BULL. AUSTRAL. MATH. SOC. 10F05
VOL. 10 (1974), 325-335. (10F35)

On rational approximations of the exponential function at rational points

Kurt Mahler

Let p, q, u, and v be any four positive integers, and let further δ be a number in the interval $0 < \delta \le 2$. In this note an effective lower bound for q will be obtained which insures that

$$\left|e^{u/v} - \frac{p}{q}\right| > q^{-(2+\delta)} .$$

In the special case when $u = v = 1$, it was shown by J. Popken, *Math. Z.* 29 (1929), 525-541, that

$$\left|e - \frac{p}{q}\right| > q^{-\{2+(c/\log \log q)\}} \quad \text{for} \quad q \ge C .$$

Here c and C are two positive absolute constants which, however, were not determined explicity. A similarly non-effective result was given in my paper, *J. reine angew. Math.* 166 (1932), 118-150.

The method of this note depends again on the classical formulae by Hermite which I applied also *op. cit.*

1.

Denote by m and n two non-negative integers and put

$$F(w) = \frac{w^m(w-1)^n}{m!n!} \quad \text{and} \quad F(z; w) = \sum_{k=0}^{\infty} z^{-k-1}\left(\frac{d}{dw}\right)^k F(w) .$$

A simple calculation shows that

Received 19 December 1973.

$$F(z;\ 0) = \sum_{k=m}^{m+n} z^{-k-1}\binom{k}{m}\frac{(-1)^{m+n-k}}{(m+n-k)!} \quad \text{and} \quad F(z;\ 1) = \sum_{k=n}^{m+n} z^{-k-1}\binom{k}{n}\frac{1}{(m+n-k)!} .$$

Put therefore

$$P(z) = (m+n)!z^{m+n+1}F(z;\ 0) \quad \text{and} \quad Q(z) = (m+n)!z^{m+n+1}F(z;\ 1) .$$

Then

$$P(z) = \sum_{k=m}^{m+n} k!(-1)^{m+n-k}\binom{k}{m}\binom{m+n}{k}z^{m+n-k} , \quad Q(z) = \sum_{k=n}^{m+n} k!\binom{k}{n}\binom{m+n}{k}z^{m+n-k} .$$

By these formulae, $P(z)$ and $Q(z)$ are polynomials in z of the degrees n and m, and with integral coefficients divisible by $m!$ and $n!$, respectively.

It also follows from the definitions of $F(w)$ and $F(z;\ w)$ that

$$\int_0^1 F(w)e^{-zw}dw = F(z;\ 0) - F(z;\ 1)e^{-z} .$$

Hence, on putting

$$R(z) = (m+n)!z^{m+n+1}e^{z}\int_0^1 F(w)e^{-zw}dw ,$$

we obtain Hermite's identity

(1) $$P(z)e^{z} - Q(z) = R(z) .$$

2.

From now on denote by r a positive integer. The identity (1) will be used only in the two special cases when either

(A) $m = r - 1$, $n = r$, or

(B) $m = r$, $n = r - 1$.

Thus in either case $m + n = 2r - 1$, and the functions $P(z)$, $Q(z)$, and $R(z)$ take the following special forms.

Case A:

$$P_A(z) = \sum_{k=r-1}^{2r-1} k!(-1)^{k-1}\binom{k}{r-1}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$Q_A(z) = \sum_{k=r}^{2r-1} k!\binom{k}{r}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$R_A(z) = \binom{2r-1}{r} z^{2r} e^z \int_0^1 w^{r-1}(w-1)^r e^{-zw} dw ,$$

and

Case B:

$$P_B(z) = \sum_{k=r}^{2r-1} k!(-1)^{k-1}\binom{k}{r}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$Q_B(z) = \sum_{k=r-1}^{2r-1} k!\binom{k}{r-1}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$R_B(z) = \binom{2r-1}{r} z^{2r} e^z \int_0^1 w^r(w-1)^{r-1} e^{-zw} dw .$$

By these formulae, $P_A(z)$ and $Q_B(z)$ have the exact degree r, and $P_B(z)$ and $Q_A(z)$ have the exact degree $r - 1$; further both $R_A(z)$ and $R_B(z)$ vanish at $z = 0$ to the order $2r$. Further, by (1),

$$P_A(z)e^z - Q_A(z) = R_A(z) \quad \text{and} \quad P_B(z)e^z - Q_B(z) = R_B(z) .$$

Therefore the determinant

$$D(z) = \begin{vmatrix} P_A(z), & Q_A(z) \\ P_B(z), & Q_B(z) \end{vmatrix} = \begin{vmatrix} P_A(z), & -R_A(z) \\ P_B(z), & -R_B(z) \end{vmatrix}$$

is a polynomial in z of the exact degree $2r$ which has at $z = 0$ a zero of order $2r$. This determinant can therefore be written as

$$D(z) = dz^{2r}$$

where d is a constant distinct from zero. Thus

(2) $$D(z) \neq 0 \quad \text{if} \quad z \neq 0 .$$

3.

All four polynomials $P_A(z)$, $Q_A(z)$, $P_B(z)$, $Q_B(z)$ have integral coefficients divisible by $(r-1)!$, and they have the degrees r or $r-1$ in z . Denote by u and v two positive integers and put in the preceding formulae

$$z = u/v \ .$$

Let further

$$U_A = \frac{v^r}{(r-1)!} P_A(u/v) \ , \quad V_A = \frac{v^r}{(r-1)!} Q_A(u/v) \ , \quad W_A = \frac{v^r}{(r-1)!} R_A(u/v) \ ,$$

and

$$U_B = \frac{v^r}{(r-1)!} P_B(u/v) \ , \quad V_B = \frac{v^r}{(r-1)!} Q_B(u/v) \ , \quad W_B = \frac{v^r}{(r-1)!} R_B(u/v) \ .$$

Then, by (2), U_A, V_A, U_B, V_B are integers of determinant

(3) $$U_A V_B - U_B V_A \neq 0 \ .$$

We require upper estimates for these six quantities and therefore introduce the two maxima

$$X = \max\left(|U_A|, \ |V_A|, \ |U_B|, \ |V_B|\right) \quad \text{and} \quad Y = \max\left(|W_A|, \ |W_B|\right) \ .$$

In the sum

$$\sum_{k=0}^{2r-1} \binom{2r-1}{k} = 2^{2r-1}$$

the two terms with $k = r - 1$ and $k = r$ are identical and hence satisfy the inequality

$$\binom{2r-1}{r-1} = \binom{2r-1}{r} \leq 2^{2r-2} \ .$$

It follows that also

$$\frac{(2r-1)!}{(r-1)!} = r! \binom{2r-1}{r-1} \leq 2^{2r-2} r! \ .$$

Further, in the sums defining U_A, V_A, U_B , and V_B , the factors

$$k! \ , \quad \binom{k}{r} \ , \text{ and } \binom{k}{r-1}$$

assume their largest possible values when $k = 2r - 1$. Hence trivially,

$$X \leq \frac{(2r-1)!}{(r-1)!} \binom{2r-1}{r-1}^2 (u+v)^r ,$$

and therefore

(4) $$X \leq 2^{6r-6} r!(u+v)^r .$$

Next, when w lies in the interval $0 \leq w \leq 1$,

$$\max\left(w^{r-1}(w-1)^r, w^r(w-1)^{r-1}\right) \leq \left(w(w-1)\right)^{r-1} \leq 4^{-(r-1)} \quad \text{and} \quad e^{-zw} \leq 1 .$$

The integrals for W_A and W_B imply therefore that

$$Y \leq \frac{v^r}{(r-1)!} 2^{2r-2}(u/v)^{2r} e^{u/v} 4^{-(r-1)}$$

and hence that

(5) $$Y \leq \frac{e^{u/v}}{(r-1)!} \left(u^2/v\right)^r .$$

4.

These upper estimates will now be applied to the rational approximations of $e^{u/v}$. For this purpose, denote by p and q any two positive integers and put

$$qe^{u/v} - p = d .$$

An explicit lower estimate for $|d|$ can be obtained by the following considerations.

Since the determinant (3) is distinct from zero, the same is true for at least one of the two determinants

$$\begin{vmatrix} U_A, & V_A \\ q, & p \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} U_B, & V_B \\ q, & p \end{vmatrix} .$$

Denote then by C the suffix A or B for which

$$\begin{vmatrix} U_C, & V_C \\ q, & p \end{vmatrix} \neq 0 .$$

The two equations

$$U_C e^{u/v} - V_C = W_C \quad \text{and} \quad q e^{u/v} - p = d$$

imply that

$$pU_C - qV_C = dU_C - qW_C \ .$$

Here the left-hand side is an integer distinct from zero and is thus at least of absolute value 1 . Thus the following deduction can be made.

LEMMA. *If the integer* r *can be chosen such that*

(6) $$|2qW_C| \le 1 \ ,$$

then also

(7) $$|2dU_C| \ge 1 \ .$$

5.

By the definition of Y and by its upper estimate (5),

$$|2qW_C| \le \frac{2e^{u/v}}{(r-1)!} \left(u^2/v\right)^r . q \ .$$

Assume now that r satisfies the inequality

$$(r-1)! \ge 2e^{u/v}\left(u^2/v\right)^r q \ ,$$

or, equivalently, the inequality

(8) $$e^r r! \ge 2re^{u/v}\left(eu^2/v\right)^r q \ .$$

Then the condition (6) holds, and it follows from (4) and (7) that

(9) $$|d| \ge \left(2^{6r-5} r! (u+v)^r\right)^{-1} \ .$$

To simplify these formulae, denote by ε a constant in the interval

$$0 < \varepsilon \le \frac{1}{2} \ ,$$

so that

$$(1+2\varepsilon)\left(1 - \frac{\varepsilon}{2}\right) = 1 + \frac{3\varepsilon}{2} - \varepsilon^2 = 1 + \varepsilon + \varepsilon\left(\frac{1}{2} - \varepsilon\right) \ge 1 - \varepsilon \ .$$

Further assume from now on that both

(10) $$q \geq \left(2re^{u/v}\left(eu^2/v\right)^r\right)^{1/\varepsilon}$$

and

(11) $$e^r r! \geq q^{1+\varepsilon} .$$

Then the inequality (8) likewise is satisfied. Now almost trivially,

$$r! > r^r e^{-r} .$$

The hypothesis (11) may then be replaced by the following stronger one,

(12) $$r^r \geq q^{1+\varepsilon} .$$

In order to satisy this condition, assume that q, in addition to the condition (10), also has the property that

(13) $$\log\log\log q \leq \frac{\varepsilon}{2}\log\log q ,$$

and then define r as a function of q by the equation

(14) $$r = \left[\frac{(1+2\varepsilon)\log q}{\log\log q}\right] + 1 .$$

Then

$$r > \frac{(1+2\varepsilon)\log q}{\log\log q}$$

and therefore

$$\log r > \log\log q - \log\log\log q$$

because $\log(1+2\varepsilon)$ is positive. It follows that

$$r\log r > \frac{(1+2\varepsilon)\log q}{\log\log q}\left(1 - \frac{\varepsilon}{2}\right)\log\log q \geq (1+\varepsilon)\log q .$$

This shows that (12) is a consequence of the two formulae (13) and (14).

6.

Also the condition (13) may be replaced by a simpler one. If for the moment c is any positive constant and x a positive variable, the function

$$x^{-c}\log x$$

assumes its maximum at $x = e^{1/c}$ so that for all x,

$$\log x \le (ce)^{-1} x^{c} ;$$

hence for $c = 1/2$,

(15) $$\log x \le (2/e) x^{1/2} .$$

On putting $x = \log \log q$ in (13), this inequality takes the form

$$\log x \le \frac{\varepsilon}{2} x$$

and is thus certainly satisfied if

$$(2/e) x^{1/2} \le \frac{\varepsilon}{2} x \text{ , that is, if } x \ge \left(\frac{4}{e\varepsilon}\right)^2 .$$

The condition (13) is thus a consequence of the simpler condition

(16) $$q \ge e^{e^{(4/e\varepsilon)^2}} .$$

We have just replaced the condition (13) for q by (16). As a next simplification, the other condition (10) for q will now be replaced by two conditions in which the integer r no longer occurs.

Evidently,

$$2r \le e^{r}$$

for all positive integers r. Hence the inequality (10) is certainly satisfied if

(17) $$q^{\varepsilon} \ge e^{u/v} \left(e^{2} u/v\right)^{r} .$$

Here, by (14),

$$r \le \frac{(1+2\varepsilon)\log q}{\log \log q} + 1 .$$

Further

$$0 < \varepsilon \le \tfrac{1}{2} , \quad 1 + 4\varepsilon \le 3 , \quad \left(\frac{4}{e\varepsilon}\right)^2 > 8 .$$

Since

$$\log\log q \geq \left(\frac{4}{e\varepsilon}\right)^2 ,$$

it follows therefore easily that

$$r < \frac{3\log q}{\log\log q} ,$$

and so also

$$\left(e^2u/v\right)^r < q^{\frac{3\log\left(e^2u/v\right)}{\log\log q}}$$

Hence the single inequality (17) for q may be replaced by the pair of conditions

(18) $$q \geq e^{\left(e^2u/v\right)^{6/\varepsilon}} \quad \text{and} \quad q \geq e^{2(u/v)/\varepsilon} .$$

It depends on the size of u/v which of these two conditions is the stronger one.

7.

By what so far has been proved, the three conditions (16) and (18) for q , together with the definition (14) of r as a function of q , imply the inequalitiy (9) for d . This inequality still contains the parameter r , which will now be eliminated from it.

We found already that

$$r \leq \frac{(1+2\varepsilon)\log q}{\log\log q} + 1 .$$

Here $\log(1+x) < x$ for positive x , so that

$$\log r \leq \log\log q + \log(1+2\varepsilon) - \log\log\log q + \log\left(1 + \frac{\log\log q}{(1+2\varepsilon)\log q}\right) \leq$$

$$\leq \log\log q - \log\log\log q + 2\varepsilon + \frac{\log\log q}{\log q} .$$

Here, by (16),

$$\log\log\log q \geq 2\log\left(\frac{4}{e\varepsilon}\right) \geq 2\log\frac{8}{e} \geq 2 \geq 4\varepsilon$$

and

$$\frac{\log\log q}{\log q} \le \left(\frac{4}{e\varepsilon}\right)^2 e^{-(4/e\varepsilon)^2} \le \left(\frac{4}{e\varepsilon}\right)^2 \left(\frac{1}{6}\left(\frac{4}{e\varepsilon}\right)^6\right)^{-1} \le 6\left(\frac{e\varepsilon}{4}\right)^4 < \varepsilon .$$

It follows that

$$\varepsilon \frac{\log q}{\log\log q} > 1 ,$$

whence

(19)
$$r < \frac{(1+3\varepsilon)\log q}{\log\log q} .$$

Further

$$\log r \le \log\log q - 4\varepsilon + 2\varepsilon - \varepsilon < \log\log q ,$$

so that

$$r\log r < (1+3\varepsilon)\log q$$

and finally

(20)
$$r! \le r^r < q^{1+3\varepsilon} .$$

On substituting this lower estimate for $r!$ in (9), it follows that

$$|d| > 32\left(\{64(u+v)\}^r q^{1+3\varepsilon}\right)^{-1} .$$

Here, by (19),

$$\{64(u+v)\}^r < q^{\frac{(1+3\varepsilon)\log\{64(u+v)\}}{\log\log q}} \le q^{\frac{5\log\{64(u+v)\}}{2\log\log q}} .$$

In order to simplify this estimate, add to the previous conditions (16) and (18) for q the following new one,

(21)
$$q \ge e^{\{64(u+v)\}^{5/(2\varepsilon)}} .$$

The lower bound for $|d|$ takes then the simple form

$$|d| > 32q^{-(1+4\varepsilon)} .$$

Here it is convenient to put

$$4\varepsilon = \delta .$$

Then the result just obtained may be formulated as follows.

THEOREM. *Let* δ *be a constant in the interval* $0 < \delta \le 2$, *and let*

p, q, u, and v be four positive integers where q is restricted by the conditions

$$q \geq e^{e^{(1/e\delta)^2}}, \quad q \geq e^{\left(e^2 u/v\right)^{24/\delta}}, \quad q \geq e^{8(u/v)/\delta}, \quad q \geq e^{\{64(u+v)\}^{10/\delta}}.$$

Then

$$\left|e^{u/v} - \frac{p}{q}\right| > q^{-(2+\delta)}.$$

The conditions for q in this theorem are stronger than necessary, and it would in particular be possible to replace the first condition by a weaker one. However, such a change would probably complicate the proof and the final result. My aim was to establish an effective lower bound for $\left|e^{u/v} - \frac{p}{q}\right|$ which does not contain any unknown constants.

References

[1] Kurt Mahler, "Zur Approximation der Exponentialfunktion und des Logarithmus, Teil I", *J. Reine angew. Math.* **166** (1932), 118-136 (1931).

[2] Kurt Mahler, "Zur Approximation der Exponentialfunktion und des Logarithmus, Teil II", *J. reine angew. Math.* **166** (1932), 137-150.

[3] J. Popken, "Zur Transzendenz von e", *Math. Z.* **29** (1929), 525-541.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.