THE
MATHEMATICAL GAZETTE

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LONDON
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXXVIII | DECEMBER, 1954 | NO. 326

A PROBLEM IN ELEMENTARY GEOMETRY.

BY KURT MAHLER

RECENTLY, in connection with some work on Diophantine approximations, I encountered the following problem on triangles.

Let T be a triangle with vertices A, B, C which are, respectively, inner points of the sides a, b, c of a second triangle t. Is it always possible to move T into a new position where its vertices are inner points of t?

I give here an affirmative answer to the problem and prove, moreover, that it suffices to apply to T an arbitrarily small rotation about a suitably chosen point of the plane. I am indebted to my Manchester colleagues for a number of simplifications of this solution, arrived at when discussing the problem with them.

In the proof, several cases will be distinguished. We always denote by α, β, γ the perpendiculars to the lines a, b, c at A, B, and C, respectively ; these perpendiculars will be thought of as extending to infinity in both directions.

Case 1. α and β intersect at a point D which does not also lie on γ (Fig. 1 and 2).

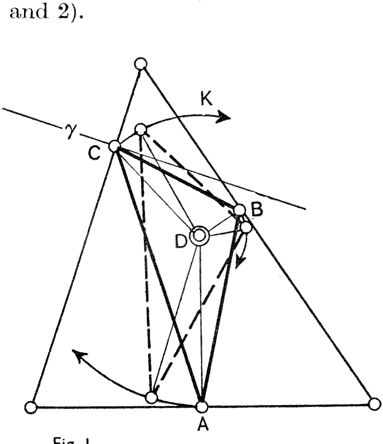

Fig. 1

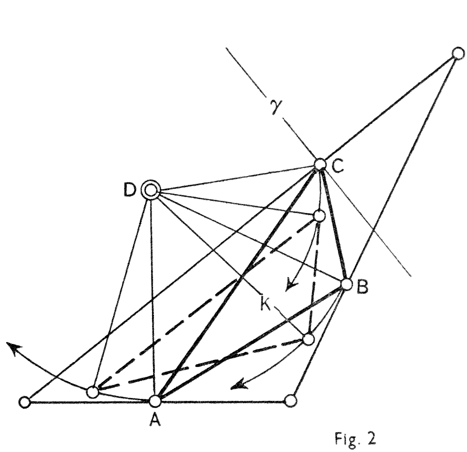

Fig. 2

If T is rotated about D by an arbitrarily small angle, C describes a small arc of the circle K with centre at D. The side c of t is not a tangent to K; hence the direction of the rotation can be chosen such that the new position of C lies inside t. This rotation has thus the required property since it obviously transports also A and B into inner points of t.

Case 2 : All three lines α, β, γ intersect at a point D in the interior of t (Fig. 3).

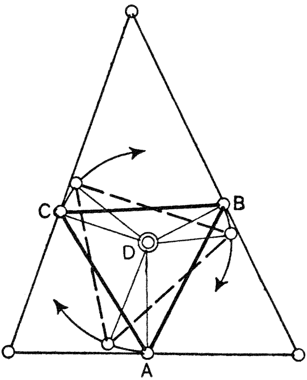

Fig. 3

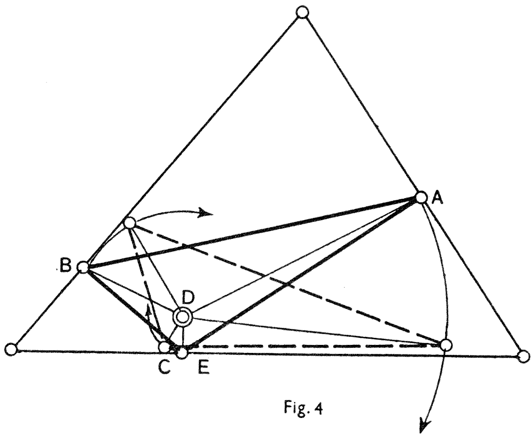

Fig. 4

Evidently every sufficiently small rotation of T about D has the required property.

Case 3 : The three lines α, β, γ intersect at a point E which lies, say, on the side c of t (Fig. 4).

The hypothesis implies that E coincides with C, and that α is the side AC and β the side BC of T. Select a point D on γ arbitrarily near to E and inside T. The smaller angle at A of AD with the side a is less than, but arbitrarily near to, 90°. Hence there is a rotation of T about D such that A, after first leaving t, is changed into an inner point of t, and that both B and C become likewise inner points of t. Moreover, the angle of this rotation can be made arbitrarily small by taking D sufficiently near to $C = E$.

Case 4 : The three lines α, β, γ intersect at a point E which lies outside t, say on the outer perpendicular γ (Fig. 5).

Draw the circle, K say, that passes through C and E and has as its centre the midpoint of the line segment CE; further select a point D arbitrarily near to E on K. Evidently K touches the side c of t on the outside of the triangle. If P is any point on K which does not lie on the small arc DE of this circle, the angle $\angle DPE$ has the constant value $\angle DCE$, $= \zeta$ say. Further the points A and B are separated from D and E by the tangent c of K. It follows therefore that the two angles $\angle DAE$, $= \xi$ say, and $\angle DBE$, $= \eta$ say, are both smaller than ζ. We can then select an angle ϕ which is smaller than 2ζ, but greater than both 2ξ and 2η; moreover, ϕ will be arbitrarily small if D was chosen sufficiently near to E. We rotate now T about D by the angle ϕ in such a direction that A and B first leave t and afterwards become inner points of t; then, at the same time, C has likewise been changed into an inner point of t. This concludes the proof.

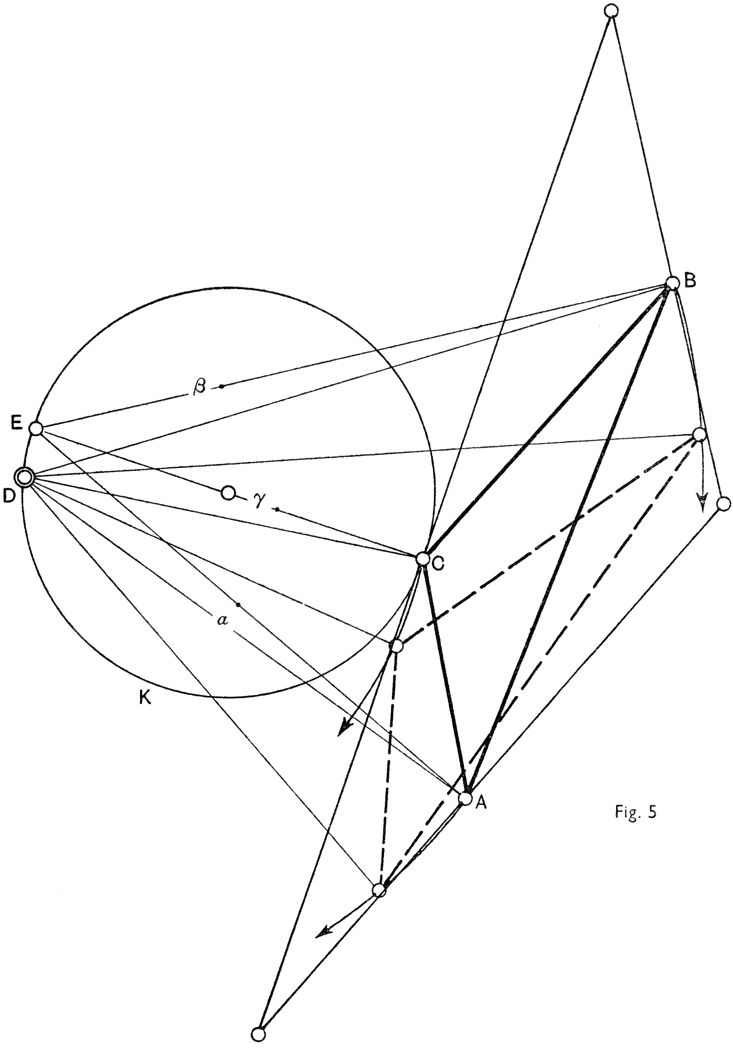

Fig. 5

It may be noted that the theorem has no obvious generalization to polygons of more than three sides. Thus there exist rectangles with their vertices on the sides of a square Q that cannot be moved into any new position where their vertices are inner points of Q.

The theorem may be extended to simplices in more dimensions.

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