ON SOME INEQUALITIES FOR POLYNOMIALS IN SEVERAL VARIABLES

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In the theory of transcendental numbers, frequent use is made of a certain inequality which establishes a lower bound for the height of a product of polynomials in terms of the heights of the factors. A particularly general and accurate form of this inequality was proved by A. O. Gelfond [**1**; 168–173]. In the present note I give a new proof for Gelfond's formula and also show a similar, but simpler, inequality for the length of a product of polynomials.

1. Let

$$f(z_1, \ldots, z_n) = \sum_{h_1=0}^{m_1} \ldots \sum_{h_n=0}^{m_n} a_{h_1 \ldots h_n} z_1^{h_1} \ldots z_n^{h_n}$$

be any polynomial in n variables $x_1, \ldots, x_n$ with arbitrary real or complex coefficients. For shortness put

$$H(f) = \max_{\substack{h_1=0,1,\ldots,m_1 \\ \vdots \\ h_n=0,1,\ldots,m_n}} |a_{h_1 \ldots h_n}|, \quad L(f) = \sum_{h_1=0}^{m_1} \ldots \sum_{h_n=0}^{m_n} |a_{h_1 \ldots h_n}|,$$

and

$$M(f) = \begin{cases} \exp \int_0^1 dt_1 \ldots \int_0^1 dt_n \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| & \text{if } f(z_1, \ldots, z_n) \not\equiv 0, \\ 0 & \text{if } f(z_1, \ldots, z_n) \equiv 0. \end{cases}$$

These expressions $H(f)$, $L(f)$ and $M(f)$ will be called the *height*, the *length* and the *measure* of f; unless f is identically zero, they have positive values. We note that, for $N = 1, 2, \ldots, n$, the integration over t_N in the definition of $M(f)$ may always be omitted when f does not actually depend on the corresponding variable z_N. Furthermore,

$$M(f) = |f|.$$

if f is a constant.

Let $k_1, \ldots, k_N$, for $N = 1, 2, \ldots, n$, run independently over the integers

$$k_1 = 0, 1, \ldots, m_1; \ \ldots; \ k_N = 0, 1, \ldots, m_N,$$

and put

$$f_{k_1 \ldots k_N}(z_{N+1}, \ldots, z_n) = \sum_{h_{N+1}=0}^{m_{N+1}} \ldots \sum_{h_n=0}^{m_n} a_{k_1 \ldots k_N h_{N+1} \ldots h_n} z_{N+1}^{h_{N+1}} \ldots z_n^{h_n} \text{ if } N < n,$$

$$f_{k_1 k_2 \ldots k_n} = a_{k_1 k_2 \ldots k_n} \quad \text{if } N = n.$$

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It is then obvious that

$$f(z_1, \ldots, z_n) = \sum_{k_1=0}^{m_1} f_{k_1}(z_2, \ldots, z_n) z_1^{k_1} \text{ for } N = 1, \tag{1}$$

and

$$f_{k_1 \ldots k_{N-1}}(z_N, \ldots, z_n) = \sum_{k_N=0}^{m_N} f_{k_1 \ldots k_N}(z_{N+1}, \ldots, z_n) z_N^{k_N} \text{ for } N > 1. \tag{2}$$

2. In a recent note **[3]** I proved that if

$$F(z) = \sum_{k=0}^{m} A_k z^k$$

is an arbitrary polynomial in a single variable, then

$$|A_k| \leqslant \binom{m}{k} M(F) \quad (k = 0, 1, \ldots, m).$$

On combining this result with the identities (1) and (2), we deduce immediately that

$$M(f_{k_1}) \leqslant \binom{m_1}{k_1} M(f) \quad (N = 1),$$

$$M(f_{k_1 \ldots k_N}) \leqslant \binom{m_N}{k_N} M(f_{k_1 \ldots k_{N-1}}) \quad (N = 2, 3, \ldots, n-1),$$

$$|a_{k_1 \ldots k_n}| \leqslant \binom{m_n}{k_n} M(f_{k_1 \ldots k_{n-1}}) \quad (N = n).$$

These formulae evidently imply the basic inequality

$$|a_{k_1 \ldots k_n}| \leqslant \binom{m_1}{k_1}\binom{m_2}{k_2} \cdots \binom{m_n}{k_n} M(f) \text{ for all suffixes } k_1, \ldots, k_n. \tag{3}$$

We thus obtain an upper bound for the absolute values of the coefficients of a polynomial in terms of its measure.

3. It is now easy to establish both upper and lower bounds for the height and the length of a polynomial in terms of its measure. We begin with the formulae for the length, which are rather simpler.

On summing in (3) over all suffixes $k_1, \ldots, k_n$, it follows that

$$L(f) \leqslant 2^{m_1+m_2+\ldots+m_n} M(f). \tag{4}$$

This inequality is best possible, with equality *e.g.* when

$$f_1(z, \ldots, z_n) = (1+z_1)^{m_1} \ldots (1+z_n)^{m_n}.$$

Also, trivially,

$$|f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| \leqslant L(f)$$

for all real $t_1, \ldots, t_n$. Hence, in the other direction,

$$M(f) \leqslant L(f). \tag{5}$$

Also this inequality is best possible since equality holds when f is a monomial.

4. Slightly less good estimates connect the measure of f with its height. It is easy to prove by induction for m that

$$\binom{m}{k} \leqslant 2^{m-1} \text{ if } m \geqslant 1.$$

Hence the basic inequality (3) implies that

$$H(f) \leqslant 2^{m_1+m_2+\ldots+m_n-\nu(f)} M(f) \tag{6}$$

where the symbol $\nu(f)$ is to denote the number of variables $z_1, \ldots, z_n$ that occur in f at least to the first degree. Equality can never hold in (6) if any one of the degrees $m_1, \ldots, m_n$ exceeds 1.

For an estimate in the opposite direction, we apply the well-known inequality (Hardy-Littlewood-Pólya [**2**; 137–138])

$$M(f) \leqslant \left\{\int_0^1 dt_1 \ldots \int_0^1 dt_n \, |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})|^2\right\}^{\frac{1}{2}}.$$

Here, by the explicit expression for f and by Parseval's equation,

$$\int_0^1 dt_1 \ldots \int_0^1 dt_n \, |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})|^2 = \sum_{k_1=0}^{m_1} \ldots \sum_{k_n=0}^{m_n} |a_{k_1 \ldots k_n}|^2$$
$$\leqslant (m_1+1) \ldots (m_n+1) H(f)^2,$$

so that

$$M(f) \leqslant \{(m_1+1) \ldots (m_n+1)\}^{\frac{1}{2}} H(f). \tag{7}$$

Here equality can hold only for constant polynomials.

5. From now on let f be written as a product

$$f(z_1, \ldots, z_n) = \prod_{l=1}^{s} f_l(z_1, \ldots, z_n)$$

of other polynomials in $z_1, \ldots, z_n$. Denote by $m_{l1}, \ldots, m_{ln}$ the degrees of f_l in $z_1, \ldots, z_n$, respectively, and by $\nu(f_l)$ the number of variables $z_1, \ldots, z_n$, that occur in f_l at least to the first degree. It is then obvious that

$$m_1 = \sum_{l=1}^{s} m_{l1}, \ldots, m_n = \sum_{l=1}^{s} m_{ln}, \text{ and } \nu(f) \leqslant \sum_{l=1}^{s} \nu(f_l).$$

Also, from the definition of the measure in terms of logarithms,

$$M(f) = \prod_{l=1}^{s} M(f_l).$$

Therefore, from (4),

$$\prod_{l=1}^{s} L(f_l) \leqslant \prod_{l=1}^{s} \{2^{m_{l1}+\dots+m_{ln}} M(f_l)\} = 2^{m_1+\dots+m_n} M(f),$$

whence, by (5),

$$\prod_{l=1}^{s} L(f_l) \leqslant 2^{m_1+m_2+\dots+m_n} L(f). \tag{I}$$

The inequality in the opposite direction

$$L(f) \leqslant \prod_{l=1}^{s} L(f_l),$$

is nearly trivial.

In the same way, from (6),

$$\prod_{l=1}^{s} H(f_l) \leqslant \prod_{l=1}^{s} \{2^{m_{l1}+\dots+m_{ln}-\nu(f_l)} M(f_l)\} \leqslant 2^{m_1+\dots+m_n-\nu(f)} M(f),$$

whence, by (7),

$$\prod_{l=1}^{s} H(f_1) \leqslant 2^{m_1+m_2+\dots+m_n-\nu(f)} \{(m_1+1)\dots(m_n+1)\}^{\frac{1}{2}} H(f). \tag{II}$$

In the opposite direction it is nearly obvious that

$$H(f) \leqslant 2^{m_1+m_2+\dots+m_n} \prod_{l=1}^{s} H(f_l).$$

While (I) seems to be new, (II) is essentially Gelfond's formula. He has shown that on the right-hand side the basis 2 cannot be replaced by a smaller number. Except in trivial cases, neither of the inequalities (I) and (II) is best possible. It would therefore have great interest to find the exact maxima of

$$L(f)^{-1} \prod_{l=1}^{s} L(f_l) \text{ and } H(f)^{-1} \prod_{l=1}^{s} H(f_l)$$

as functions of the degrees $m_1, \dots, m_n$.

References.

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3. K. Mahler, "An application of Jensen's formula to polynomials", *Mathematika* 7 (1960), 98–100.

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