THE p-TH COMPOUND OF A SPHERE

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LET G_n: $|X| \leqslant 1$ be the unit sphere in R_n, and let $\Gamma_n^{(p)} = [G_n]^{(p)}$ be its pth compound in R_N, where $N = \binom{n}{p}$. By definition, $\Gamma_n^{(p)}$ is the convex hull of the set $\Sigma_n^{(p)} = \langle G_n \rangle^{(p)}$ which consists of all compound points

$$\Xi = [X^{(1)}, X^{(2)}, \ldots, X^{(p)}] \quad \text{where } X^{(1)}, X^{(2)}, \ldots, X^{(p)} \in G_n.$$

Our aim is to give more explicit definitions of $\Sigma_n^{(p)}$ and $\Gamma_n^{(p)}$.

1. THEOREM 1. *$\Sigma_n^{(p)}$ is the intersection of the N-dimensional unit sphere G_N: $|\Xi| \leqslant 1$ with the Grassmann manifold $\mathbf{\Omega}(n, p)$.*

Proof. Let $\Xi \neq O$ be an arbitrary point of $\Sigma_n^{(p)}$. There exist then p linearly independent points $Y^{(1)}, Y^{(2)}, \ldots, Y^{(p)}$ such that

$$Y^{(1)}, Y^{(2)}, \ldots, Y^{(p)} \in G_n \quad \text{and} \quad \Xi = [Y^{(1)}, Y^{(2)}, \ldots, Y^{(p)}].$$

These points may be replaced by p others that are mutually orthogonal, as follows.

Put $X^{(1)} = Y^{(1)}$, and define $X^{(2)} = Y^{(2)} + \lambda_{21} X^{(1)}$ where the constant λ_{21} is chosen such that $X^{(1)}X^{(2)} = X^{(1)}Y^{(2)} + \lambda_{21} X^{(1)}X^{(1)} = 0$; this is possible because $X^{(1)} = Y^{(1)} \neq O$ since $Y^{(1)}, Y^{(2)}, \ldots, Y^{(p)}$ are independent. Assume now that, for some k with $2 \leqslant k \leqslant p$, we have already defined $k-1$ points

$$X^{(1)} = Y^{(1)}, \quad X^{(2)} = Y^{(2)} + \lambda_{21} X^{(1)}, \quad X^{(3)} = Y^{(3)} + \lambda_{31} X^{(1)} + \lambda_{32} X^{(2)}, \quad \ldots,$$
$$X^{(k-1)} = Y^{(k-1)} + \lambda_{k-11} X^{(1)} + \lambda_{k-12} X^{(2)} + \ldots + \lambda_{k-1k-2} X^{(k-2)}$$

satisfying the orthogonality conditions

$$X^{(i)}X^{(j)} = 0 \quad \text{if } 1 \leqslant i < j \leqslant k-1.$$

Then a further point $X^{(k)}$ of the form

$$X^{(k)} = Y^{(k)} + \lambda_{k1} X^{(1)} + \lambda_{k2} X^{(2)} + \ldots + \lambda_{kk-1} X^{(k-1)}$$

may be determined such that also

$$X^{(1)}X^{(k)} = X^{(2)}X^{(k)} = \ldots = X^{(k-1)}X^{(k)} = 0,$$

because, for $1 \leqslant i \leqslant k-1$, the equations

$$X^{(i)}X^{(k)} = X^{(i)}Y^{(k)} + \lambda_{ki} X^{(i)}X^{(i)} = 0$$

have a solution λ_{ki} since $X^{(i)}X^{(i)} > 0$.

By this construction, we obtain p independent points $X^{(1)}, X^{(2)}, \dots, X^{(p)}$ such that

$$X^{(k)} = Y^{(k)} + \lambda_{k1} X^{(1)} + \lambda_{k2} X^{(2)} + \dots + \lambda_{kk-1} X^{(k-1)} \quad (1 \leqslant k \leqslant p),$$

$$X^{(i)} X^{(j)} = 0 \quad (1 \leqslant i < j \leqslant p).$$

By these equations,

$$Y^{(k)} Y^{(k)} = \lambda_{k1}^2 X^{(1)} X^{(1)} + \lambda_{k2}^2 X^{(2)} X^{(2)} + \dots + \lambda_{kk-1}^2 X^{(k-1)} X^{(k-1)} + X^{(k)} X^{(k)}$$

and therefore $\quad X^{(k)} X^{(k)} \leqslant Y^{(k)} Y^{(k)} \leqslant 1 \quad (1 \leqslant k \leqslant p),$

so that also $X^{(1)}, X^{(2)}, \dots, X^{(p)} \in G_n$. It is further obvious that

$$[X^{(1)}, X^{(2)}, \dots, X^{(p)}] = [Y^{(1)}, Y^{(2)}, \dots, Y^{(p)}] = \Xi.$$

Write $\quad |X^{(1)}| = t_1, \quad |X^{(2)}| = t_2, \quad \dots, \quad |X^{(p)}| = t_p,$

so that $t_1 > 0$, $t_2 > 0, \dots, t_p > 0$. Since the points $X^{(i)}$ are orthogonal in pairs, an orthogonal transformation $X \to X' = \Omega X$ of R_n exists for which

$$X^{(1)} = t_1 \Omega U_1, \quad X^{(2)} = t_2 \Omega U_2, \quad \dots, \quad X^{(p)} = t_p \Omega U_p.$$

Here $\quad U_1 = (1, 0, \dots, 0), \quad U_2 = (0, 1, \dots, 0), \quad \dots, \quad U_n = (0, 0, \dots, 1)$

are the n unit points on the coordinate axes in R_n. Therefore

$$0 < t_1 \leqslant 1, \quad 0 < t_2 \leqslant 1, \quad \dots, \quad 0 < t_p \leqslant 1,$$

since, as already shown, $|X^{(i)}| \leqslant 1$; hence also

$$0 < t \leqslant 1, \quad \text{where } t = t_1 t_2 \dots t_p.$$

We have therefore the result that every point $\Xi \neq O$ of $\Sigma_n^{(p)} = \langle G_n \rangle^{(p)}$ can be written as

$$\Xi = t\, \Omega^{(p)} \Xi_0 \quad (0 < t \leqslant 1),$$

where Ξ_0 is the special point

$$\Xi_0 = [U_1, U_2, \dots, U_p]$$

in R_N which has just one coordinate equal to 1 and all the others 0, while $\Omega^{(p)}$ denotes the pth compound of Ω. It is well known that the compounds of orthogonal transformations are again orthogonal. The result just proved means therefore that

$$|\Xi| \leqslant 1 \quad \text{if } \Xi \in \Sigma_n^{(p)},$$

hence that $\Sigma_n^{(p)} \subseteq G_N \cap \mathbf{\Omega}(n, p)$.

We finally show that the converse, $G_N \cap \mathbf{\Omega}(n, p) \subseteq \Sigma_n^{(p)}$, also holds. Let Ξ be any point satisfying

$$|\Xi| \leqslant 1 \quad \text{and} \quad \Xi \in \mathbf{\Omega}(n, p).$$

Then Ξ may again be written as a compound

$$\Xi = [X^{(1)}, X^{(2)}, \dots, X^{(p)}]$$

of p points $X^{(1)}$, $X^{(2)}, \dots$, $X^{(p)}$ that are orthogonal in pairs. Put again

$$|X^{(1)}| = t_1, \quad |X^{(2)}| = t_2, \quad \dots, \quad |X^{(p)}| = t_p, \quad t = t_1 t_2 \dots t_p;$$

then $t = |\Xi| \leqslant 1$. Now

$$[s_1 X^{(1)}, s_2 X^{(2)}, \dots, s_p X^{(p)}] = [X^{(1)}, X^{(2)}, \dots, X^{(p)}] \quad \text{if } s_1 s_2 \dots s_p = 1.$$

There is no loss of generality in assuming that

$$0 \leqslant t_1 \leqslant 1, \quad 0 \leqslant t_2 \leqslant 1, \quad \dots, \quad 0 \leqslant t_p \leqslant 1,$$

i.e. that $X^{(1)}$, $X^{(2)}, \dots$, $X^{(p)} \in G_n$. Hence $\Xi \in \Sigma_n^{(p)}$, as asserted. This concludes the proof.

2. THEOREM 2. *$\Gamma_n^{(p)}$ is the convex hull of the intersection of the Grassmann manifold $\mathbf{\Omega}(n,p)$ with the spherical surface C_N: $|\Xi| = 1$.*

Proof. By the last theorem, $\Gamma_n^{(p)}$ is the convex hull of

$$\Sigma_n^{(p)} = G_N \cap \mathbf{\Omega}(n,p).$$

As the origin O lies in all three sets $\Gamma_n^{(p)}$, G_N, and $\mathbf{\Omega}(n, p)$, this means that every point Ξ of $\Gamma_n^{(p)}$ belongs either to the interior, or to the boundary, of a certain simplex S' with vertices at O, Ξ'_1, $\Xi'_2, \dots$, Ξ'_N, where the points Ξ'_K are elements of $G_N \cap \mathbf{\Omega}(n,p)$. To each vertex Ξ'_K there is a number $t_K \geqslant 1$ such that the point

$$\Xi_K = t_K \Xi'_K \quad (K = 1, 2, \dots, N)$$

lies on C_N and therefore also on $C_N \cap \mathbf{\Omega}(n,p)$. The new simplex with the vertices O, Ξ_1, $\Xi_2, \dots$, Ξ_N evidently contains S' as a subset, hence has Ξ as an element. But this means exactly that $\Gamma_n^{(p)}$ coincides with the convex hull of $C_N \cap \mathbf{\Omega}(n,p)$, as was to be proved.

3. As an application of Theorem 2, let us determine the compound body $\Gamma_4^{(2)} = [G_4]^{(2)}$, for which we write more simply Γ.

Let $X^{(1)} = (x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$ and $X^{(2)} = (x_0^{(2)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$ be two general points in R_4, and let $\Xi = (\xi_1, \xi_2, \dots, \xi_6) = [X^{(1)}, X^{(2)}]$ be their compound point in R_6, with the coordinates numbered in such a way that

$$\xi_1 = x_0^{(1)} x_1^{(2)} - x_0^{(2)} x_1^{(1)}, \qquad \xi_2 = x_0^{(1)} x_2^{(2)} - x_0^{(2)} x_2^{(1)}, \qquad \xi_3 = x_0^{(1)} x_3^{(2)} - x_0^{(2)} x_3^{(1)},$$

$$\xi_4 = x_2^{(1)} x_3^{(2)} - x_2^{(2)} x_3^{(1)}, \qquad \xi_5 = x_3^{(1)} x_1^{(2)} - x_3^{(2)} x_1^{(1)}, \qquad \xi_6 = x_1^{(1)} x_2^{(2)} - x_1^{(2)} x_2^{(1)};$$

note the change of sign of ξ_5. Then $\mathbf{\Omega}(4, 2) = \mathbf{\Omega}$, say, becomes the cone

$$\xi_1 \xi_4 + \xi_2 \xi_5 + \xi_3 \xi_6 = 0,$$

and $C_6 = C$, say, is the spherical surface

$$\xi_1^2 + \xi_2^2 + \dots + \xi_6^2 = 1.$$

Our aim is to determine the convex hull of $C \cap \mathbf{\Omega}$. This will be done by evaluating first the tac function and then the distance function of this convex hull Γ.

4. Let $\mathsf{H} = (\eta_1, \eta_2, \ldots, \eta_6)$ be an arbitrary point not O in R_6. The tac function $\Theta(\mathsf{H})$ of Γ is given by

$$\Theta(\mathsf{H}) = \max \sum_{H=1}^{6} \xi_H \, \eta_H,$$

where $\Xi = (\xi_1, \xi_2, \ldots, \xi_6)$ runs over all points of $C \cap \mathbf{\Omega}$, thus over all points that satisfy the two equations

$$\sum_{H=1}^{6} \xi_H^2 = 1, \qquad \sum_{H=1}^{3} \xi_H \, \xi_{H+3} = 0.$$

This maximum problem can be solved as follows. Put

$$a = \sum_{H=1}^{6} \eta_H^2, \qquad b = 2 \sum_{H=1}^{3} \eta_H \, \eta_{H+3}.$$

The maximum is attained at a stationary point of the function

$$\Phi = \sum_{H=1}^{6} \xi_H \, \eta_H - \tfrac{1}{2}\lambda\Big(\sum_{H=1}^{6} \xi_H^2 - 1\Big) - \mu \sum_{H=1}^{3} \xi_H \, \xi_{H+3},$$

where λ and μ are the Lagrange parameters. On differentiating with respect to the variables ξ_H, we obtain the equations

$$\eta_H - \lambda\xi_H - \mu\xi_{H+3} = 0 \quad (H = 1, 2, \ldots, 6), \tag{1}$$

in which the index $H+3$ is understood (mod 6). From these equations,

$$\lambda = \sum_{H=1}^{6} \xi_H \, \eta_H, \qquad \mu = \sum_{H=1}^{6} \xi_{H+3} \, \eta_H. \tag{2}$$

At the maximum, λ is evidently positive.

For convenience, put

$$\vartheta = \frac{\mu}{\lambda}, \qquad \alpha = \frac{a}{b}.$$

From (1), we obtain the values

$$\lambda\xi_1 = \frac{\eta_1 - \vartheta\eta_4}{1-\vartheta^2}, \qquad \lambda\xi_2 = \frac{\eta_2 - \vartheta\eta_5}{1-\vartheta^2}, \qquad \lambda\xi_3 = \frac{\eta_3 - \vartheta\eta_6}{1-\vartheta^2},$$

$$\lambda\xi_4 = \frac{-\vartheta\eta_1 + \eta_4}{1-\vartheta^2}, \qquad \lambda\xi_5 = \frac{-\vartheta\eta_2 + \eta_5}{1-\vartheta^2}, \qquad \lambda\xi_6 = \frac{-\vartheta\eta_3 + \eta_6}{1-\vartheta^2}.$$

Therefore

$$(\eta_1 - \vartheta\eta_4)(-\vartheta\eta_1 + \eta_4) + (\eta_2 - \vartheta\eta_5)(-\vartheta\eta_2 + \eta_5) + (\eta_3 - \vartheta\eta_6)(-\vartheta\eta_3 + \eta_6) = 0,$$

which is equivalent to

$$b - 2a\vartheta + b\vartheta^2 = 0 \quad \text{or} \quad \vartheta^2 - 2\alpha\vartheta + 1 = 0.$$

Therefore ϑ has one of the two values

$$\vartheta = \alpha \pm \sqrt{(\alpha^2-1)}.$$

Next

$$\lambda^2(1-\vartheta^2)^2 = \sum_{H=1}^{3}(\eta_H-\vartheta\eta_{H+3})^2+\sum_{H=1}^{3}(-\vartheta\eta_H+\eta_{H+3})^2$$
$$= a-2b\vartheta+a\vartheta^2,$$

so that

$$\lambda^2 = \frac{a-2b\vartheta+a\vartheta^2}{(1-\vartheta^2)^2} = b\frac{(1+\vartheta^2)\alpha-2\vartheta}{(1+\vartheta^2)^2-4\vartheta^2} = b\frac{2\alpha^2\vartheta-2\vartheta}{4\alpha^2\vartheta^2-4\vartheta^2} = \frac{b}{2\vartheta}.$$

Since λ, as the value of $\Theta(\mathsf{H})$, is to be as large as possible, it must have the value given by

$$\vartheta = \alpha-\sqrt{(\alpha^2-1)},$$

and so

$$\lambda^2 = \frac{b}{2\{\alpha-\sqrt{(\alpha^2-1)}\}} = \tfrac{1}{2}b\{\alpha+\sqrt{(\alpha^2-1)}\} = \frac{a+\sqrt{(a^2-b^2)}}{2}.$$

In order to simplify this formula, we introduce the new parameters

$$\begin{aligned} \xi_H+\xi_{H+3} &= 2X_H, & \xi_H-\xi_{H+3} &= 2X_{H+3}; \\ \eta_H+\eta_{H+3} &= 2Y_H, & \eta_H-\eta_{H+3} &= 2Y_{H+3} \end{aligned} \qquad (H = 1, 2, 3).$$

Then

$$\lambda = \sum_{H=1}^{6}\xi_H\,\eta_H = 2\sum_{H=1}^{6}X_H\,Y_H,$$

and

$$a = 2\sum_{H=1}^{6}Y_H^2,$$

$$a+b = \sum_{H=1}^{3}(\eta_H+\eta_{H+3})^2 = 4\sum_{H=1}^{3}Y_H^2,$$

$$a-b = \sum_{H=1}^{3}(\eta_H-\eta_{H+3})^2 = 4\sum_{H=4}^{6}Y_H^2,$$

so that

$$\lambda^2 = \sum_{H=1}^{6}Y_H^2+2\left\{\sum_{H=1}^{3}Y_H^2\sum_{H=4}^{6}Y_H^2\right\}^{\frac{1}{2}}.$$

Thus, on extracting the square root,

$$\lambda = \left\{\sum_{H=1}^{3}Y_H^2\right\}^{\frac{1}{2}}+\left\{\sum_{H=4}^{6}Y_H^2\right\}^{\frac{1}{2}},$$

and so the final result for $\Theta(\mathsf{H})$ is

$$\Theta(\mathsf{H}) = (Y_1^2+Y_2^2+Y_3^2)^{\frac{1}{2}}+(Y_4^2+Y_5^2+Y_6^2)^{\frac{1}{2}}, \tag{3}$$

or, in explicit form,

$$\Theta(\mathsf{H}) = \{(\eta_1+\eta_4)^2+(\eta_2+\eta_5)^2+(\eta_3+\eta_6)^2\}^{\frac{1}{2}}+$$
$$+\{(\eta_1-\eta_4)^2+(\eta_2-\eta_5)^2+(\eta_3-\eta_6)^2\}^{\frac{1}{2}}.$$

5. The distance function $\Phi(\Xi)$ of Γ is given, in terms of its tac function $\Theta(\mathsf{H})$, by the equation

$$\Phi(\Xi) = \max_{\mathsf{H}\neq 0} \frac{|\Xi\mathsf{H}|}{\Theta(\mathsf{H})}.$$

We introduce again the parameters X_H and Y_H so that $\Theta(\mathsf{H})$ is given by the formula (3), and the product $\Xi\mathsf{H}$ takes the form

$$\Xi\mathsf{H} = \sum_{H=1}^{6} \xi_H \eta_H = 2\sum_{H=1}^{6} X_H Y_H.$$

Hence, with the abbreviations

$$u = +(Y_1^2+Y_2^2+Y_3^2)^{\frac{1}{2}} \quad \text{and} \quad v = +(Y_4^2+Y_5^2+Y_6^2)^{\frac{1}{2}},$$

the expression for the distance function $\Phi(\Xi)$ takes the form

$$\Phi(\Xi) = 2\max\Big|\sum_{H=1}^{6} X_H Y_H\Big|,$$

where the maximum is extended over all parameters Y_H satisfying

$$u+v \leqslant 1.$$

At the maximum, the Y_H's obviously have signs making all products $X_H Y_H$ non-negative. If Y_1, Y_2, Y_3 vary so as to leave u constant, the sum $X_1Y_1+X_2Y_2+X_3Y_3$ assumes its greatest value when

$$Y_1 = tX_1, \qquad Y_2 = tX_2, \qquad Y_3 = tX_3;$$

here the proportionality factor t is given by

$$t^2(X_1^2+X_2^2+X_3^2) = u^2.$$

Hence

$$\max(X_1Y_1+X_2Y_2+X_3Y_3) = |t|(X_1^2+X_2^2+X_3^2) = +u(X_1^2+X_2^2+X_3^2)^{\frac{1}{2}}.$$

A similar formula holds for the sum of the other three terms. Thus

$$\Phi(\Xi) = 2\max_{\substack{u\geqslant 0,\, v\geqslant 0\\ u+v\leqslant 1}} \{u(X_1^2+X_2^2+X_3^2)^{\frac{1}{2}}+v(X_4^2+X_5^2+X_6^2)^{\frac{1}{2}}\},$$

whence $$\Phi(\Xi) = 2\max\{(X_1^2+X_2^2+X_3^2)^{\frac{1}{2}},\ (X_4^2+X_5^2+X_4^2)^{\frac{1}{2}}\}.$$

Therefore, on returning to the original coordinates, the final result is that the distance function is equal to

$$\Phi(\Xi) = \max\Big\{\Big(\sum_{H=1}^{3}(\xi_H+\xi_{H+3})^2\Big)^{\frac{1}{2}},\ \Big(\sum_{H=1}^{3}(\xi_H-\xi_{H+3})^2\Big)^{\frac{1}{2}}\Big\}.$$

6. This result means that Γ consists of all points Ξ satisfying the two inequalities

$$\sum_{H=1}^{3}(\xi_H+\xi_{H+3})^2 \leqslant 1, \qquad \sum_{H=1}^{3}(\xi_H-\xi_{H+3})^2 \leqslant 1;$$

hence Γ is the intersection of two ellipsoidal cylinders. It is not difficult to deduce from this that Γ has the volume

$$V(\Gamma) = \frac{\pi^2}{72}.$$

Further, Γ contains the sphere of radius $\sqrt{\frac{1}{2}}$ and centre O, as the largest of this kind, but is itself contained in the unit sphere G_6.

I have not so far succeeded in obtaining similar formulae for higher spherical compounds $\Gamma_n^{(p)}$.

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