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On the order function of a transcendental number

by

K. MAHLER (Columbus, Ohio)

To the memory of Harold Davenport

Some forty years ago, I introduced the classification of all (real or complex) transcendental numbers into three disjoint classes S, T, and U (see the detailed treatment of this classification and of an equivalent one by J. F. Koksma in Th. Schneider [5], Kapitel III). This classification possessed the *Invariance Property*; i.e., two numbers which are algebraically dependent over the rational field $\boldsymbol{Q}$ always belong to the same class.

In the present paper, a new classification will be introduced. I associate with each transcendental number ξ a positive valued non-decreasing function $O(u\,|\,\xi)$ of an integral variable $u \geqslant 1$, called the *order function* of ξ. For such order functions, both a partial ordering and an equivalence relation will be defined, and it will be proved that if any two transcendental numbers ξ and η are algebraically dependent over $\boldsymbol{Q}$, then $O(u\,|\,\xi)$ and $O(u\,|\,\eta)$ are equivalent. We may now put any transcendental numbers into one and the same class whenever their order functions are equivalent. In this way we evidently obtain a classification of the transcendental numbers into infinitely many disjoint classes.

The order function $O(u\,|\,\xi)$ is defined in terms of the approximation properties of ξ. Unfortunately, the actual determination of $O(u\,|\,\xi)$ for a given ξ is a difficult problem, and more work on such order functions is called for.

1. The following notation will be used. We denote by V the set of all polynomials

$$p(x) = p_0 + p_1 x + \ldots + p_m x^m \quad \text{where} \quad p_m \neq 0,$$

by W the set of such polynomials with integral coefficients. The exact degree of a polynomial in V is denoted by

$$\partial_x(p) = \partial(p) = m,$$

and we further put

$$L_x(p) = L(p) = |p_0| + |p_1| + \ldots + |p_m|, \qquad A_x(p) = A(p) = 2^{\partial(p)} L(p).$$

When the variable is y, we write instead ∂_y, L_y, and Λ_y. The function $L(p)$ has the two properties

$$L(p+q) \leqslant L(p)+L(q) \quad \text{and} \quad L(pq) \leqslant L(p)L(q),$$

and analogous inequalities hold for $\Lambda(p)$. In addition, $\Lambda(p)$ has the basic property that there are for every integer $u \geqslant 1$ only *finitely many* polynomials $p(x)$ in W for which

$$\Lambda(p) \leqslant u.$$

The set of these polynomials is denoted by $W(u)$. It contains the constant polynomial 1, and when $u < u'$, then $W(u)$ is a subset of $W(u')$.

For any algebraic number ξ, denote by $P(x \mid \xi)$ the primitive irreducible polynomial with integral coefficients and positive highest coefficient for which

$$P(\xi \mid \xi) = 0.$$

We then put

$$\partial^0(\xi) = \partial(P), \qquad L^0(\xi) = L(P), \qquad \Lambda^0(\xi) = \Lambda(P).$$

In particular, $\partial^0(\xi)$ is the degree of ξ.

Next let $a(u)$ and $b(u)$ be any two positive valued non-increasing functions of u. If there exist two positive integers c and u_0 and a positive number γ such that

$$a(u^c) \geqslant \gamma b(u) \quad \text{for} \quad u \geqslant u_0,$$

then we write

$$a(u) >> b(u) \quad \text{or} \quad b(u) << a(u).$$

This relation $>>$ evidently defines a partial ordering. If, simultaneously,

$$a(u) >> b(u) \quad \text{and} \quad a(u) << b(u),$$

then we write

$$a(u) >< b(u).$$

It is clear that this sign $><$ defines an equivalence relation. With respect to this relation, the functions $a(u)$ can be distributed into disjoint classes, and then the sign $>>$ defines a partial ordering of these classes.

2. Let ξ be any real or complex number; put

$$\sigma(\xi) = \begin{cases} 1 & \text{if } \xi \text{ is real,} \\ 2 & \text{if } \xi \text{ is not real.} \end{cases}$$

For every positive integer u denote by $\Omega(u)$ the set of all polynomials $p(x)$ in $W(u)$ for which

$$p(\xi) \neq 0.$$

Thus, for all u, $\Omega(u)$ is a finite set which contains the polynomial 1, and $\Omega(u)$ is a subset of $\Omega(u')$ when $u < u'$. Therefore the minimum

$$o(u \mid \xi) = \inf_{p(x) \in \Omega(u)} |p(\xi)|$$

exists for all u, satisfies the inequality

$$0 < o(u \mid \xi) \leqslant 1,$$

and is a non-increasing function of u. In the special case when ξ is a rational integer, or an integer in an imaginary quadratic field, always

$$o(u \mid \xi) = 1.$$

On the other hand, as is easily proved, for all other ξ

$$0 < o(u \mid \xi) < 1$$

as soon as u is sufficiently large.

We also introduce the derived function

$$O(u \mid \xi) = \log\{1/o(u \mid \xi)\} = \sup_{p(x) \in \Omega(u)} \log |1/p(\xi)| \tag{1}$$

which we call the *order function* of ξ. This function is non-negative and non-decreasing for all u; it vanishes identically if ξ is a rational integer or an integer in an imaginary quadratic field, and otherwise is positive as soon as u is sufficiently large.

We shall use the notations

$$\xi >> \eta \quad \text{if} \quad O(u \mid \xi) >> O(u \mid \eta),$$
$$\xi >< \eta \quad \text{if} \quad O(u \mid \xi) >< O(u \mid \eta).$$

Evidently $\xi >> \eta$ defines a partial ordering, and $\xi >< \eta$ an equivalence relation, on the set of all real and complex numbers.

3. A result due to R. Güting [3] allows to formulate an upper estimate for the order function when ξ is algebraic.

Let ξ be an algebraic number, and let $p(x)$ be a polynomial in W. Then either

$$p(\xi) = 0,$$

or

$$|p(\xi)| \geqslant \frac{\max(1, |\xi|)^{\partial(p)}}{L^0(\xi)^{\partial(p)/\sigma(\xi)} L(p)^{\{\partial^0(\xi)/\sigma(\xi)\}-1}}.$$

Assume here, in particular, that $p(x)$ lies in $\Omega(u)$. Then the first case is excluded, and $\Lambda(p) = 2^{\partial(p)} L(p)$ does not exceed u. Hence there exist two positive numbers c_1 and c_2 independent of u and $p(x)$ such that

$$|p(\xi)| \geqslant c_1 u^{-c_2} \quad \text{if} \quad p(x) \in \Omega(u).$$

We can express this result in the following form.

THEOREM 1. *If ξ is an algebraic number, then*

$$O(u\,|\,\xi) << \log u.$$

4. Consider next the case when m is a given positive integer, and ξ either is transcendental, or it is algebraic but of a degree greater than m. We shall construct polynomials $p(x)$ in W, with degrees not greater than m, for which $|p(\xi)|$ is small and $\Lambda(p)$ does not exceed a given value u.

The easiest method of finding such polynomials uses an inequality from the theory of positive definite quadratic forms

$$F(x_1, \ldots, x_n) = \sum_{h=1}^{n}\sum_{k=1}^{n} F_{hk} x_h x_k \qquad (F_{hk} = F_{kh}).$$

Denote by

$$D_F = \begin{vmatrix} F_{11} & \ldots & F_{1n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ F_{n1} & \ldots & F_{nn} \end{vmatrix} > 0$$

the discriminant of F. On writing the form as the sum of the squares of n linear forms and applying Minkowski's theorem on linear forms, it can easily be proved that there exist to F integers $x_1^0, \ldots, x_n^0$ not all zero such that

(2) $$F(x_1^0, \ldots, x_n^0) \leqslant n D_F^{1/n}.$$

Depending on whether ξ is real or not, two different cases of this estimate will be applied.

5. Firstly, let ξ be real. Put $n = m+1$, and denote by s and t two parameters such that

$$s \geqslant \max(1, |\xi|)^{-m/(m+1)}, \qquad t = (m+1)(m+2)^{1/\{2(m+1)\}}\max(1, |\xi|)^{m/(m+1)} s$$

and hence

$$t \geqslant (m+1)(m+2)^{1/\{2(m+1)\}}.$$

Take for F the positive definite quadratic form

$$F(x_0, x_1, \ldots, x_m) = s^{2(m+1)}(x_0 + x_1\xi + \ldots + x_m\xi^m)^2 + x_0^2 + x_1^2 + \ldots + x_m^2$$

which is easily seen to have the discriminant

$$D_F = 1 + s^{2(m+1)}(1 + \xi^2 + \ldots + \xi^{2m})$$
$$\leqslant 1 + s^{2(m+1)}(m+1)\max(1, |\xi|)^{2m} \leqslant s^{2(m+1)}(m+2)\max(1, |\xi|)^{2m}.$$

By the property (2), there exists then a polynomial

$$p(x) = p_0 + p_1 x + \ldots + p_m x^m$$

with integral coefficients not all zero such that

$$s^{2(m+1)}p(\xi)^2+p_0^2+p_1^2+\ldots+p_m^2 \leqslant (m+1)s^2(m+2)^{1/(m+1)}\max(1,|\xi|)^{2m/(m+1)} = t^2/(m+1).$$

Since $p(x)\not\equiv 0$, and since ξ is not algebraic at most of degree m, this implies that

$$0<|p(\xi)|<(m+1)^{1/2}(m+2)^{1/\{2(m+1)\}}\max(1,|\xi|)^{m/(m+1)}s^{-m} \leqslant (m+1)^{(2m+1)/2}(m+2)^{1/2}\max(1,|\xi|)^m t^{-m}$$

and therefore

(3) $$0<|p(\xi)|<(m+2)^{m+1}\max(1,|\xi|)^m t^{-m}.$$

It further follows that also

$$0<p_0^2+p_1^2+\ldots+p_m^2<t^2/(m+1),$$

whence, by Cauchy's inequality,

(4) $$0<L(p)<t.$$

Secondly, let ξ be a non-real complex number, and assume now that the parameters s and t are such that

$$s\geqslant \max(1,|\xi|)^{-2m/(m+1)}, \qquad t=(m+1)(m+2)^{1/(m+1)}\max(1,|\xi|)^{2m/(m+1)}s,$$

hence that

$$t\geqslant(m+1)(m+2)^{1/(m+1)}.$$

The case $m=1$ is now trivial and will be excluded.

We split the powers

$$\xi^k, =\lambda_k+i\mu_k \quad \text{say} \quad (k=0,1,\ldots,m),$$

into their real and imaginary parts. The positive definite quadratic form

$$F(x_0,x_1,\ldots,x_m)=s^{m+1}|x_0+x_1\xi+\ldots+x_m\xi^m|^2+x_0^2+x_1^2+\ldots+x_m^2$$

in $x_0,x_1,\ldots,x_m$ can easily be shown to have the discriminant

$$D_F=1+s^{m+1}\sum_{k=0}^{m}(\lambda_k^2+\mu_k^2)+s^{2(m+1)}\sum_{0\leqslant k_1<k_2\leqslant m}(\lambda_{k_1}\mu_{k_2}-\lambda_{k_2}\mu_{k_1})^2$$

where evidently

$$D_F\leqslant s^{2(m+1)}(m+2)^2\max(1,|\xi|)^{4m}.$$

We find thus just as in the real case that there exists a polynomial

$$p(x)=p_0+p_1x+\ldots+p_mx^m$$

with integral coefficients not all zero such that

$$s^{m+1}|p(\xi)|^2+p_0^2+p_1^2+\ldots+p_m^2\leqslant(m+1)s^2(m+2)^{2/(m+1)}\max(1,|\xi|)^{4m/(m+1)}.$$

As in the real case, this inequality implies that simultaneously

(5) $$0 < |p(\xi)| < (m+1)^{m/2}(m+2)^{1/2}\max(1, |\xi|)^m t^{-(m-1)/2}$$

and

(6) $$0 < L(p) < t.$$

On combining the two results (3), (4) and (5), (6), we have thus proved:

Let $m \geqslant \sigma(\xi)$, and also $m < \partial^0(\xi)$ if ξ is algebraic; let further

(7) $$t \geqslant (m+2)^{(m+2)/(m+1)}.$$

Then there exists a polynomial $p(x)$ with integral coefficients satisfying

(8) $$\partial(p) \leqslant m, \quad 0 < L(p) < t, \quad \textit{hence also} \quad \Lambda(p) < 2^m t,$$

and

(9) $$0 < |p(\xi)| < (m+2)^{(m+1)/\sigma(\xi)}\max(1, |\xi|)^m t^{-\{(m+1)/\sigma(\xi)\}+1}.$$

6. Assume now, firstly, that ξ is algebraic but is neither rational nor lies in an imaginary quadratic field. Choose $m = \sigma(\xi)$, and allow t to tend to infinity. We obtain then infinitely many distinct polynomials $p(x)$ with integral coefficients for which

$$0 < |p(\xi)| < \begin{cases} 3^2\max(1, |\xi|)t^{-1} < 2\cdot 3^2\max(1, |\xi|)\Lambda(p)^{-1} & \text{if } \xi \text{ is real,} \\ 4^{3/2}\max(1, |\xi|)^2 t^{-1/2} < 2^4\max(1, |\xi|)^2\Lambda(p)^{-1/2} & \text{if } \xi \text{ is not real.} \end{cases}$$

Thus, in either case, for all sufficiently large u,

$$O(u\,|\,\xi) \geqslant c_3\log u$$

where $c_3 > 0$ depends only on ξ. Hence, by Theorem 1, we find as a first result.

THEOREM 2. *If ξ is algebraic, but is neither a rational number nor lies in an imaginary quadratic field, then*

$$O(u\,|\,\xi) >< \log u.$$

This result remains valid in the excluded case provided ξ is not an algebraic *integer*.

Secondly, let ξ be transcendental. We now choose

$$t = 2^m.$$

Then, for sufficiently large m, the condition (7) is satisfied, and

$$\Lambda(p) < 4^m.$$

Further

$$0 < |p(\xi)| < (m+2)^{(m+1)/\sigma(\xi)}\max(1, |\xi|)^m t^{-\{(m+1)/\sigma(\xi)\}+1} < 2^{-m^2/3}$$

as soon as m is sufficiently large because $\sigma(\xi) \leqslant 2$.

This means that for every sufficiently large positive integer there exists a polynomial $p(x) \not\equiv 0$ with integral coefficients for which both

$$0 < |p(\xi)| < e^{-c_4(\log u)^2} \quad \text{and} \quad A(p) < u.$$

Here $c_4 > 0$ is a certain absolute constant. From this result, the following theorem follows at once.

THEOREM 3. *If ξ is transcendental, then*

$$O(u \mid \xi) >> (\log u)^2.$$

7. We proceed now to the study of the order functions of two transcendental numbers ξ and η which are algebraically dependent over the rational field $\boldsymbol{Q}$.

By this hypothesis, there exists a primitive irreducible polynomial

$$A(x, y) = \sum_{h=0}^{M} \sum_{k=0}^{N} A_{hk} x^h y^k \not\equiv 0$$

with rational integral coefficients and, say, of the exact degrees $M \geqslant 1$ in x and $N \geqslant 1$ in y, such that

$$A(\xi, \eta) = 0.$$

From this we shall deduce that $\xi >< \eta$.

Put

$$A_h(y) = \sum_{k=0}^{N} A_{hk} y^k \quad (h = 0, 1, \ldots, M),$$

so that

$$A(x, y) = \sum_{h=0}^{M} A_h(y) x^h.$$

By the hypothesis,

$$A_M(y) \not\equiv 0,$$

and

(10) $$\max_{0 \leqslant h \leqslant M} \partial_y(A_h) = N.$$

We shall use the notation

$$C = \max_{0 \leqslant h \leqslant M} L_y(A_h).$$

8. The equation $A(\xi, \eta) = 0$ can be written in the form

$$A_M(\eta)\xi^M = -\{A_0(\eta) + A_1(\eta)\xi + \ldots + A_{M-1}(\eta)\xi^{M-1}\}.$$

We multiply this formula repeatedly by ξ and each time eliminate the term in ξ^M on the right-hand side by means of the formula. We so obtain an infinite sequence of equations

(11) $$A_M(\eta)^k \xi^k = \sum_{h=1}^{M-1} a_{hk}(\eta)\xi^h \quad (k = 0, 1, 2, \ldots).$$

Here the $a_{hk}(y)$ denote certain polynomials in y with integral coefficients which are defined by the initial values

$$(12)\qquad a_{hk}(y) = \begin{cases} A_M(y)^k & \text{if} \quad h = k \\ 0 & \text{if} \quad h \neq k \end{cases} \quad \text{and} \quad k = 0, 1, \ldots, M-1,$$

and, for $k = M, M+1, M+2, \ldots,$ by the recursive formulae

$$(13)\quad a_{h,k+1}(y) = \begin{cases} -A_0(y)\,a_{M-1,k}(y) & \text{if } h = 0, \\ -A_h(y)\,a_{M-1,k}(y) + A_M(y)a_{h-1,k}(y) & \text{if } h = 1, 2, \ldots, M-1. \end{cases}$$

From these formulae and from (10),

$$(14)\qquad \partial_y(a_{hk}) \leqslant kN \quad \text{for all } h \text{ and } k.$$

Further, for all h, by (12),

$$L_y(a_{hk}) \leqslant C^k \quad \text{if} \quad k = 0, 1, \ldots, M-1,$$

and by (13),

$$L_y(a_{h,k+1}) \leqslant 2C \max_{0 \leqslant h \leqslant M-1} L_y(a_{hk}) \quad \text{if} \quad k \geqslant M-1.$$

It follows therefore by induction for k that

$$(15)\qquad L_y(a_{hk}) \leqslant (2C)^k \quad \text{for all } h \text{ and } k.$$

It is convenient to replace the last formulae by slightly different ones. Denote by m any positive integer not less than $M-1$. The formulae (11) imply that also

$$(16)\qquad A_M(\eta)^m \xi^k = \sum_{h=0}^{M-1} B_{hk}(\eta)\xi^k \quad (k = 0, 1, \ldots, m)$$

where the $B_{hk}(y)$ denote new polynomials in y with integral coefficients defined by

$$(17)\qquad B_{hk}(y) = A_M(y)^{m-k} a_{hk}(y).$$

Therefore, by (14) and (15),

$$(18)\qquad \partial_y(B_{hk}) \leqslant mN \quad \text{and} \quad L_y(B_{hk}) \leqslant (2C)^m \quad \text{for all } h \text{ and } k.$$

9. Let

$$p(x) = p_0 + p_1 x + \ldots + p_m x^m, \quad \text{where} \quad p_m \neq 0,$$

be any polynomial in x with integral coefficients, of the exact degree

$$\partial_x(p) = m.$$

Here it is assumed that

$$m \geqslant M-1.$$

Therefore, by (16),

$$A_M(\eta)^m p(\xi) = \sum_{h=0}^{M-1} \sum_{k=0}^{m} p_k B_{hk}(\eta)\,\xi^h,$$

say

$$(19) \qquad A_M(\eta)^m p(\xi) = \sum_{h=0}^{M-1} b_h(\eta)\,\xi^h.$$

Here we have put

$$(20) \qquad b_h(y) = \sum_{k=0}^{m} p_k B_{hk}(y) \qquad (h = 0, 1, \ldots, M-1),$$

so that also the $b_h(y)$ are polynomials in y with integral coefficients. From the estimates (18), it follows immediately that

$$(21) \quad \partial_y(b_h) \leqslant mN \quad \text{and} \quad L_y(b_h) \leqslant (2C)^m L_x(p) \qquad (h = 0, 1, \ldots, M-1).$$

Denote now by $q(y)$ the resultant relative to x of the two polynomials

$$A(x, y) = A_0(y) + A_1(y)x + \ldots + A_M(y)x^M$$

and

$$A^*(x, y) = b_0(y) + b_1(y)x + \ldots + b_{M-1}(y)x^{M-1}.$$

This resultant is given explicitly by the determinant

$$(22) \qquad q(y) = \begin{vmatrix} A_0(y) & A_1(y) & \ldots & A_M(y) & \ldots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \ldots & A_0(y) & A_1(y) & \ldots & A_M(y) \\ b_0(y) & b_1(y) & \ldots & b_{M-1}(y) & \ldots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \ldots & b_0(y) & b_1(y) & \ldots & b_{M-1}(y) \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} M-1 \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} M \text{ rows} \end{matrix}$$

Hence $q(y)$ is a polynomial with integral coefficients. By (10) and (21),

$$\partial_y(q) \leqslant (M-1)\cdot mN + M\cdot mN$$

and therefore

$$(23) \qquad \partial_y(q) \leqslant m(2M-1)N.$$

It follows further from the trivial estimate for a determinant and from (21) that

$$L_y(q) \leqslant (2M-1)!\,(2C)^{m(M-1)}\{(2C)^m L_x(p)\}^M$$

and hence

$$(24) \qquad L_y(q) \leqslant (2M-1)!\,(2C)^{m(2M-1)} L_x(p)^M.$$

10. Next multiply the 2nd, 3rd, ..., $(2M-1)$st columns of the determinant for $q(y)$ by the factors

$$x, x^2, \ldots, x^{2M-2},$$

respectively, and add to the first column. The new first column becomes then

$$A(x, y), A(x, y)x, \ldots, A(x, y)x^{M-2}, A^*(x, y), A^*(x, y)x, \ldots, A^*(x, y)x^{M-1}.$$

Here put

$$x = \xi \quad \text{and} \quad y = \eta.$$

Then

$$A(\xi, \eta) = 0 \quad \text{and} \quad A^*(\xi, \eta) = A_M(\eta)^m p(\xi),$$

whence

$$q(\eta) = A_M(\eta)^m p(\xi)\cdot q^*(\xi, \eta), \tag{25}$$

where $q^*(\xi, \eta)$ denotes the determinant obtained from that defining $q(y)$ by replacing its first column by the new column

$$0, 0, \ldots, 0, 1, \xi, \xi^2, \ldots, \xi^{M-1}$$

and substituting η for y. Thus $q^*(\xi, \eta)$ can be written as a polynomial in ξ of the form

$$q^*(\xi, \eta) = q_0^*(\eta) + q_1^*(\eta)\xi + \ldots + q_{M-1}^*(\eta)\xi^{M-1}. \tag{26}$$

Here, for $h = 0, 1, \ldots, M-1$, the $q_h^*(y)$ denote the cofactors of the last M elements of the first column of the determinant for $q(y)$. They are thus polynomials in y with integral coefficients. Just as for (23) and (24), we find the estimates

$$\partial_y(q_h^*) \leqslant 2m(M-1)N \quad \text{and} \quad L_y(q_h^*) \leqslant (2M-2)!(2C)^{2m(M-1)}L_x(p)^{M-1} \tag{27}$$
$$(h = 0, 1, \ldots, M-1).$$

11. The resultant $q(y)$ does not vanish identically because $A(x, y)$ is irreducible and has the exact degree M in x, while $A^*(x, y)$ has at most the degree $M-1$ in this variable. The transcendency of η implies then that

$$q(\eta) \neq 0.$$

By (23) and (24),

$$\Lambda_y(q) \leqslant 2^{m(2M-1)N}(2M-1)!(2C)^{m(2M-1)}L_x(p)$$

and also

$$\Lambda_x(p) = 2^m L_x(p).$$

Hence there exist two positive integers C_1 and Γ_1 depending only on C, M, and N, and so only on the polynomial $A(x, y)$, such that

$$\Lambda_y(q) \leqslant \Lambda_x(p)^{C_1} \quad \text{if} \quad \Lambda_x(p) \geqslant \Gamma_1. \tag{28}$$

Next put

$$|A_M(\eta)| = c_5, \quad \max(1, |\xi|) = c_6, \quad \text{and} \quad \max(1, |\eta|) = c_7.$$

By (26) and (27),

$$|q^*(\xi, \eta)| \leqslant M c_6^{M-1}(2M-2)!(2C)^{2m(M-1)} L_x(p)^{M-1} c_7^{2m(M-1)N},$$

so that, by (25),

$$\left|\frac{q(\eta)}{p(\xi)}\right| \leqslant c_5 M c_6^{M-1}(2M-2)!(2C)^{2m(M-1)} L_x(p)^{M-1} c_7^{2m(M-1)N}.$$

By this inequality, there exist two further positive integers C_2 and Γ_2 which depend only on the polynomial $A(x, y)$ and on the two numbers ξ and η such that

(29) $$|q(\eta)| \leqslant \Lambda_x(p)^{C_2}|p(\xi)| \quad \text{if} \quad \Lambda_x(p) \geqslant \Gamma_2.$$

12. Assume now that the parameter u is not less than

$$\Gamma = \max(\Gamma_1, \Gamma_2).$$

Further choose in $\Omega(u)$ a polynomial $p(x)$ satisfying the equation

$$\log|1/p(\xi)| = O(u \mid \xi).$$

By this choice,

$$\Lambda_x(p) \leqslant u.$$

Further, by (28),

(30) $$\Lambda_y(q) \leqslant u^{C_1},$$

and by (29),

(31) $$|q(\eta)| \leqslant |p(\xi)| u^{C_2}.$$

We found already, in the proof of Theorem 3, that

$$\log|1/p(\xi)| > c_4(\log u)^2,$$

where $c_4 > 0$ was a certain absolute constant. Hence, if Γ_0 is a sufficiently large positive integer, then, by (31),

(32) $$\log|1/q(\eta)| \geqslant \tfrac{1}{2}\log|1/p(\xi)| = O(u \mid \xi)/2 \quad \text{if} \quad u \geqslant \Gamma_0.$$

On the other hand, $q(\eta) \neq 0$, and so, by (30), $q(y)$ belongs to the set $\Omega(u^{C_1})$. But then, necessarily,

$$O(u^{C_1} \mid \eta) \geqslant \log|1/q(\eta)|,$$

so that, by (32), we arrive finally at the estimate

$$O(u^{C_1} \mid \eta) \geqslant \tfrac{1}{2} O(u \mid \xi) \quad \text{if} \quad u \geqslant \Gamma_0.$$

Naturally, on interchanging ξ and η, we also obtain an analogous estimate

$$O(u^{C_1^*} \mid \xi) \geqslant \tfrac{1}{2} O(u \mid \eta) \quad \text{if} \quad u \geqslant \Gamma_1^*,$$

where C_1^* and Γ_1^* are two further positive integers.

We have thus established the following *Invariance Property.*

THEOREM 4. *Let ξ and η be two transcendental numbers which are algebraically dependent over the rational field* $\boldsymbol{Q}$. *Then*

$$O(u \mid \xi) >< O(u \mid \eta) \quad \textit{and therefore} \quad \xi >< \eta.$$

13. Denote by $\mathscr{T}$ the set of all transcendental numbers. Let us then subdivide $\mathscr{T}$ into disjoint subsets or *classes* $\Xi, H, Z, \ldots$ by putting numbers ξ and η into the same class if and only if $\xi >< \eta$. Thus, by what has just been proved, *numbers which are algebraically dependent over* $\boldsymbol{Q}$ *belong always to the same class.*

There are evidently non-countably many positive valued non-decreasing functions $a(u), b(u), \ldots$ of the integer $u \geqslant 1$ no two of which stand in the relation

$$a(u) >< b(u),$$

but it is not evident which of these functions are order functions of transcendental numbers. It is further clear that there exist transcendental numbers ξ (e.g. Liouville numbers) for which $O(u \mid \xi)$ tends arbitrarily rapidly to infinity; but it does not seem to be easy to find the exact size of these order functions. Thus the following two problems remain open.

PROBLEM 1. *Do there exist non-countably many distinct classes* $\Xi, H, Z, \ldots$? (1)

PROBLEM 2. *Let $a(u)$ be any positive valued non-decreasing function of the integer $u \geqslant 1$. To establish necessary and sufficient conditions for the existence of a number $\xi \in \mathscr{T}$ such that*

$$a(u) >< O(u \mid \xi).$$

In addition to the equivalence relation $><$ we had also defined an order relation $>>$ for both functions and numbers, and it is easily seen that it can be extended to classes. With respect to this order relation, the following two questions arise.

PROBLEM 3. *Does there exist a pair of numbers ξ and η in $\mathscr{T}$ such that neither $\xi >> \eta$ nor $\xi << \eta$?*

PROBLEM 4. *Does there exist a number $\zeta \in \mathscr{T}$ such that*

$$\xi >> \zeta \quad \textit{for all} \quad \xi \in \mathscr{T}\ ?$$

(1) Note added on January 12, 1971. S. Świerczkowski has recently proved that the answer is affirmative.

The following metrical question also has some interest.

PROBLEM 5. *To decide whether there exist, and if so, to determine, two positive valued non-decreasing functions* $a(u)$ *and* $b(u)$ *of the integer* $u \geq 1$ *such that*

(i) $O(u \mid \xi) << a(u)$ *for almost all real numbers* $\xi \in \mathscr{T}$,

and

(ii) $O(u \mid \xi) << b(u)$ *for almost all complex numbers* $\xi \in \mathscr{T}$,

and that, in addition, $a(u)$ *and* $b(u)$ *increase as slowly as possible.*

I conjecture that this problem has the solution

$$a(u) >< (\log u)^2, \qquad b(u) >< (\log u)^2.$$

The actual determination of $O(u \mid \xi)$ for any given $\xi \in \mathscr{T}$ presents a difficult problem which has as yet not even be solved for the two classical transcendental numbers e and π. For the order functions of these two numbers the best *lower* bounds known seem to be those given in Theorem 3.

The best *upper* bounds known at present are those due to N. I. Feldman ([1] and [2]) which state that

$$O(u \mid e) << (\log u)^3 (\log\log u)^3,$$

$$O(u \mid \pi) << (\log u)^2 (\log\log u)^3.$$

We had defined the order function $O(u \mid \xi)$ in terms of the functional

$$\Lambda(p) = 2^{\partial(p)} L(p).$$

No essentially different results are obtained if 2 is here replaced by any other constant greater than 1. It may, however, be useful to consider other functionals.

Just as in Koksma's approach ([4]) to my old classification, one can replace the order function $O(u \mid \xi)$ by a new function

$$O^*(u \mid \xi) = \sup_{\alpha \in \Omega^*(u)} \log \{1/|\xi - \alpha|\}$$

where $\Omega^*(u)$ denotes the set of all algebraic numbers α for which

$$\alpha \neq \xi \quad \text{and} \quad \Lambda^0(\alpha) \leq u.$$

However, both Koksma's work and a recent paper by Wirsing ([6]) suggest that the results will be completely analogous to those for $O(u \mid \xi)$.

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OHIO STATE UNIVERSITY
Columbus, Ohio

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