ON THE MINIMA OF COMPOUND QUADRATIC FORMS

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In my paper on *Compound Convex Bodies*,*) I obtained certain approximation theorems by using geometrical methods. In this paper, it will be my aim to derive similar theorems by means of properties of quadratic forms. These properties deserve an interest in themselves and seem to be new.

1. As in my paper on Compound Convex Bodies, let $1 \leqq p \leqq n$, let $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ be any p points in R_n, and let $\Xi = [X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$ be their *compound point* in R_N where $N = \binom{n}{p}$. Explicitly, the coordinates $\xi_1, \xi_2, \ldots, \xi_N$ of Ξ are defined as the N p-th order minors of the $p \times n$ matrix formed by the coordinates of $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$. The ordering of these minors is arbitrary, but fixed once for all. The compound point Ξ is different from O provided $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ are linearly independent.

2. Next let $a = (a_{hk})$ be any n-th order quadratic matrix. We then denote by $\alpha = a^{(p)} = (a^{(p)}_{HK})$ the *p-th compound of* a, i. e. that matrix of order N the elements of which are all the N^2 minors of order p of the original matrix a; the ordering of the two indices of these minors shall be the same as in 1. From the theory of matrices and determinants it is known that *the p-th compound of the product of two or more matrices is equal to the product of their p-th compounds*,

$$(ab\ldots)^{(p)} = a^{(p)}b^{(p)}\ldots,$$

and that *the determinant* $||\alpha||$ *of the p-th compound of a is given by*

$$||\alpha|| = ||a^{(p)}|| = ||a||^P \textit{ where } P = \binom{n-1}{p-1}.$$

In particular, the p-th compound of a *non-singular* matrix is again non-singular.

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3. Let

$$F(X) = \sum_{h=1}^{n} \sum_{k=1}^{n} a_{hk} x_h x_k \quad (a_{hk} = a_{kh})$$

be a positive-definite quadratic form in R_n, and let

$$\Phi(\Xi) = \sum_{H=1}^{N} \sum_{K=1}^{N} \alpha_{HK} \xi_H \xi_K = \sum_{H=1}^{N} \sum_{K=1}^{N} a_{HK}^{(p)} \xi_H \xi_K \quad (\alpha_{HK} = \alpha_{KH})$$

be its p-th compound form (concomitant, Begleitform) in R_N, the coefficient matrix $a^{(p)}$ being defined as in the last section. Denote further by

$$A = \|a\| \quad \text{and} \quad \mathsf{A} = A^{(p)} = \|\alpha\| = \|a^{(p)}\|$$

the discriminants of $F(X)$ and $\Phi(\Xi)$, respectively. Hence

$$\mathsf{A} = A^P ,$$

by what was said before.

The following statement follows immediately from the product rule for compound matrices. Denote by

$$X \to X' = \Omega X \quad \text{and} \quad \Xi \to \Xi' = \Omega^{(p)}\Xi$$

a non-singular affine transformation of R_n, and its p-th compound in R_N, respectively. Then the new quadratic forms

$$G(X) = F(\Omega X) \quad \text{and} \quad \Psi(\Xi) = \Phi(\Omega^{(p)}\Xi)$$

stand to one-another in the relation that $\Psi(\Xi)$ is again the p-th compound of $G(X)$.

To apply this result, let $F_0(X)$ and $\Phi_0(\Xi)$ be the unit forms

$$F_0(X) = \sum_{h=1}^{n} x_h^2 \quad \text{and} \quad \Phi_0(\Xi) = \sum_{H=1}^{N} \xi_H^2$$

in R_n and R_N, respectively, which trivially are positive-definite. So, by hypothesis, is $F(X)$; hence there exists a non-singular affine transformation Ω such that

$$F(X) = F_0(\Omega X) .$$

It is easily verified that $\Phi_0(\Xi)$ is the p-th compound of $F_0(X)$; so necessarily also

$$\Phi(\Xi) = \Phi_0(\Omega^{(p)}\Xi) .$$

This formula makes it evident that also $\Phi(\Xi)$ is positive-definite. It has thus been proved that *the compounds of positive-definite forms are likewise positive-definite.*

4. Denote again by $F(X)$ and $\Phi(\Xi)$ a positive-definite form in R_n and its p-th compound in R_N. Next let L_0 and Λ_0 be the lattices of all points with integral coordinates in R_n and R_N, respectively. We denote by

$$m_1, m_2, \ldots, m_n$$

Minkowski's successive minima of $F(X)$ in L_0, and by

$$\mu_1, \mu_2, \ldots, \mu_N$$

those of $\Phi(\Xi)$ in Λ_0. Thus n independent points $X_1, X_2, \ldots, X_n$ of L_0, and N independent points $\Xi_1, \Xi_2, \ldots, \Xi_N$ of Λ_0, exist which have the following characteristic properties. First,

$$F(X_h) = m_h \text{ for } h = 1, 2, \ldots, n\,, \quad \Phi(\Xi_H) = \mu_H \text{ if } H = 1, 2, \ldots, N\,;$$

secondly,

$$F(X) \geqq m_1 \text{ if } X \neq O \text{ is in } L_0\,, \quad \Phi(\Xi) \geqq \mu_1 \text{ if } \Xi \neq O \text{ is in } \Lambda_0\,;$$

and third, for $h = 1, 2, \ldots, n-1$ and $H = 1, 2, \ldots, N-1$,

$$F(X) \geqq m_{h+1} \text{ if } X \in L_0 \text{ is independent of } X_1, X_2, \ldots, X_h\,,$$
$$\Phi(\Xi) \geqq \mu_{H+1} \text{ if } \Xi \in \Lambda_0 \text{ is independent of } \Xi_1, \Xi_2, \ldots, \Xi_H\,.$$

It was shown by MINKOWSKI in the „Geometrie der Zahlen" that the successive minima satisfy the inequalities,

$$A \leqq m_1 m_2 \ldots m_n \leqq \delta_n A\,, \tag{1}$$

$$\mathsf{A} \leqq \mu_1 \mu_2 \ldots \mu_N \leqq \delta_N \mathsf{A}\,. \tag{2}$$

Here δ_n and δ_N are defined by

$$\delta_n = \Delta(G_n)^{-2}\,, \quad \delta_N = \Delta(G_N)^{-2}\,,$$

where $\Delta(G_n)$ and $\Delta(G_N)$ denote the determinants of the critical lattices of the unit spheres

$$G_n\colon |X| \leqq 1 \quad \text{and} \quad G_N\colon |\Xi| \leqq 1$$

in n and N dimensions, respectively. There are well-known lower and upper bounds for $\Delta(G_n)$ and $\Delta(G_N)$, due to MINKOWSKI, BLICHFELDT, HLAWKA and others; but for our purpose it suffices to know that δ_n and δ_N are positive constants depending only on n and N and not on the special quadratic forms under consideration.

5. There exist $N = \binom{n}{p}$ distinct systems of p indices $\nu_1, \nu_2, \ldots, \nu_p$ satisfying

$$1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n\,.$$

With each such system we associate the product of minima

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p}\,.$$

Denote by $M_1, M_2, \ldots, M_N$ all these products arranged in order of increasing size,

$$M_1 \leqq M_2 \leqq \ldots \leqq M_N\,.$$

Then, evidently,

$$M_1 M_2 \dots M_N = (m_1 m_2 \dots m_n)^P . \tag{3}$$

We shall prove a set of simple inequalities connecting the N minima μ_H with the N products M_H.

6. This proof is based on an algebraic identity. Denote by

$$F(X, Y) = F(Y, X) = \sum_{h=1}^{n} \sum_{k=1}^{n} a_{hk} x_h y_k$$

the bilinear form belonging to $F(X) = F(X, X)$. Let again $X^{(1)}, X^{(2)}, \dots, X^{(p)}$ be p linearly independent points in R_N, and let

$$\Xi = [X^{(1)}, X^{(2)}, \dots, X^{(p)}]$$

be their compound. Then, identically in $X^{(1)}, X^{(2)}, \dots, X^{(p)}$,

$$\begin{vmatrix} F(X^{(1)}, X^{(1)}), & F(X^{(1)}, X^{(2)}), & \dots, & F(X^{(1)}, X^{(p)}) \\ F(X^{(2)}, X^{(1)}), & F(X^{(2)}, X^{(2)}), & \dots, & F(X^{(2)}, X^{(p)}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ F(X^{(p)}, X^{(1)}), & F(X^{(p)}, X^{(2)}), & \dots, & F(X^{(p)}, X^{(p)}) \end{vmatrix} = \Phi(\Xi) . \tag{4}$$

For assume, first, that $F(X) = F_0(X)$ and hence $\Phi(\Xi) = \Phi_0(\Xi)$ are the unit forms of X and Ξ, as defined in 3. Then $F_0(X, Y) = XY$ is simply the inner product of X and Y, and the identity (4) holds in this special case because it coincides with the well-known formula for the determinant of the product of a rectangular matrix into its transposed. But then (4) is true in general, as follows from the equations

$$F(X, Y) = F_0(\Omega X, \Omega Y) , \quad \Phi(\Xi) = \Phi_0(\Omega^{(p)}\Xi) ,$$
$$[\Omega X^{(1)}, \Omega X^{(2)}, \dots, \Omega X^{(p)}] = \Omega^{(p)}\Xi ,$$

where the affine transformation Ω is defined as in 3.

7. The determinant on the left-hand side of (4) is symmetrical. We therefore construct the corresponding quadratic form

$$Q(Z) = \sum_{\varrho=1}^{p} \sum_{\sigma=1}^{p} F(X^{(\varrho)}, X^{(\sigma)})\, z_\varrho z_\sigma ;$$

here $Z = (z_1, z_2, \dots, z_p)$ may be any point in R_p. This quadratic form can be written as an inner product,

$$Q(Z) = \sum_{\varrho=1}^{p} \sum_{\sigma=1}^{p} (z_\varrho \Omega X^{(\varrho)})(z_\sigma \Omega X^{(\sigma)}) = YY ,$$

where

$$Y = z_1 \Omega X^{(1)} + z_2 \Omega X^{(2)} + \dots + z_p \Omega X^{(p)} .$$

The points $X^{(1)}, X^{(2)}, \dots. X^{(p)}$, by hypothesis, are independent, and the same is therefore true for the transformed points $\Omega X^{(1)}, \Omega X^{(2)}, \dots \Omega X^{(p)}$. Hence

$Y \neq O$ and $YY > 0$ unless $z_1 = z_2 = \ldots = z_p = 0$. This means that the quadratic form $Q(Z)$ is positive-definite. But then its discriminant, i. e. the determinant on the left-hand side of (4), cannot exceed the product

$$F(X^{(1)}, X^{(1)})\, F(X^{(2)}, X^{(2)}) \ldots F(X^{(p)}, X^{(p)}) = F(X^{(1)})\, F(X^{(2)}) \ldots F(X^{(p)})$$

of its diagonal elements, and so it follows from (4) that always

$$\Phi(\Xi) \leqq F(X^{(1)})\, F(X^{(2)}) \ldots F(X^{(p)}) \text{ if } \Xi = [X^{(1)}, X^{(2)}, \ldots, X^{(p)}]\,. \tag{5}$$

8. The n points $X_1, X_2, \ldots, X_n$ in which the successive minima of $F(X)$ in L_0 were attained are, by hypothesis, independent. Let us form the compounds

$$\mathsf{H}(\nu) = [X_{\nu_1}, X_{\nu_2}, \ldots, X_{\nu_p}] \tag{6}$$

of all N sets of p distinct ones of these points; as before, the ν's run over all sets of indices for which $1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n$. *The so defined points* $\mathsf{H}(\nu)$ *in* R_N *are then also independent*, as may be proved in the following way.

Since $X_1, X_2, \ldots, X_n$ are independent, every point in R_n is of the form

$$s_1X_1 + s_2X_2 + \ldots + s_nX_n$$

with real coefficients. Such a representation holds thus, in particular, for each one of the n unit points

$$E_1 = (1, 0, \ldots, 0),\; E_2 = (0, 1, \ldots, 0),\; \ldots,\; E_n = (0, 0, \ldots, 1)$$

in R_n. On the other hand, we obtain all N unit points in R_N by forming all the N compounds

$$[E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_p}]$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ run over the distinct set of p indices with

$$1 \leqq \lambda_1 < \lambda_2 < \ldots < \lambda_p \leqq n\,.$$

These compounds may then be written as linear combinations

$$\sum_{(\nu)} \sigma(\nu)\, \mathsf{H}(\nu)\,,$$

with real cooefficients $\sigma(\nu)$, of the compounds (6), and they form a basis of R_N. This proves the assertion.

Evidently *the points* $\mathsf{H}(\nu)$ *belong to the lattice* Λ_0, i. e. they have integral coordinates.

9. With each lattice point $\mathsf{H}(\nu)$ associate the product

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p}\,.$$

Further denote these points also by

$$\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_N\,, \tag{7}$$

where H_H is that point $\mathsf{H}(\nu)$ for which $M(\nu) = M_H$.

The basic inequality (5) implies that

$$\Phi(\mathsf{H}(\nu)) \leqq m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} = M(\nu) .$$

In the new notation, this is equivalent to

$$\Phi(\mathsf{H}_H) \leqq M_H \quad (H = 1, 2, \ldots, N) . \tag{8}$$

Here, by hypothesis,

$$M_1 \leqq M_2 \leqq \ldots \leqq M_N . \tag{9}$$

On the other hand, it will not, in general, be true that the values $\Phi(\mathsf{H}_H)$ are likewise arranged in order of increasing size. Therefore denote by

$$\mathsf{H}_1^*, \mathsf{H}_2^*, \ldots, \mathsf{H}_N^*$$

such a permutation of the points (7) for which

$$\Phi(\mathsf{H}_1^*) \leqq \Phi(\mathsf{H}_2^*) \leqq \ldots \leqq \Phi(\mathsf{H}_N^*) . \tag{10}$$

Then, by the characteristic properties of the successive minima μ_H,

$$\Phi(\mathsf{H}_H^*) \geqq \mu_H \quad (H = 1, 2, \ldots, N) .$$

It is further clear from (8), (9), and (10) that

$$\Phi(\mathsf{H}_H^*) \leqq \max \{\Phi(\mathsf{H}_1), \Phi(\mathsf{H}_2), \ldots, \Phi(\mathsf{H}_H)\} \leqq M_H .$$

Hence the last inequality implies that

$$\mu_H \leqq M_H \quad (H = 1, 2, \ldots, N) . \tag{11}$$

10. On combining this upper bound for μ_H with former inequalities, we may deduce also a lower bound.

By (2) and (11),

$$\mu_H \geqq \mathsf{A}\{\prod_{\substack{K=1 \\ K \neq H}}^{N} \mu_K\}^{-1} \geqq \mathsf{A}\{\prod_{\substack{K=1 \\ K \neq H}}^{N} M_K\}^{-1} = \mathsf{A} M_H \{\prod_{K=1}^{N} M_K\}^{-1} .$$

Here, by (1) and (3),

$$\prod_{K=1}^{N} M_K = (m_1 m_2 \ldots m_n)^P \leqq (\delta_n A)^P ,$$

while further

$$\mathsf{A} = A^P , \quad \delta_n = \Delta(G_n)^{-2} .$$

Therefore, finally,

$$\mu_H \geqq \mathsf{A} M_H (\delta_n A)^{-P} = \Delta(G_n)^{2P} M_H \quad (H = 1, 2, \ldots, N) . \tag{12}$$

In (11) and (12), the following result is contained.

Theorem 1: *Let* $1 \leqq p \leqq n-1$, $N = \binom{n}{p}$, *and* $P = \binom{n-1}{p-1}$; *let further* $\Delta(G_n)$ *denote the lattice determinant of the n-dimensional unit sphere* G_n: $|X| \leqq 1$.

Let $F(X)$ and $\Phi(\Xi)$ be a positive-definite quadratic form in R_n, and its p-th compound in R_N, respectively. Let further

$$m_1, m_2, \ldots, m_n \text{ and } \mu_1, \mu_2, \ldots, \mu_N$$

be the successive minima of $F(X)$ and $\Phi(\Xi)$ in the lattices L_0 and Λ_0 of all points with integral coordinates in R_n and R_N, respectively. Finally let $M_1, M_2, \ldots, M_N$ be the set of all products

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)$$

arranged in order of increasing size. Then

$$\Delta(G_n)^{2P} M_H \leqq \mu_H \leqq M_H \quad (H = 1, 2, \ldots, N).$$

11. Two simple deductions from the last theorem have some interest in themselves.

Theorem 2: *Let the hypothesis be as in the last theorem; assume furthermore that $F(X)$ and so also $\Phi(\Xi)$ are of unit discriminants. Then the first minima m_1 and μ_1 of these two forms are connected by the relation*

$$\Delta(G_n)^{2P} m_1^p \leqq \mu_1 \leqq \Delta(G_n)^{-2} m_1^{\frac{n-p}{n-1}}.$$

Proof: Since

$$m_1 \leqq m_2 \leqq \ldots \leqq m_n, \tag{13}$$

it is obvious that M_1, the smallest of the products M_H, has the explicit value

$$M_1 = m_1 m_2 \ldots m_p.$$

Therefore

$$M_1 \geqq m_1^p,$$

and so, by Theorem 1,

$$\mu_1 \geqq \Delta(G_n)^{2P} m_1^p.$$

Next, again by Theorem 1 and on account of $A = 1$,

$$m_1^{-1} \leqq m_2 m_3 \ldots m_n$$

and

$$M_1 = m_1 m_2 \ldots m_p \leqq \delta_n (m_{p+1} m_{p+2} \ldots m_n)^{-1}.$$

From this inequality, by (13),

$$m_{p+1} m_{p+2} \ldots m_n \geqq (m_2 m_3 \ldots m_n)^{\frac{n-p}{n-1}} \geqq m_1^{-\frac{n-p}{n-1}}.$$

whence, finally,

$$\mu_1 \leqq M_1 \leqq \delta_n m_1^{\frac{n-p}{n-1}} = \Delta(G_n)^{-2} m_1^{\frac{n-p}{n-1}}.$$

This completes the proof.

Theorem 2 is a *transfer principle* of the same kind as the well-known ones due to PERRON and KHINTCHINE,[1]) and, in fact, one easily sees that it contains these as very special cases.

12. Since $\binom{n}{n-p} = \binom{n}{p}$, the $(n-p)$-th compound of $F(X)$, the form

$$\Psi(\mathsf{H}) = \sum_{H=1}^{N} \sum_{K=1}^{N} a_{HK}^{(n-p)} \eta_H \eta_K \quad (a_{HK}^{(n-p)} = a_{KH}^{(n-p)})$$

say, depends likewise on just N variables. We may then interpret $\Phi(\Xi)$ and $\Psi(\mathsf{H})$ as forms in the same space R_N. It lies near to look for relations between the successive minima $\mu_1, \mu_2, \ldots, \mu_N$ of $\Phi(\Xi)$ and the successive minima $\mu_1^*, \mu_2^*, \ldots, \mu_N^*$ of $\Psi(\mathsf{H})$, both in the lattice Λ_0 of all points in R_N with integral coordinates. Again Theorem 1 leads to an answer to this question.

We earlier introduced already the products

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)\,.$$

We now associate with each such product a second product

$$M^*(\nu) = m_{\nu_{p+1}} m_{\nu_{p+2}} \ldots m_{\nu_n} \quad (1 \leqq \nu_{p+1} < \nu_{p+2} < \ldots < \nu_n \leqq n)\,,$$

the new indices being such that the sequence $\nu_1, \nu_2, \ldots, \nu_n$ forms a permutation of $1, 2, \ldots, n$. Thus

$$M(\nu)\, M^*(\nu) = m_1 m_2 \ldots m_n$$

is independent of $\nu_1, \nu_2, \ldots, \nu_n$.

Let, as before, $M_1, M_2, \ldots, M_N$ denote the products $M(\nu)$ arranged in order of increasing size,

$$M_1 \leqq M_2 \leqq \ldots \leqq M_N\,.$$

If, in this notation, $M(\nu) = M_H$, then write

$$M^*(\nu) = M^*_{N-H+1}\,.$$

Then also

$$M_1^* \leqq M_2^* \leqq \ldots \leqq M_N^*$$

and

$$M_H M^*_{N-H+1} = m_1 m_2 \ldots m_n\,.$$

By Theorem 1,

$$\Delta(G_n)^{2P} M_H \leqq \mu_H \leqq M_H\,, \tag{14}$$

and the same theorem gives also the analogous inequalities

$$\Delta(G_n)^{2P^*} M_H^* \leqq \mu_H^* \leqq M_H^*\,. \tag{15}$$

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[1]) See J. F. KOKSMA, Diophantische Approximationen, Ergeb. d. Math. IV, *4* (Berlin 1935), p. 66.

Here

$$P^* = \binom{n-1}{n-p-1} = \binom{n-1}{p}$$

and therefore

$$P + P^* = \binom{n-1}{p-1} + \binom{n-1}{p} = \binom{n}{p} = N .$$

Hence, on multiplying corresponding inequalities (14) and (15), we find that

$$\Delta(G_n)^{2N} M_H M^*_{N-H+1} \leqq \mu_H \mu^*_{N-H+1} \leqq M_H M^*_{N-H+1} .$$

Further, by (1),

$$A \leqq M_H M^*_{N-H+1} = m_1 m_2 \ldots m_n \leqq \Delta(G_n)^{-2} A .$$

Therefore, finally,

$$\Delta(G_n)^{2N} A \leqq \mu_H \mu^*_{N-H+1} \leqq \Delta(G_n)^{-2} A .$$

There is no loss of generality in assuming again that $A = 1$. The result obtained may then be expressed as follows.

Theorem 3: *Let* $1 \leqq p \leqq n-1$ *and* $N = \binom{n}{p}$; *let further* $\Delta(G_n)$ *be defined as before. Let* $\Phi(\Xi)$ *and* $\Psi(\mathsf{H})$ *be the p-th compound and the* $(n-p)$*-th compound, respectively, of the same positive-definite quadratic form* $F(X)$ *of unit discriminant in* R_n. *Let* $\mu_1, \mu_2, \ldots, \mu_N$ *and* $\mu_1^*, \mu_2^*, \ldots, \mu_N^*$ *be the successive minima of* $\Phi(\Xi)$ *and* $\Psi(\mathsf{H})$, *respectively, in the lattice* Λ_0 *of all points with integral coordinates in* R_N. *Then*

$$\Delta(G_n)^{2N} \leqq \mu_H \mu^*_{N-H+1} \leqq \Delta(G_n)^{-2} \quad (H = 1, 2, \ldots, N) .$$

The most interesting case of this theorem is that when $p = 1$. This case is closely related to the theorem on *polar convex bodies* by M. Riesz and myself; compare my paper on Compound Convex Bodies.

13. To conclude this paper, let us deduce from Theorem 1 the main result of the paper just mentioned.

Let K be any closed, bounded, symmetric, convex body in R_n, and let $\mathsf{K} = [K]^{(p)}$ be its p-th compound in R_N. By the theorem of Fritz John,*) there exists in R_n an ellipsoid E with centre at the origin such that

$$n^{-\frac{1}{2}} E \subset K \subset E . \tag{16}$$

Hence, if $\mathsf{E} = [E]^{(p)}$ denotes the p-th compound of E, then also

$$n^{-\frac{1}{2}p} \mathsf{E} \subset \mathsf{K} \subset \mathsf{E} . \tag{17}$$

It is rather difficult finding the *explicit* form of E. We introduce therefore an ellipsoid E^* which is easier to handle.

*) See R. Courant, Anniversary Volume, New York 1948, 187—204.

Let $F(X)$ be the positive definite quadratic form the square root of which is the distance function of E; this ellipsoid is thus defined by $F(X) \leqq 1$. Let $\Phi(\Xi)$ be the p-th compound form of $F(X)$, and let E^* be the ellipsoid in R_N which has $\Phi(\Xi)^{\frac{1}{2}}$ as its distance function, hence is defined by $\Phi(\Xi) \leqq 1$. As we shall prove,

$$N^{-\frac{1}{2}}\mathsf{E}^* \subset \mathsf{E} \subset \mathsf{E}^* , \tag{18}$$

whence, by (17),

$$(n^p N)^{-\frac{1}{2}}\mathsf{E}^* \subset \mathsf{K} \subset \mathsf{E}^* . \tag{19}$$

14. The relation (18) is obtained as follows. There is a non-singular affine transformation $X \to X' = \Omega X$ in R_n, with its p-th compound $\Xi \to \Xi' = \Omega^{(p)}\Xi$ in R_N, such that

$$G_n = \Omega E \quad \text{and} \quad \Gamma_n^{(p)} = \Omega^{(p)}\mathsf{E} . \tag{20}$$

Here G_n: $|X| \leqq 1$ denotes the unit sphere in R_n, and $\Gamma_n^{(p)}$ its p-th compound in R_N. The first relation (20) implies the identities

$$F(X) = F_0(\Omega X) , \quad \Phi(\Xi) = \Phi_0(\Omega^{(p)}\Xi)$$

of section 3; here $F_0(X)$ and $\Phi_0(\Xi)$ are again the quadratic unit forms. Since $\Phi_0(\Xi)^{\frac{1}{2}}$ is the distance function of the unit sphere G_N in R_N, and $\Phi(\Xi)^{\frac{1}{2}}$ is that of the ellipsoid E^*, we obtain the further relation

$$G_N = \Omega^{(p)}\mathsf{E}^* . \tag{21}$$

Since the property of being a subset is not destroyed by any affine transformation, it is obvious from (20) and (21) that the relation (18) holds if, and only if, it is true that

$$N^{-\frac{1}{2}}G_N \subset \Gamma_n^{(p)} \subset G_N . \tag{22}$$

As I have shown in my note *On the p-th Compound of a Sphere,**) $\Gamma_n^{(p)}$ is the convex hull of the set

$$\Sigma_n^{(p)} = G_N \cap \Omega(n, p) .$$

Here $\Omega(n, p)$, the Grassmann manifold in R_N, consists of all points

$$\Xi = [X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$$

where $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ run independently over R_n. In particular,

$$\Sigma_n^{(p)} \subset G_N , \quad \Gamma_n^{(p)} \subset G_N ,$$

whence the right-hand half of (22).

Next, by taking for $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ all combinations of p distinct unit points in R_n, it follows at once that $\Omega(n, p)$ and therefore also $\Sigma_n^{(p)}$ and $\Gamma_n^{(p)}$ contain the $2N$ positive and negative unit points

$$(\mp 1, 0, \ldots, 0), \ (0, \mp 1, \ldots, 0), \ \ldots, \ (0, 0, \ldots, \mp 1)$$

*) To appear in the Proceedings of the London Math. Soc.

in R_N. Hence $\Gamma_n^{(p)}$, by its definition as a convex hull, encloses the generalised octahedron

$$|\xi_1| + |\xi_2| + \ldots + |\xi_N| \leqq 1 .$$

Next, all the hyperplanes

$$\mp \xi_1 \mp \xi_2 \mp \ldots \mp \xi_N = 1$$

bounding this octahedron have a distance $N^{-\frac{1}{2}}$ from the origin. The sphere $N^{-\frac{1}{2}} G_N$ is then a subset of the octahedron and so also of $\Gamma_n^{(p)}$. This proves the left-hand half of the relation (22).

15. Let now $m_1, m_2, \ldots, m_n$ be the successive minima of K in L_0, and $\mu_1, \mu_2, \ldots, \mu_N$ those of K in Λ_0, L_0 and Λ_0 being as before the lattices of all points with integral coordinates in R_n and R_N, respectively. Let similarly $m_1^*, m_2^*, \ldots, m_n^*$ and $\mu_1^*, \mu_2^*, \ldots, \mu_N^*$ be the successive minima of E in L_0, and of E^* in Λ_0, respectively. The relations (16) and (19) lead immediately to the inequalities,

$$\begin{aligned} m_h^* &\leqq m_h \leqq n^{\frac{1}{2}} m_h^* \quad (h = 1, 2, \ldots, n) , \\ \mu_H^* &\leqq \mu_H \leqq (n^p N)^{\frac{1}{2}} \mu_H^* \quad (H = 1, 2, \ldots, N) . \end{aligned} \tag{23}$$

Next denote by $m_1', m_2', \ldots, m_n'$ and $\mu_1', \mu_2', \ldots, \mu_N'$ the successive minima of the quadratic forms $F(X)$ and $\Phi(\Xi)$ in L_0 and Λ_0, respectively. Since $F(X)^{\frac{1}{2}}$ is the distance function of E, and $\Phi(\Xi)^{\frac{1}{2}}$ is that of E^*, the equations

$$\begin{aligned} m_h' &= m_h^{*2} \quad (h = 1, 2, \ldots, n) , \\ \mu_H' &= \mu_H^{*2} \quad (H = 1, 2, \ldots, N) \end{aligned} \tag{24}$$

hold.

We finally apply Theorem 1. Denote by $M_1', M_2', \ldots, M_N'$ the products

$$M'(\nu) = m_{\nu_1}' m_{\nu_2}' \ldots m_{\nu_p}' \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)$$

arranged in order of increasing size, and define corresponding products $M_1^*, M_2^*, \ldots, M_N^*$ and $M_1, M_2, \ldots, M_N$ for the two sets of minima $m_1^*, m_2^*, \ldots, m_n^*$ and $m_1, m_2, \ldots, m_n$. It is then trivial, by (24), that

$$M_H' = M_H^{*2} \quad (H = 1, 2, \ldots, N) . \tag{25}$$

Next associate with each product

$$M^*(\nu) = m_{\nu_1}^* m_{\nu_2}^* \ldots m_{\nu_p}^*$$

the analogous product

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} .$$

The first set of formulae (23) implies that

$$M^*(\nu) \leqq M(\nu) \leqq n^{\frac{1}{2}p} M^*(\nu) ,$$

and it is therefore evident that also

$$M_H^* \leqq M_H \leqq n^{\frac{1}{2}p} M_H^* \quad (H = 1, 2, \ldots, N) , \tag{26}$$

because both sets of products M_H^* and M_H are numbered in order of increasing size.

16. We finally apply Theorem 1 which, at once, gives

$$\Delta(G_n)^{2P} M'_H \leqq \mu'_H \leqq M'_H .$$

By (24), these inequalities are equivalent to

$$\Delta(G_n)^P M_H^* \leqq \mu_H^* \leqq M_H^* .$$

On using now the second set of formulae (23) together with the formulae (26) we find that

$$n^{-\frac{1}{2}p} \Delta(G_n)^P M_H \leqq \Delta(G_n)^P M_H^* \leqq \mu_H^* \leqq \mu_H \leqq (n^p N)^{\frac{1}{2}} \mu_H^* \leqq$$
$$\leqq (n^p N)^{\frac{1}{2}} M_H^* \leqq (n^p N)^{\frac{1}{2}} M_H .$$

The following result has thus been proved.

Theorem 4: *Let* $1 \leqq p \leqq n-1$, $N = \binom{n}{p}$, *and* $P = \binom{n-1}{p-1}$; *let further* $\Delta(G_n)$ *be as before. Let* K *be any closed, bounded, symmetric, convex body in* R_n, *and let* $\mathsf{K} = [K]^{(p)}$ *be its p-th compound in* R_N. *Let further*

$$m_1, m_2, \ldots, m_n \quad \textit{and} \quad \mu_1, \mu_2, \ldots, \mu_N$$

be the successive minima of K *and* K *in the lattices* L_0 *and* Λ_0, *respectively. Let finally* $M_1, M_2, \ldots, M_N$ *be the* N *products*

$$M(\nu) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} \quad (1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)$$

numbered in order of increasing size. Then

$$n^{-\frac{1}{2}p} \Delta(G_n)^P M_H \leqq \mu_H \leqq (n^p N)^{\frac{1}{2}} M_H \quad (H = 1, 2, \ldots, N) .$$

This is essentially the Theorem 3 of my paper On Compound Convex Bodies, which stated that

$$c_7 M_H \leqq \mu_H \leqq M_H ,$$

with a positive constant c_7 depending only on n and p. By combining the two theorems, we obtain immediately the slightly improved result that

$$n^{-\frac{1}{2}p} \Delta(G_n)^P M_H \leqq \mu_H \leqq M_H .$$

The old theorem was in so far more general that, instead of the minima of K and K in L_0 and Λ_0, the minima of these two bodies in any pair of lattices L and $\Lambda = [L]^{(p)}$ were considered; here $[L]^{(p)}$ denotes the p-th compound lattice of L. There is no difficulty in extending also Theorem 4 to this more general case, and no new ideas are involved.

О МИНИМУМАХ КВАДРАТИЧЕСКИХ СОПРОВОЖДАЮЩИХ ФОРМ

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В своей работе „*Compound Convex Bodies*" я получил геометрическим путем несколько аппроксимационных теорем. Здесь я на основании свойств квадратичных форм доказываю несколько родственных теорем, имеющих самостоятельный интерес.

I. Пусть $1 \leqq p \leqq n - 1$, $N = \binom{n}{p}$. Если $X^{(1)}, \ldots, X^{(p)}$ — точки n-мерного пространства R_n, то пусть $\Xi = [X^{(1)}, \ldots, X^{(p)}]$ означает их сопровождающую точку в R_n, т. е. ту точку, N координатами которой являются определители p-го порядка $n \times p$-матрицы, составленной из координат точек $X^{(1)}, \ldots, X^{(p)}$. Эти определители можно брать в произвольном, но раз навсегда установленном порядке.

II. Пусть

$$F(X) = \sum_{h, k = 1}^{n} a_{hk} x_h x_k , \quad (a_{hk} = a_{kh})$$

— действительная квадратичная форма (в последующем изложении всегда положительно определенная). Выражением „p-ая сопровождающая форма формы F" обозначим форму

$$\Phi(\Xi) = \sum_{H, K = 1}^{N} a_{HK}^{(p)} \xi_H \xi_K ,$$

где коэффициенты $a_{HK}^{(p)}$ равны N^2 определителям p-го порядка матрицы, составленной из коэффициентов a_{hk}. Порядок индексов такой же, как и в I. Если F положительно определенна, то и Φ будет положительно определенной формой.

Пусть теперь L_0 (соотв. Λ_0) — множество всех точек из R_n (соотв. из R_N) с целочисленными координатами. Пусть $m_1 \leqq \ldots \leqq m_n$ — последовательные минимумы Минковского формы $F(X)$ для X из L_0. Аналогично, пусть $\mu_1 \leqq \ldots \leqq \mu_N$ — минимумы $\Phi(\Xi)$ для Ξ из Λ_0. Известно, что

$$A \leqq m_1 \ldots m_n \leqq \Delta(G_n)^{-2} A , \tag{1}$$

$$\mathsf{A} \leqq \mu_1 \ldots \mu_N \leqq \Delta(G_N)^{-2} \mathsf{A} , \quad \mathsf{A} = A^P , \quad P = \binom{n-1}{p-1}, \tag{2}$$

где A, A обозначают дискриминанты форм F и Φ; $\Delta(G_n)$ есть определитель критических решеток n-мерной единичной сферы. Пусть теперь $M_1 \leqq$

$\leqq \ldots \leqq M_N$ представляют расположенные в возрастающем порядке произведения $m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p}$ $(1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)$. Тогда справедлива

Теорема 1.

$$\Delta(G_n)^{2P} M_H \leqq \mu_H \leqq M_H \quad (H = 1, 2, \ldots, N).$$

Доказательство проводится при помощи (1), (2) и неравенства

$$\Phi(\Xi) \leqq F(X^{(1)}) \ldots F(X^{(p)}), \tag{5}$$

где Ξ обозначает сопровождающую точку $[X^{(1)}, \ldots, X^{(p)}]$.

Из теоремы 1 вытекают два следствия (в теоремах 2, 3 мы предполагаем для простоты $A = 1$). Во-первых, общая теорема переноса:

Теорема 2.

$$\Delta(G_n)^{2P} m_1^p \leqq \mu_1 \leqq \Delta(G_n)^{-2} m_1^{\frac{n-p}{n-1}}.$$

Во-вторых, из теоремы 1 следует теорема, которая в простейшем случае $p = 1$ находится в тесной связи с теоремами о полярных выпуклых телах М. Риса и автора. А именно, пусть Φ будет p-ая, Ψ будет $(n-p)$-ая сопровождающая форма формы F; тогда Φ и Ψ имеют одну и ту же размерность $N = \binom{n}{p} = \binom{n}{n-p}$. Пусть $\mu_1^* \leqq \ldots \leqq \mu_N^*$ — последовательные минимумы Ψ. Тогда имеет место

Теорема 3.

$$\Delta(G_n)^{2N} \leqq \mu_H \mu_{N-H+1}^* \leqq \Delta(G_n)^{-2} \quad (H = 1, 2, \ldots, N).$$

Наконец, в последних абзацах (13—16) работы доказывается теорема, не выказывающая существенных отличий от главной теоремы работы Compound Convex Bodies:

Теорема 4. *Пусть K — ограниченное, замкнутое, симметрическое выпуклое тело в R_N, пусть K — его p-ое сопровождающее тело в R_N. Пусть, далее, $m_1 \leqq \ldots \leqq m_n$ — последовательные минимумы тела K в L_0, пусть $\mu_1 \leqq \ldots \leqq \mu_N$ — минимумы K в Λ_0. Пусть, наконец, $M_1 \leqq \ldots \leqq M_N$ — расположенные в возрастающем порядке произведения $m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p}$ $(1 \leqq \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n)$. Тогда будет*

$$n^{-\frac{1}{2}p}\Delta(G_n)^P M_H \leqq \mu_H \leqq (n^p N)^{\frac{1}{2}} M_H \quad (H = 1, 2, \ldots, N).$$

Доказательство проводится, в общих чертах, следующим образом: Согласно Ф. Джону, существует эллипсоид E, для которого $n^{-\frac{1}{2}}E \subset \subset K \subset E$. Пусть $F(X) \leqq 1$ есть неравенство, определяющее эллипсоид E; пусть Φ есть p-ая сопровождающая форма формы F и пусть E^* — эллипсоид $\Phi(\Xi) \leqq 1$ в R_N. Мы доказываем, что $(n^p N)^{-\frac{1}{2}} \mathsf{E}^* \subset \mathsf{K} \subset \mathsf{E}^*$ и используем потом теорему 1.