THE CLASSIFICATION OF TRANSCENDENTAL NUMBERS

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1. All numbers ζ considered in this article are real or complex. For polynomials

$$p(z)=p_0+p_1z+\cdots+p_mz^m, \quad \text{where } p_m\neq 0,$$

the following notation will be used.

$$\partial(p)=m, \qquad H(p)=\max_{\mu=0,\,1,\,\dots,\,m} |p_\mu|, \quad \text{and} \quad L(p)=\sum_{\mu=0}^{m} |p_\mu|$$

denote the exact *degree*, the *height*, and the *length* of $p(z)$, respectively. We further put

$$\Lambda(p)=2^{\partial(p)}L(p) \quad \text{and} \quad M(p)=\prod_{\mu=0}^{m}(2+|p_\mu|).$$

If V denotes the set of all polynomials $p(z)\neq 0$ with rational integral coefficients and v is any positive integer, it is obvious that either of the inequalities $\Lambda(p)\leqq v$ or $M(p)\leqq v$ is satisfied by at most *finitely many* elements of V.

Consider now the set $\boldsymbol{C}$ of all real or complex numbers ζ. Our aim is to subdivide $\boldsymbol{C}$ into subsets or *classes* which are disjoint and have the following *invariance property*.

Any two numbers in distinct classes are algebraically independent over the rational number field $\boldsymbol{Q}$.

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Here the subdivision of $\boldsymbol{C}$ is to depend solely on the approximation properties of ζ, and the number of distinct classes should by preference be large.

2. A first such classification with the invariance property, but into only four classes, was found by me about 40 years ago. A detailed account of this classification, and of the almost equivalent one by J. F. Koksma, can be found in the book on transcendental numbers by Th. Schneider (1957).

This classification is obtained as follows. Put successively

$$w_m(v \mid \zeta) = \inf |p(\zeta)|,$$

where the lower bound extends over all polynomials $p(z)$ satisfying

$$p(z) \in V, \quad \partial(p) \leqq m, \quad H(p) \leqq v, \quad \text{and} \quad p(\zeta) \neq 0;$$

$$w_m(\zeta) = \limsup_{v \to \infty} \frac{\log\{1/w_m(v \mid \zeta)\}}{\log v}, \qquad w = w(\zeta) = \limsup \frac{w_m(\zeta)}{m}.$$

Let further the symbol $\mu = \mu(\zeta)$ denote ∞ if $w_m(\zeta)$ is finite for all suffixes m, and otherwise let it be equal to the smallest suffix m for which $w_m(\zeta) = \infty$. Thus at least one of the two numbers w and μ is always equal to ∞.

Therefore the complex numbers split into the following four disjoint classes:

Class A:	ζ satisfies	$w=0$	and	$\mu=\infty$.
Class S:	ζ satisfies	$0<w<\infty$	and	$\mu=\infty$.
Class T:	ζ satisfies	$w=\infty$	and	$\mu=\infty$.
Class U:	ζ satisfies	$w=\infty$	and	$\mu<\infty$.

It can now be proved that: (i) the class A consists exactly of all algebraic numbers, hence the transcendental numbers are distributed amongst the classes S, T, and U; and (ii) the invariance property holds, i.e. numbers in different classes are algebraically independent over $\boldsymbol{Q}$.

One can also show that *almost all numbers are S-numbers*, a result greatly strengthened by V. Sprindžuk (1967). There are noncountably many U-numbers, e.g. all Liouville numbers; these are simply characterised by $\mu = 1$. Until recently it was not known whether there exist any T-numbers, but this existence has now been established by W. Schmidt (1971), although as yet no actual T-number seems to be known.

By way of example, e is an S-number, while π is either an S-number or a T-number.

3. I come now to a new classification (Mahler, 1971) which leads to a sub-

division of $\boldsymbol{C}$ into *infinitely many* disjoint classes with the invariance property. In this classification, we need to consider polynomials in V of independently variable degree and height (or rather length).

This classification depends on the following partial ordering of monotone nondecreasing functions.

If $a(v)>0$ and $b(v)>0$ are any two nondecreasing functions of $v \geqq 1$ for which there exist three positive numbers c, v_0, and γ such that

$$a(v^c) \geqq \gamma b(v) \quad \text{for} \quad v \geqq v_0,$$

then we write

$$a(v) \gg b(v) \quad \text{or} \quad b(v) \ll a(v).$$

If simultaneously

$$a(v) \gg b(v) \quad \text{and} \quad a(v) \ll b(v),$$

then we write

$$a(v) >< b(v).$$

This sign $><$ evidently defines an *equivalence relation.*

With each element ζ of $\boldsymbol{C}$ we associate now an *order function*

$$O(v \mid \zeta) = \sup \log \{1/|p(\zeta)|\}$$

where the upper bound is extended over all polynomials $p(z)$ in V for which

$$\Lambda(p) \leqq v, \qquad p(\zeta) \neq 0.$$

Since they behave slightly differently, it is convenient to exclude from the consideration all those ζ which are either rational integers, or are integers in any imaginary quadratic field. With this restriction, the following results hold.

$$O(v \mid \zeta) >< \log v \qquad \text{if } \zeta \text{ is algebraic;}$$

$$O(v \mid \zeta) \gg (\log v)^2 \qquad \text{if } \zeta \text{ is transcendental;}$$

$$O(v \mid \zeta) >< O(v \mid \zeta') \quad \text{if } \zeta, \zeta' \text{ are algebraically dependent over } \boldsymbol{Q}.$$

Thus, if numbers ζ, ζ' with equivalent order functions are put into one and the same class, then the invariance property holds.

The actual determination of the order function of a number is, of course, a very difficult problem. I mention, by way of example, the following relations.

$$O(v \mid e) \ll (\log v)^3 (\log \log v)^3, \qquad O(v \mid \pi) \ll (\log v)^2 (\log \log v)^3,$$

which are implicit in work by N. I. Fel′dman (1951 and 1963). It is interesting to see that in the second formula the upper estimate comes close to the lower estimate $(\log v)^2$.

In my paper on the order function I raised a number of questions. One of these questions has in the meantime been solved by Świerczkowski in an unpublished note; he proved that there are noncountably many inequivalent order functions and hence also as many classes in this classification.

It is not known which monotonic functions are equivalent to order functions and which can be the order function of almost all real or almost all complex numbers. It is also unknown whether the order functions can be strictly ordered.

4. I conclude this article by suggesting a still different kind of classification; however, I do not know whether it has the invariance property, or rather how the classification has to be defined so that this property holds.

The important recent work by W. Schmidt (1970) and A. Baker (1965) suggests that instead of $O(v \mid \zeta)$ one should associate with ζ the function

$$R(v \mid \zeta) = \sup \log \{1/|p(\zeta)|\},$$

where the upper bound is now extended over all polynomials $p(z)$ in V for which $M(p) \leqq v$, $p(\zeta) \neq 0$. It seems highly probable that also for these functions R an equivalence relation can be found which preserves the invariance property. I dare to conjecture that the ideas of Schmidt could be used to settle this question.

So far we have only discussed classifications based on the values of a single variable polynomial $p(z)$ at the given point $z = \zeta$. A more powerful kind of classification would consider simultaneous approximations by sets of polynomials. I have little doubt that the modern general transfer theorems in the geometry of numbers of convex bodies are the right tool for attacking such problems.

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