frac{1}{2}\cdot\frac{\alpha}{2^n} < \varphi$. Hence the line segment AA_1 lies between the two tangents at A of the circular arcs ACB and ADB, hence lies outside the circle



ACB, except for the single point A. By just the same reasoning, the line segment $A_{2^n-1}B$ lies outside the circular arc ACB, except for the single point B.

Secondly, let M be an arbitrary point on the circular arc Γ. The radius QM meets the arc ADB in a unique point, M^* say, and the arc ACB in a unique point, N say. Since, by hypothesis, the arc ADB is separated from Q by the arc ACB, the three collinear points QNM^* follow one another in just this order. By Lemma 6, $\overline{NM^*}$ is smallest when M lies at one of the two endpoints M_0 and M_{2^n-1} of Γ. Further, from the construction, the distance $\overline{MM^*}$ is independent of the special choice of M, namely equal to the difference:

$$\overline{QA} - \varrho$$

of the radii of the two concentric arcs ADB and Γ. Hence also $\overline{MN}$ is smallest when M lies at either endpoint M_0 or M_{2^n-1} of Γ. Since, as we saw before, both M_0 and M_{2^n-1} are separated from Q by the arc ACB, the same must then be true for all points M on Γ.

We have proved, in this way, that the broken line

$$A_0A_1A_2\ldots A_{2^n}$$

lies everywhere between the two arcs ACB and ADB, and has the same endpoints A and B. The asserted inequality

$$\widehat{ACB} < \widehat{ADB}$$

is now obtained as follows:

Denote, for $\nu = 0, 1, \ldots, 2^n-1$, by M^*_ν the point on the arc ADB collinear with Q and M_ν. Then the line segment $A_\nu A_{\nu+1}$ is perpendicular to the line $QM_\nu M^*_\nu$. Therefore, by Lemma 2,

$$\overline{A_\nu M_\nu} < \widehat{A_\nu M^*_\nu},\ \overline{M_\nu A_{\nu+1}} < \widehat{M^*_\nu A_{\nu+1}} \qquad (\nu = 0, 1, \ldots, 2^n-1),$$

whence

$$\overline{A_\nu A_{\nu+1}} < \widehat{A_\nu A_{\nu+1}} \qquad (\nu = 0, 1, \ldots, 2^n-1).$$

On adding now over all values of ν, we find that the broken line

$$A_0A_1A_2\ldots A_{2^n}$$

is shorter than the circular arc $\widehat{ADB}$.

Next denote, for $\nu = 0,1, \ldots, 2^n$, by P_ν the point where the radius OA_ν meets the circular arc ACB; in particular, $P_0 = A$ and $P_{2^n} = B$. By Lemma 4, we have

$$\widehat{P_\nu P_{\nu+1}} < \overline{A_\nu A_{\nu+1}} \qquad (\nu = 0, 1, \ldots, 2^n-1)$$

On adding again over all values of ν, we find this time that the broken line

$$A_0A_1A_2\ldots A_{2^n}$$

is greater than the arc $\widehat{ACB}$

On combining these two inequalities for the broken line, the assertion follows at once.

It would be of interest to simplify this proof.

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