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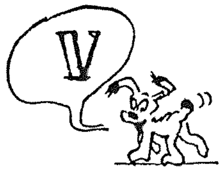
Arithmétix



Sommaire

ON A SPECIAL TRANSCENDENTAL NUMBER

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Extract. - Let $f(z) = \prod_{j=0}^{\infty} (1 - z^{2^j})$. Denote by s and t two integers such that $0 < s < t$. In this paper a measure of transcendency for the number $f(s/t)$ is determined.

On the two similar functions

$$f(z) = \prod_{j=0}^{\infty} (1 - z^{2^j}) \quad \text{and} \quad g(z) = \prod_{j=0}^{\infty} (1 + z^{2^j}) = (1 - z)^{-1}$$

the first one is transcendental and the second one rational. This property has an arithmetic analogue. Let s and t be two integers satisfying $0 < s < t$. Then $f(s/t)$ is a transcendental and $g(s/t)$ a rational number. Some fifty years ago I proved a very general result in which the property of $f(s/t)$ is contained as a special case (see Mahler 1930).

In the present paper I establish a measure of transcendency for $f(s/t)$. I use algebraic approximation formulae for $f(z)$ which are analogous to those for the exponential function in Hermite's classical proof of the transcendency of e (Hermite 1873). The proof is based on the non-vanishing of a certain determinant, and the method has perhaps a slight interest, even if the result itself has not.

1. - The infinite product

$$f(z) = \prod_{j=0}^{\infty} (1-z^{2^j})$$

defines a regular function on the unit disk

$$U : |z| < 1$$

in the complex plane. When z tends along a radius to any 2^jth root of unity, $f(z)$ tends to zero. Since these roots of unity lie everywhere dense on the unit circle $|z| = 1$, this circle is a natural boundary for $f(z)$ and hence $f(z)$ is a transcendental function of z. For if there were a regular point on the unit circle, $f(z)$ would be identically zero, contrary to $f(0) = 1$.

We may compare $f(z)$ with the similar product

$$g(z) = \prod_{j=0}^{\infty} (1+z^{2^j}) = \frac{1}{1-z}$$

which defines a rational function of z.

It is clear that $f(z)$ and all its powers

$$f(z)^k = \sum_{j=0}^{\infty} f_{jk} z^j \qquad (k = 0, 1, 2, \dots)$$

have rational integral Taylor coefficients f_{jk}.

2. - Let

$$a_k(z) = \sum_{h=0}^{m} a_{hk} z^h \qquad (k = 0, 1, \dots, n)$$

be $n+1$ polynomials at most of degree m, with coefficients a_{hk} which have yet to be chosen. Form the n functions

$$r_k(z) = a_o(z) f(z)^k - a_k(z) \qquad (k = 1, 2, \dots, n)$$

and write them as power series

$$r_k(z) = \sum_{j=0}^{\infty} r_{jk} z^j \qquad (k = 1, 2, \dots, n) .$$

It is easily shown that the new coefficients r_{jk} are linear forms in the $(m+1)(n+1)$ numbers a_{hk} with rational integral coefficients. It is therefore possible to find $(m+1)(n+1)$ integers a_{hk} not all zero such that $(m+1)(n+1)-1$ of the coefficients r_{jk} are zero.

In particular, put

$$I = \left[\frac{(m+1)(n+1)-1}{n}\right] = \left[\frac{mn+m+n}{n}\right] = m+1+\left[\frac{m}{n}\right] .$$

Then $nI \leq (m+1)(n+1)-1$, and hence there exist $n+1$ polynomials $a_k(z)$ with integral coefficients not all zero such that the nI linear equations

$$(1): \qquad r_{jk} = 0 \quad \text{for} \quad 0 \leq j \leq I-1 , \quad k = 1, 2, \dots , n$$

are satisfied.

THEOREM 1. - _If_ $m \geq n$, _then none of the polynomials_

$$a_k(z) \qquad\qquad (k = 0, 1, \dots , n)$$

vanishes identically.

Proof. - The hypothesis implies that $I \geq m+2$, hence that

$$r_{jk} = 0 \quad \text{for} \quad 0 \leq j \leq m+1 , \quad k = 1, 2, \dots , n .$$

Therefore each of the n functions $r_k(z)$ has a zero at least of order $m+1$ at $z = 0$.

We show now that if one of the polynomials $a_k(z)$, say the polynomial $a_\varkappa(z)$, is identically zero, then all these $n+1$ polynomials vanish identically, which is false.

If, firstly, $\varkappa = 0$, then for $k = 1, 2, \dots , n$ the function $r_k(z) \equiv -a_k(z)$ can only then have a zero at least of order $m+1$ at $z = 0$ if it vanishes identically ; for $a_k(z)$ is a polynomial at most of degree m.

Secondly, let $1 \leq \varkappa \leq n$. Now $r_\varkappa(z) \equiv a_0(z)\, f(z)^\varkappa$ has a zero at least of order $m+1$ at $z = 0$, and since $f(0) = 1$, the same is true of the polynomial $a_0(z)$ which must therefore vanish identically. But then, by the first case, again all the polynomials $a_k(z)$ vanish identically. This concludes the proof.

3. - From its definition, $f(z)$ satisfies for every positive integer ℓ the functional equation

$$f(z) = (1-z)(1-z^2)\dots(1-z^{2^{\ell-1}})\, f(z^{2^\ell}) .$$

This functional equation remains valid in the trivial case $\ell = 0$ when it reduces to the identity

$$f(z) = f(z) .$$

It is obvious that for $\ell = 0, 1, 2, \dots,$

$$r_k(z^{2^\ell}) = a_o(z^{2^\ell})\, f(z^{2^\ell})^k - a_k(z^{2^\ell}) \qquad (k = 1, 2, \dots, n) .$$

Therefore, on putting

$$a_k^{(\ell)}(z) = ((1-z)(1-z^2)\dots(1-z^{2^{\ell-1}}))^k\, a_k(z^{2^\ell}) \qquad \left(\begin{matrix} k=0, 1, \dots n \\ \ell = 0, 1, 2, \dots \end{matrix}\right),$$

where

$$a_k^{(o)}(z) = a_k(z) \qquad (k = 0, 1, \dots, n),$$

it follows that

$$(2) \qquad ((1-z)(1-z^2)\dots(1-z^{2^{\ell-1}}))^k\, r_k(z^{2^\ell}) = a_o^{(\ell)}(z)\, f(z)^k - a_k^{(\ell)}(z) \qquad \left(\begin{matrix} k = 1, 2, \dots, n \\ \ell = 0, 1, 2, \dots \end{matrix}\right).$$

If it is again assumed that $m \geq n$, then by Theorem 1 none of the polynomials $a_k(z)$ vanishes identically, and hence also

$$(3) \qquad a_k^{(\ell)}(z) \not\equiv 0 \qquad \left(\begin{matrix} k = 0, 1, \dots, n \\ \ell = 0, 1, 2, \dots \end{matrix}\right).$$

From these polynomials form now the determinant of order $n+1$,

$$D(z) = \begin{vmatrix} a_o^{(0)}(z) & a_1^{(0)}(z) & \dots & a_n^{(0)}(z) \\ a_o^{(1)}(z) & a_1^{(1)}(z) & \dots & a_n^{(1)}(z) \\ \vdots & \vdots & & \vdots \\ a_o^{(n)}(z) & a_1^{(n)}(z) & \dots & a_n^{(n)}(z) \end{vmatrix} .$$

This determinant has the following property.

THEOREM 2. - *If* $m \geq n$, *then* $D(z)$ *is not identically zero*.

4. - The proof of Theorem 2 depends on two lemmas of which the first one is well known.

LEMMA 1. - *Associate with each permutation*

$$\Pi = \begin{pmatrix} 0 & 1 & & n \\ k_o & k_1 & \dots & k_n \end{pmatrix}$$

the sum

$$\sigma(\Pi) = 0 . k_o + 1 . k_1 + \dots + n . k_n$$

and denote by Π_o *the special permutation*

$$\Pi_o = \begin{pmatrix} 0 & 1 & 2 & & n \\ n & n-1 & n-2 & \dots & 0 \end{pmatrix}.$$

Then

$$\sigma(\Pi) > \sigma(\Pi_o) \quad \text{if} \quad \Pi \neq \Pi_o .$$

For a proof see Item 368 of Hardy, Littlewood, and Pólya, "Inequalities", Cambridge 1934.

LEMMA 2. - *Denote by*

$$A_k^{(\ell)}(z) \qquad\qquad (k, \ell = 0, 1, \dots, n)$$

a set of $(n+1)^2$ *polynomials such that*

$$A_n^{(0)}(1) \neq 0 , A_{n-1}^{(1)}(1) \neq 0, \dots, A_o^{(n)}(1) \neq 0 .$$

Then the determinant

$$\Delta(z) = \begin{vmatrix} A_o^{(0)}(z) & A_1^{(0)}(z) & A_2^{(0)}(z) & & A_n^{(0)}(z) \\ A_o^{(1)}(z) & (z-1)A_1^{(1)}(z) & (z-1)^2 A_2^{(1)}(z) & \cdots & (z-1)^n A_n^{(1)}(z) \\ A_o^{(2)}(z) & (z-1)^2 A_1^{(2)}(z) & (z-1)^4 A_2^{(2)}(z) & \cdots & (z-1)^{2n} A_n^{(2)}(z) \\ \vdots & \vdots & \vdots & & \vdots \\ A_o^{(n)}(z) & (z-1)^n A_1^{(n)}(z) & (z-1)^{2n} A_2^{(n)}(z) & \cdots & (z-1)^{n^2} A_n^{(n)}(z) \end{vmatrix}$$

is not identically zero.

Proof. - Let the notation be as in Lemma 1. To each of the $(n+1)!$ permutations Π there corresponds a term

$$T(\Pi) = \pm (z-1)^{\sigma(\Pi)} A_{k_o}^{(0)}(z) A_{k_1}^{(1)}(z) \dots A_{k_n}^{(n)}(z)$$

of $\Delta(z)$, and the determinant is the sum of all these terms. In particular, the second diagonal of the determinant gives the term

$$T(\Pi_o) = \pm (z-1)^{\sigma(\Pi_o)} A_n^{(0)}(z) A_{n-1}^{(1)}(z) \dots A_o^{(n)}(z) .$$

By the hypothesis this diagonal term is divisible exactly by $(z-1)^{\sigma(\Pi_o)}$ and by no higher power of $z-1$; on the other hand, Lemma 1 implies that all other terms are divisible by a higher power of $z-1$. Hence they cannot cancel the diagonal term, and therefore $\Delta(z)$ does not vanish identically.

5. - Theorem 2 can now be proved as follows.

By Theorem 1, on account of $m \geq n$, none of the polynomials $a_k(z)$ vanishes identically, and hence the same is true for all the polynomials $a_k^{(\ell)}(z)$. For each suffix k denote by e_k the largest non-negative integer such that

$$a_k(z) \text{ is divisible by } (z-1)^{e_k},$$

and put

$$a_k(z) = (z-1)^{e_k} b_k(z) \qquad (k = 0, 1, \ldots, n) .$$

The polynomials $b_k(z)$ have then the property

$$b_k(1) \neq 0 \qquad (k = 0, 1, \ldots, n) .$$

For all suffixes k and ℓ,

$$a_k(z^{2^{\ell}}) = (2^{2^{\ell}}-1)^{e_k} b_k(z^{2^{\ell}}) = (z-1)^{e_k}(z^{2^{\ell}-1} + z^{2^{\ell}-2} + \ldots + z^2 + z + 1)^{e_k} b_k(z^{2^{\ell}}),$$

where neither of the factors

$$z^{2^{\ell}-1} + z^{2^{\ell}-2} + \ldots + z^2 + z + 1 \quad \text{and} \quad b_k(z^{2^{\ell}})$$

vanishes at $z = 1$. Further the product

$$(1-z)(1-z^2)\ldots(1-z^{2^{\ell-1}}) = \pm (z-1)^{\ell} (1+z)(1+z+z^2+z^3)\ldots(1+z+z^2+\ldots+z^{2^{\ell-1}-1})$$

is divisible by $(z-1)^{\ell}$, but by no higher power of $z-1$.

It follows that

$$a_k^{(\ell)}(z) = a_k(z^{2^{\ell}})((1-z)(1-z^2)\ldots(1-z^{2^{\ell-1}}))^k$$

can be written in the form

$$a_k^{(\ell)}(z) = (z-1)^{e_k + k\ell} A_k^{(\ell)}(z)$$

where the new polynomials $A_k^{(\ell)}(z)$ satisfy for all suffixes k and ℓ the inequality

$$A_k^{(\ell)}(1) = 0 .$$

Hence these polynomials in particular satisfy the weaker conditions of Lemma 2. It is further clear that

$$D(z) = (z-1)^{e_o + e_1 + \ldots + e_n} \Delta(z)$$

where $\Delta(z)$ is the determinant of Lemma 2. Since this lemma may be applied, the assertion of the theorem follows immediately.

6. - The determinant $D(z)$ can be generalised, as follows.

Denote by L any non-negative integers and put

$$D^{(L)}(z) = \begin{vmatrix} a_0^{(L)}(z) & a_1^{(L)}(z) & \cdots & a_n^{(L)}(z) \\ a_0^{(L+1)}(z) & a_1^{(L+1)}(z) & \cdots & a_n^{(L+1)}(z) \\ \vdots & \vdots & & \vdots \\ a_0^{(L+n)}(z) & a_1^{(L+n)}(z) & \cdots & a_n^{(L+n)}(z) \end{vmatrix} .$$

This determinant is connected by a simple formula with the determinant $D(z)$. For by definition,

$$a_k^{(\ell)}(z) = a_k(z^{2^\ell})((1-z)(1-z^2)\dots(1-z^{2^{\ell-1}}))^k$$

and therefore

$$a_k^{(L+\ell)}(z) = a_k(z^{2^{L+\ell}})((1-z)(1-z^2)\dots(1-z^{2^{L-1}})\times(1-z^{2^L})(1-z^{2^{L+1}})\dots(1-z^{2^{L+\ell-1}}))^k ,$$

so that

$$a_k^{(L+\ell)}(z) = a_k^{(\ell)}(z^{2^L})((1-z)(1-z^2)\dots(1-z^{2^{L-1}}))^k \quad \left(\begin{matrix} k = 0, 1, \dots, n \\ \ell = 0, 1, \dots, n \end{matrix}\right).$$

All elements of $D^{(L)}(z)$ in the row of suffix k have the common factor

$$((1-z)(1-z^2)\dots(1-z^{2^{L-1}}))^k ,$$

from which it follows that

$$D^{(L)}(z) = D(z^{2^L})((1-z)(1-z^2)\dots(1-z^{2^{L-1}}))^{n(n+1)/2} . \tag{4}$$

This identity implies the following result.

THEOREM 3. - *Let* z *be a number satisfying*

$$0 < |z| < 1 .$$

Then there exists a positive integer $L_0 = L_0(z)$ *such that*

$$D^{(L)}(z) \neq 0 \quad \textit{for} \quad L \geq L_0 .$$

Proof. - The product

$$((1-z)(1-z^2)\dots(1-z^{2^{L-1}}))^{n(n+1)/2}$$

on the right-hand side of (4) is certainly distinct from zero. Further, by Theorem 2, the determinant $D(z)$ does not vanish identically, hence as a polynomial has the form

$$D(z) = \sum_{j=u}^{v} D_j z^j$$

where u and $v>u$ are two non-negative integers, and where the lowest coefficient D_u is not equal to zero. Hence, as L tends to infinity,

$$D(z^{2^L}) \sim D_u z^{2^L \cdot u}$$

does not vanish as soon as L is sufficiently large. The assertion is therefore an immediate consequence of (4).

7. - From now on let s and $t>s$ be two positive integers, and let x be the positive rational number

$$x = s/t \,, \quad \text{so that} \quad 0<x<1 \,.$$

Hence the function value

$$f(x) = f(x/t) \,, = f \quad \text{say},$$

exists and from its definition as a product satisfies the inequalities

$$0<f<1 \,.$$

It has been known for half a century that f is transcendental (Mahler 1930). We shall establish a measure of transcendency for f.

Since this product will occur often, define $y^{(\ell)}$ by

$$y^{(0)} = 1 \,, \quad y^{(\ell)} = (1-x)(1-x^2)\dots(1-x^{2^{\ell-1}}) \qquad (\ell = 1,2,3,\dots).$$

Thus $y^{(\ell)}$ is a rational number, with the denominator

$$t^{1+2+4+\ldots+2^{\ell-1}} = t^{2^\ell-1},$$

and satisfies the inequalities

$$f < y^{(\ell)} < 1 \qquad (\ell=0,1,2,\ldots).$$

The polynomial values

$$a_k^{(\ell)}(x) \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots}$$

are rational numbers ; we require upper estimates for their numerators and denominators.

The original polynomials

$$a_k(z) \qquad (k=0,1,\ldots,n)$$

are at most of degree m and have integral coefficients which do not depend on ℓ. Denote by $c \geq 1$ the maximum of the absolute values of these coefficients.

It is obvious that the numbers

$$t^{m.2^\ell} a_k(x^{2^\ell}) \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots}$$

are integers ; since

$$1+x+x^2+\ldots = 1/(1-x) = t/(t-s) \leq t,$$

they satisfy the inequalities

$$|t^{m.2^\ell} a_k(x^{2^\ell})| \leq ct.t^{m.2^\ell} \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots}.$$

Now by § 5,

$$a_k^{(\ell)}(x) = y^{(\ell)k} a_k(x^{2^\ell}) \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots}.$$

Therefore also the products

$$t^{(m+n)2^\ell} a_k^{(\ell)}(x), = A_k^{(\ell)} \text{ say} \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots},$$

are integers, and here

$$|A_k^{(\ell)}| \leq ct^{(m+n)2^\ell} \qquad \binom{k=0,1,\ldots,n}{\ell=0,1,2,\ldots}.$$

8. - The equations (2) imply that

$$y^{(\ell)k}\, r_k(x^{2^\ell}) = a_o^{(\ell)}(x) f^k - a_k^{(\ell)}(x).$$

On putting

$$R_k^{(\ell)} = t^{(m+n)2^\ell} y^{(\ell)k} \times r_k(x^{2^\ell}) \qquad \binom{k=1,2,\ldots,n}{\ell=0,1,2,\ldots},$$

we obtain the basic system of equations

$$(5) \qquad R_k^{(\ell)} = A_o^{(\ell)} f^k - A_k^{(\ell)} \qquad \binom{k=1,2,\ldots,n}{\ell=0,1,2,\ldots}.$$

Upper estimates for the left-hand sides of these equations can be derived from the power series

$$r_k(z) = \sum_{j=0}^{\infty} r_{jk} z^k \qquad (k=1,2,\ldots,n)$$

which converge for $z \in U$ and where by the construction in § 2,

$$r_{jk} = 0 \quad \text{for} \quad 0 \leq j \leq I-1 \quad \text{and} \quad k = 1,2,\ldots,n .$$

Here

$$I = m+1+[m/n],$$

and as before it is asssumed that

$$m \geq n .$$

By the convergence of the series for the functions $r_k(z)$ there exists a positive constant C which depends only on m and n such that for all sufficiently small $|z| > 0$

$$|r_k(z)| < C|z|^I \qquad (k=1,2,\ldots,n).$$

Hence there exists a positive integer ℓ_o which depends only on m, n, s and t such that

$$|r_k(x^{2^\ell})| < C(s/t)^{2^\ell I} \quad \text{if} \quad \ell \geq \ell_o \qquad (k=1,2,\ldots,n).$$

Therefore for the same k and ℓ

$$|R_k^{(\ell)}| = |t^{(m+n)2^\ell} y^{(\ell)k} r_k(x^{2^\ell})| < C(s/t)^{2^\ell I} t^{(m+n)2^\ell} = Cs^{2^\ell I} t^{(m+n-I)2^\ell} \qquad \binom{k=1,2,\ldots,n}{\ell \geq \ell_o}.$$

Since $0 < s < t$, s can be written as a power

$$s = t^{\theta}, \quad \text{where} \quad 0 < \theta < 1 .$$

Then

$$s^{2^{\ell} I}\, t^{(m+n-I)2^{\ell}} = t^{(m+n+\theta-I)2^{\ell}} .$$

Here

$$I = m + 1 + [m/n] > m + (m/n)$$

and therefore

$$-(m+n+\theta-I) = I-m-n-\theta > (m/n)-n-1 .$$

Hence we obtain the estimate

$$|R_k^{(\ell)}| < Ct^{-((m/n)-n-1)2^{\ell}} \qquad \left(\begin{matrix} k=1,2,\dots,n \\ \ell \geq \ell_o \end{matrix}\right).$$

Assume from now on that

$$m = 2n(n+1) .$$

Then the earlier condition $m \geq n$ is satisfied, and

$$(m/n)-n-1 = n+1 .$$

The last estimate assumes thus the simpler form

$$|R_k^{(\ell)}| < Ct^{-(n+1)2^{\ell}} \qquad \left(\begin{matrix} k=1,2,\dots,n \\ \ell \geq \ell_o \end{matrix}\right).$$

9.- Let now $X_o, X_1, \dots, X_n$ be any $n+1$ integers such that

$$X = |X_o| + |X_1| + \dots + |X_n| \geq 1 .$$

We want a lower estimate for the expression

$$Z = X_o + X_1 f + \dots + X_n f^n$$

By (5),

$$A_o^{(\ell)} Z = \sum_{k=0}^{n} A_k^{(\ell)} X_k + \sum_{k=1}^{n} R_k^{(\ell)} X_k .$$

Here the first sum

$$\Sigma^{(\ell)} = \sum_{k=0}^{n} A_k^{(\ell)} X_k$$

is an integer and hence is either equal to 0 or has an absolute value at least 1 .

From the estimate above,

$$\left| \sum_{k=1}^{n} R_k^{(\ell)} X_k \right| < CXt^{-(n+1)2^{\ell}} .$$

Hence, if $\Sigma^{(\ell)} \neq 0$, then from the earlier estimate for $A_k^{(\ell)}$,

$$|Z| > c^{-1} t^{-(m+n)2^{\ell}} (1 - CXt^{-(n+1)2^{\ell}}) .$$

In order to satisfy here the condition $\Sigma^{(\ell)} \neq 0$, we apply Theorem 3 for $z = x$. Let $L_o = L_o(x)$ be the integer in this theorem ; without loss of generality $L_o \geq \ell_o$. Then

$$D^{(L)}(x) \neq 0 \quad \text{for} \quad L \geq L_o .$$

It follows that at least one of the $n+1$ linear forms in $X_o, X_1, \ldots, X_n$ defined by

$$\Sigma^{(\ell)} = \sum_{k=0}^{n} A_k^{(\ell)} X_k , \quad \text{where } \ell = L, L+1, \ldots, L+n ,$$

does not vanish, and so the lower estimate for Z may be applied for this suffix ℓ .

Denote then by L^* the smallest integer $\geq \max(L_o, \ell_o)$ for which

$$CXt^{-(n+1)2^{L^*}} \leq 1/2 .$$

There is then an integer ℓ^* between L^* and L^*+n such that $\Sigma^{(\ell^*)} \neq 0$ and therefore

$$|Z| > (2c)^{-1} t^{-(m+n)2^{\ell^*}} \geq (2c)^{-1} t^{-(m+n)2^{L^*+n}} .$$

Let us now assume that the integer X is already so large that

$$CXt^{-(n+1)\max(L_o, \ell_o)} > 1/2 .$$

Then $L^* > \max(L_o, \ell_o)$, and it follows from the definition of L^* that

$$CXt^{-(n+1)2L^*-1} > 1/2 ,$$

hence that

$$t^{-(n+1)2^{L^*}} > (2CX)^{-2} .$$

Hence finally

$$|Z| > (2c)^{-1} (2CX)^{-((m+n)/(n+1))2^{n+1}} .$$

Here

$$(m+n)/(n+1) = (2n(n+1)+n)/(n+1) = 2n+1 - (n+1)^{-1} .$$

The factor

$$X^{+(n+1)^{-1} 2^{n+1}}$$

in the last inequality for Z takes care of the constants as soon as X is sufficiently large, and hence we arrive at the following result.

THEOREM 4. - <u>Let</u> s <u>and</u> t <u>be two integers satisfying</u> $0 < s < t$; <u>let</u> n <u>be any positive integer</u> ; <u>and let</u> $X_o, X_1, \ldots, X_n$ <u>be any</u> $n+1$ <u>integers such that</u>

$$X = |X_o| + |X_1| + \ldots + |X_n|$$

<u>is greater than a certain integer which depends only on</u> s, t <u>and</u> n. <u>Then</u>

$$|X_o + X_1 f(s/t) + \ldots + X_n f(s/t)^n| > X^{-(2n+1)2^{n+1}} .$$

This inequality proves again the transcendency of the number $f(s/t)$ and shows in fact that it is either an S-number or a T-number.

There is no difficulty in replacing the factor $2n+1$ in the exponent by a smaller one. However, this has little interest because there does not seem to be any simple way of improving on the much larger factor 2^{n+1}.

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