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On Some Irrational Decimal Fractions

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Let $g \geqslant 2$ be a fixed integer, and let $a(g)$ be the decimal fraction

$$a(g) = 0.(1)(g)(g^2)(g^3)...$$

where (g^n) is to mean the number g^n written in decimal form. Thus

$$a(2) = 0.1\text{–}2\text{–}4\text{–}8\text{–}16\text{–}32\text{–}\ldots \qquad \text{and} \qquad a(3) = 0.1\text{–}3\text{–}9\text{–}27\text{–}82\text{–}243\text{–}\ldots .$$

Our aim is to prove the following theorem.

All the numbers $a(g)$ are irrational.

When g is relatively prime to 10 or divisible by 10, the assertion is almost trivial. In the first case, by Euler's theorem,

$$g^{4 \times 10^{m-1}} \equiv 1 \ (\text{mod } 10^m),$$

so that the decimal fraction $a(g)$ contains arbitrarily long sequences of digits 0, while there are again and again digits distinct from 0; the decimal fraction is therefore not periodic and hence is irrational. Also in the second case there are arbitrarily long sequences of digits 0, and the same proof applies.

We need then only discuss the two cases where g has the form

$$g = 2^r h \qquad \text{or} \qquad g = 5^r h;$$

here r is a positive integer, and h is a positive integer relatively prime to 10. Now the assertion is less obvious, and the proof is more difficult.

We shall assume that the assertion is false for $a(g)$, hence that this number is rational, hence that $a(g)$ is a periodic decimal fraction. Let the period be

$$A_1 A_2 \cdots A_u = P,$$

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where $A_1, A_2, \ldots, A_u$ are certain u digits 0, 1,..., 9. From some suffix onward, the digits of $a(g)$ thus repeat the period P indefinitely.

First let $g = 2^r h$, and let m be a positive integer. Since g is relatively prime to 5,

$$g^{4 \times 5^{m-1}} \equiv 1 \pmod{5^m},$$

while trivially

$$g^{4 \times 5^{m-1}} \equiv 0 \pmod{2^m}.$$

It follows that the 10-adic limit

$$L = \overset{10}{\lim_{m \to \infty}} \; g^{4 \times 5^{m-1}}$$

exists and has the 2-adic component 0 and the 5-adic 1. Since these two components are distinct rational numbers, L cannot be a rational 10-adic number and hence is irrational. Its canonic series is then necessarily non-periodic.

Let this canonic series be

$$L = \sum_{k=0}^{\infty} L_k 10^k,$$

where the coefficients L_k are again digits 0, 1,..., 9. It is evident that arbitrarily long sequences of digits

$$L_r L_{r-1} \cdots L_1 L_0$$

occur in the decimal fraction for $a(g)$. Here choose for r so large a multiple of the period length u that this sequence is not a repetition of periods P; this is possible because of the irrationality of L. We obtain then a contradiction to the assumed rationality of $a(g)$.

For the second case $g = 5^r h$ the same proof can be applied provided the two primes 2 and 5 interchanged.

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