ON THE LATTICE DETERMINANTS OF TWO PARTICULAR POINT SETS

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It is well known that the star domain

$$K: \quad |xy| \leqslant 1$$

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has the determinant $\Delta(K)=\sqrt{5}$. Further, for the ray domain

$$R_0:\quad 0 \leqslant xy \leqslant 1, \quad x \geqslant 0, \quad y \geqslant 0,$$

forming the part of K in the first quadrant, B. Segre [4] and I [1] have shown that $\Delta(R_0)=1$. In this note I determine the determinant of the ray domain

$$R:\quad |xy| \leqslant 1, \quad x \geqslant 0,$$

consisting of the intersection of K with the half-plane $x \geqslant 0$, and more generally that of its subset

$$R_t:\quad |xy| \leqslant 1, \quad x \geqslant t,$$

where t is an arbitrary non-negative constant. Since

$$K \supset R \supset R_t,$$

it is obvious that

$$\Delta(R_t) \leqslant \Delta(R) \leqslant \sqrt{5}.$$

Surprisingly enough, it can be proved that here the equality signs hold, so that

$$\Delta(R_t)=\Delta(R)=\sqrt{5}.$$

For let Λ be an arbitrary R_t-admissible lattice; it evidently suffices to show that $d(\Lambda) \geqslant \sqrt{5}$. In every rectangle

$$P_\tau:\quad |x| \leqslant t, \quad |y| \leqslant \tau,$$

Λ has at most finitely many points. Hence a τ with $0<\tau \leqslant t^{-1}$ can be chosen such that $O=(0, 0)$ is the only lattice point contained in P_τ. But then Λ is clearly an admissible lattice of the star domain

$$K_\tau:\quad |xy| \leqslant 1, \quad |y| \leqslant \tau.$$

Further $\Delta(K_\tau)=\sqrt{5}$, as follows at once from Theorem 10 of my paper [2] on putting $F(X)=\max(|xy|^{\frac{1}{2}}, \tau^{-1}|y|)$. Therefore

$$d(\Lambda) \geqslant \Delta(K_\tau)=\sqrt{5},$$

as asserted.

Although the set R_t covers an arbitrarily *small* part of the halfplane $x \geqslant 0$, its determinant has just been shown to be positive and constant. I now give an example of a ray set, also of positive constant determinant, but filling an arbitrarily *large* portion of this halfplane $x \geqslant 0$.

Let now $t>0$; denote by S_t the set of all points (x, y) for which

$$\text{either } 0 \leqslant x \leqslant t, \text{ or } x>t \text{ and } |xy| \leqslant 1.$$

Then we shall prove that

$$\Delta(S_t)=\frac{3+\sqrt{5}}{2}.$$

For let Λ be any S_t-admissible lattice. This lattice contains points different from the origin in the parallel strip $|x|<t$, because this strip is convex, symmetric in the origin, and of infinite area. Since the lattice is S_t-admissible, such points necessarily lie on the y-axis. There exists then also a point $(0, a)$ of Λ of smallest positive a, and this point is therefore primitive. Next we can select a second point (b, c) of Λ such that $(0, a)$ and (b, c) together form a basis of Λ; the lattice consists thus of the points

$$(bv,\ au+cv) \qquad (u,\ v=0,\ \pm 1,\ \pm 2,\ \ldots).$$

There is no loss of generality in assuming that b is also positive. Since Λ is S_t-admissible, this means that

$$b \geqslant t$$

and that further

$$|bv(au+cv)| \geqslant 1 \quad \text{if} \quad u=0,\ \pm 1,\ \pm 2,\ \ldots;\quad v=1,\ 2,\ 3,\ \ldots.$$

Put
$$\xi = -\frac{c}{a}.$$

Since $d(\Lambda)=ab$, then

$$\left|\frac{u}{v}-\xi\right| \geqslant \frac{1}{d(\Lambda)\,v^2} \quad \text{if} \quad u=0,\ \pm 1,\ \pm 2,\ \ldots;\quad v=1,\ 2,\ 3,\ \ldots.$$

Now a theorem of A. V. Prasad [3] states that for every real ξ, integers u, $v \geqslant 1$ can always be chosen such that

$$\left|\frac{u}{v}-\xi\right| \leqslant \frac{2}{(3+\sqrt{5})\,v^2}.$$

The last inequality implies then that

$$d(\Lambda) \geqslant \frac{3+\sqrt{5}}{2},$$

and therefore
$$\Delta(S_t) \geqslant \frac{3+\sqrt{5}}{2}.$$

Here the sign of equality holds. For select any $b \geqslant t$ and put

$$a=\frac{3+\sqrt{5}}{2b}, \quad c=\frac{1-\sqrt{5}}{2}\,a,$$

so that
$$d(\Lambda)=ab=\frac{3+\sqrt{5}}{2}, \quad \xi=-\frac{c}{a}=\frac{\sqrt{5}-1}{2}.$$

It is obvious that Λ contains no points (x, y) for which

$$0<x<t.$$

Further, by Prasad's theorem, the inequality

$$\frac{u}{v}-\frac{\sqrt{5}-1}{2} \leqslant \frac{C}{v^2}$$

has no solutions in integers u, $v \geqslant 1$ if

$$C < \frac{2}{3+\sqrt{5}}.$$

This implies that there are no such integers for which

$$|xy| = |bv(au+cv)| < 1,$$

and hence that Λ is S_t-admissible. This concludes the proof.

References.

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