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Integers With Digits 0 or 1

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Abstract. Let $g \geqslant 2$ be a given integer and $\mathscr{L}$ the set of nonnegative integers which may be expressed in base g employing only the digits 0 or 1. Given an integer $k > 1$, we study congruences $l \equiv a \pmod{k}$, $l \in \mathscr{L}$ and show that such a congruence either has infinitely many solutions, or no solutions in $\mathscr{L}$. There is a simple criterion to distinguish the two cases. The casual reader will be intrigued by our subsequent discussion of techniques for obtaining the smallest nontrivial solution of the cited congruence.

1. Functional Equations. Let $g \geqslant 2$ be an integer, and let $\mathscr{L}$ be the language of all nonnegative integers which, in their base g representation, employ only the digits 0 or 1. It is easy to see that a generating function $L(X)$ for $\mathscr{L}$ is given by

$$L(X) = \sum_{h \in \mathscr{L}} X^h = \prod_{n=0}^{\infty} (1 + X^{g^n})$$

and it follows readily that L has the functional equation

$$L(X) = (1 + X)L(X^g).$$

Indeed, denote by $\mathscr{P}_t$ the subset of words of $\mathscr{L}$ of at most t digits. Then $\mathscr{P}_t$ has generating function

$$P_t(X) = (1 + X)(1 + X^g) \cdots (1 + X^{g^{t-1}}).$$

Iterating the original functional equation shows that $L(X)$ has the functional equations

$$L(X) = P_t(X)L(X^{g^t}), \qquad t = 1, 2, \ldots.$$

We now show how to 'divide by k'. Let k be a positive integer. In the sums below, ζ runs through the k zeros of $X^k - 1$. Then,

$$k^{-1} \sum_{\zeta} \left(\zeta^{-a} \sum_{h \in \mathscr{L}} (\zeta X)^h \right) = X^a L_a(X^k), \qquad a = 0, 1, \ldots, k-1,$$

where

$$L_a(X) = \sum_{h \in \mathscr{L}_a} X^{(h-a)/k},$$

and

$$\mathscr{L}_a = \{ h \in \mathscr{L} : h \equiv a \pmod{k} \}.$$

Let G be any positive integer and consider a sum

$$\sum_{\zeta} \left(\zeta^{-a} (\zeta X)^l \sum_{h \in \mathscr{L}} (\zeta X)^{hG} \right) = \sum_{\zeta} \sum_{h \in \mathscr{L}} \zeta^{hG-(a-l)} X^{hG+l}.$$

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The surviving terms are those with $h \in \mathscr{L}$ and

$$Gh \equiv a - l \pmod{k}.$$

Set $(G, k) = D$. The congruence has no solution unless D divides $a - l$ in $\mathbb{Z}$. If $D \mid (a - l)$, then the congruence has D distinct solutions mod k. If c is one solution, then the D solutions are $c + jk' \pmod{k}$, $j = 0, 1, \ldots, D - 1$, where we have set $k' = k/D$. Further, set $G' = G/D$. Denote by c the solution to the congruence so that $0 \leqslant Gc - (a - l) < G'k$ and set $rk = Gc - (a - l)$. Then the sum we are considering becomes

$$kX^a \sum_{j=0}^{D-1} X^{(r+jG')k} L_{c+jk'}(X^{kG}),$$

where the suffixes $c + jk'$ are to be interpreted mod k so as to lie in $\{0, 1, \ldots, k - 1\}$.

Fix t and set $G = g^t$. Then, we have shown that for each $a = 0, 1, \ldots, k - 1$,

$$L_a(X) = \sum_{l \in \mathscr{P}_t:\ l \equiv a \pmod{D}} X^{r_l} \sum_{j=0}^{D-1} X^{jG'} L_{c_l + jk'}(X^G).$$

Here $c_l \equiv (a - l)/G \pmod{k'}$ and $0 \leqslant kr_l = Gc_l - (a - l) < G'k$; and the suffixes $c_l + jk'$ are to be interpreted mod k.

What of all this? It is plain that if for some t there is no $l \in \mathscr{P}_t$ so that $a \equiv l \pmod{D}$, where $D = (g^t, k)$, then $L_a(X) = 0$, so $\mathscr{L}_a$ is empty. But little else seems obvious. In fact, however, we are essentially finished:

Evidently, either $(g, k) = 1$, in which case we set $m = 1$, or there is an $m > 0$ so that $(g^{m-1}, k) < (g^m, k) = (g^{m+1}, k)$. In either case, we set $(g^m, k) = D$. We note that for all $t \geqslant m$, we have $(g^t, k) = D$. Moreover, with $k' = k/D$ we have $(g, k') = 1$. Hence, there are integers $t \geqslant m$ so that $g^t \equiv 1 \pmod{k'}$. Below, suppose for convenience that t has this property. Then $G = g^t \equiv 1 \pmod{k'}$, so $c_l \equiv a - l \pmod{k'}$ and $kr_l = (G - 1)(a - l) + iG'k$, with the integer i so chosen that $0 \leqslant r_l < G'$. Our choice of t makes it easier to explicitly survey the functional equations.

THEOREM. *Let $g \geqslant 2$ and $k \geqslant 1$ be integers, and let $\mathscr{L}$ be the set of nonnegative integers which in their base g representation employ only the digits* 0 *or* 1. *For each $a = 0, 1, \ldots, k - 1$, denote by $\mathscr{L}_a$ the subset of those $h \in \mathscr{L}$ satisfying the congruence $h \equiv a \pmod{k}$. If $(g, k) = 1$, set $m = 1$ and $D = 1$. Otherwise, there is a unique positive integer m, such that $(g^{m-1}, k) < (g^m, k) = (g^{m+1}, k)$, and we write $(g^m, k) = D$. Let $\mathscr{P}_m$ be the subset of elements of $\mathscr{L}$ of at most m digits. Then, $\mathscr{L}_a$ is infinite if and only if there is an $l \in \mathscr{P}_m$ so that $a \equiv l \pmod{D}$. Otherwise, $\mathscr{L}_a$ is empty. In particular (since the condition is empty if $D = 1$), each $\mathscr{L}_a$ $(a = 0, 1, \ldots, k - 1)$ is infinite if $(g, k) = 1$.*

Proof. Take $l \in \mathscr{L}$. Since $D \mid g^m$, there is no loss of generality, when studying $l \pmod{D}$, in supposing that $l \in \mathscr{P}_m$. But if $l \equiv a \pmod{k}$, then, because $D \mid k$, a fortiori $l \equiv a \pmod{D}$. Hence, plainly, $\mathscr{L}_a$ is indeed empty if there is no $l \in \mathscr{P}_m$ such that $a \equiv l \pmod{D}$.

Conversely, suppose that the criterion is satisfied for a but that $\mathscr{L}_a$ is finite. We shall show that then all $\mathscr{L}_a$ are finite, which is absurd because $\mathscr{L} = \bigcup_{a=0}^{k-1} \mathscr{L}_a$ and $\mathscr{L}$ is infinite. Firstly, suppose $(g, k) = 1$, and, as suggested, choose t such that $g^t \equiv 1 \pmod{k}$. Recall that the series L_c have nonnegative coefficients (indeed only the coefficients 0 or 1), so that L_a a polynomial implies that each L_c, with $c \equiv a - l \pmod{k}$ and $l \in \mathscr{P}_t$, is a polynomial. Since $1 \in \mathscr{P}_t$, in particular L_{a-1} is a polynomial. Iterating this remark (and, of course, interpreting the suffix mod k) implies that every L_a is a polynomial ($a = 0, 1, \ldots, k-1$), which is a contradiction. We now return to the general case, noticing that we have already shown that $\bigcup_{j=0}^{D-1}\mathscr{L}_{a+jk'}$ is infinite, for this is a congruence subset mod k' of $\mathscr{L}$ and $(g, k') = 1$. But L_a a polynomial implies that there is a c so that each of the $L_{c+jk'}$ ($j = 0, 1, \ldots, D-1$) is a polynomial and this already contradicts the remark just made.

Before mentioning some examples, we prove a simple auxiliary result.

LEMMA. *Distinct elements of $\mathscr{P}_m$ are incongruent modulo D.*

Proof. If $l \neq l'$, then reading from the right, $l - l'$ has a first nonzero digit, say its nth digit, the coefficient of g^{n-1}. Set $D_i = (g^i, k)$ and note that $1 = D_0 < D_1 < \cdots < D_m = D$. Evidently, $l - l' = \pm g^{n-1} \pmod{D_n}$. Thus $l \not\equiv l' \pmod{D}$, seeing that $D_n \mid D$, but $D_{n-1} < D_n$, so $g^{n-1} \not\equiv 0 \pmod{D_n}$.

Example 1. Take $g = 6$, $k = 15$. Here, $m = 1$, $D = 3$. For $\mathscr{L}_a$ not to be empty, we require that there be an $l \in \mathscr{P}_1$ with $a \equiv l \pmod 3$, which is $a \equiv 0$ or $1 \pmod 3$. Hence, the congruence subsets $\mathscr{L}_2(6; 15)$, $\mathscr{L}_5(6; 15)$, $\mathscr{L}_8(6; 15)$, $\mathscr{L}_{11}(6; 15)$ and $\mathscr{L}_{14}(6; 15)$ are empty; the other $\mathscr{L}_a(6; 15)$ are infinite.

Example 2. Take $g = 6$, $k = 45$. Here, $m = 2$, $D = 9$. We require that there be an $l \in \mathscr{P}_2$ with $a \equiv l \pmod 9$. The elements of $\mathscr{P}_2(6)$ are $0, 1, 6$ and 7. Thus the 25 congruence subsets $\mathscr{L}_a(6; 45)$ with $a \equiv 2, 3, 4, 5$ or $8 \pmod 9$ are empty.

Example 3. Take $g = 6$, $k = 351 = 13 \times 27$. Here, $m = 3$, $D = 27$, and noting that $g^2 \equiv 9 \pmod{27}$, the elements of $\mathscr{P}_3$ modulo 27 are $0, 1, 6, 7, 9, 10, 15$ and 16. Hence there are $13(27 - 8) = 247$ subsets $\mathscr{L}_a(6; 351)$ that are empty.

Example 4. On the other hand, take g, k so that $D = 2^m$. There are 2^m elements in $\mathscr{P}_m$ and, by the lemma, they are distinct modulo D. In this case every subset $\mathscr{L}_a(g; k)$ is infinite, notwithstanding $D > 1$.

2. The Smallest Nontrivial Element of a Congruence Subset of $\mathscr{L}$. In the previous section we expressed the generating functions $L_a(X)$ as sums of series

$$X^{r_l} L_{c_l}(X^G).$$

We chose $0 \leqslant r_l < G$ and interpreted $c_l \bmod k$. We might equivalently have chosen $0 \leqslant c_l < k$ and have interpreted $r_l \bmod G$. In either case, $kr_l = Gc_l - (a - l)$. It is easy to see that $L_c(X)$ has nonzero constant term if and only if $c \in \mathscr{L}$, $0 \leqslant c < k$. Hence, the terms of degree less than $G = g^t$ in $L_a(X)$ are given by X^{r_l} for those l so that $c_l \in \mathscr{L}$ and $0 \leqslant c_l < k$.

Example 5. Take $g = 10$, $k = 9$ and $a = 0$. Here, $m = 1$, $D = 1$. Moreover, $10^t \equiv 1 \pmod 9$ for all $t = 1, 2, \ldots$. The only elements of $\mathscr{L}$ less than $k = 9$ are 0 and 1. But $c = 0$ yields only $r = 0$, which is trivial. So, consider $1 = c_l \equiv 0 - l \pmod 9$. The smallest $l \in \mathscr{L}$ satisfying this congruence is $111\,11111 = (10^8 - 1)/9$, and it is an element of $\mathscr{P}_8$. In fact, $10^8 \equiv 1 \pmod 9$, so 8 is a 'convenient' value for t. We have $9r_l = 10^8 \times 1 + (10^8 - 1)/9$, so $r_l = (10^9 - 1)/9^2 = 12345679$. The smallest nontrivial element of $\mathscr{L}_0(10; 9)$ thus is $9 \times 12345679 = 1111\,11111$. Note that only $l = 0$ and $l = (10^8 - 1)/9$ in $\mathscr{P}_8$ yield c_l with $c_l \in \mathscr{L}$.

Example 6. Take $g = 10$, $k = 36$ and $a = 0$. Here, $m = 2$, $D = 4$; so $k' = 9$. As above, all $t = 2, 3, \ldots$ are convenient. The only elements of $\mathscr{L}$ less than $k = 36$ are $0, 1, 10$ and 11. Consider $11 = c_l \equiv 0 - l \pmod 9$ and $l \equiv 0 \pmod 4$. The smallest $l \in \mathscr{L}$ satisfying these congruences is $1111\,11100 = 10^2(10^7 - 1)/9$ (obviously

$m = 2$ implies $10^2 \mid l$), and it is an element of $\mathscr{P}_9$. We have $36r_l = 10^9 \times 11 + 10^2(10^7 - 1)/9$, so $r_l = 10^2(10^9 - 1)/4 \cdot 9^2 = 3086\ 41975$. It is easy to check that the only $l \in \mathscr{P}_9$ yielding $c_l \in \mathscr{L}$ are $l = 0$ and $l = 10^2(10^7 - 1)/9$. So the smallest nontrivial element of $\mathscr{L}_0(10; 36)$ is $36 \times 3096\ 41975 = 1\ 11111\ 11100$.

Example 7. Take $g = 7$, $k = 66$ and $a = 0$. Here, $m = 1$, $D = 1$ and $7^{10} \equiv 1 \pmod{66}$ with only multiples of 10 being convenient values of t. The only elements of $\mathscr{L}$ less than 66 are $0, 1, 7, 8, 49, 50, 56$ and 57. We note

n	0	1	2	3	4	5	6	7	8	9
$7^n \pmod{66}$	1	7	−17	13	25	−23	−29	−5	31	19.

One might notice that $1011111)_7 = 120450 = 66 \times 1825$ thus chancing upon the smallest nontrivial element of $\mathscr{L}_0(7; 66)$. But this is unsatisfying. We accordingly forget about 'convenient' t and, using the hint just provided, we look at the functional equation for $L_0(X)$ with $t = 6$. Of the $2^6 = 64$ elements of $\mathscr{P}_6(7)$, there happened to be 6, so that with $7^6 c_l \equiv -l \pmod{66}$, we obtain $c_l \in \mathscr{L}$. The relevant pairs l, c_l are 0, $c_l = 0$ and $11111)_7$, $c_l = 1$; $1\ 01011)_7$, $c_l = 50$; $101101)_7$, $c_l = 56$; $1\ 10001)_7$, $c_l = 57$; and $1\ 11100)_7$, $c_l = 50$. In each case, we have $66r_l = 7^6 c_l + l$ yielding, as smallest nontrivial element of $\mathscr{L}_0(7; 66)$, the element $(6 \times 7^6 + 7^5 - 1)/6 = 66 \times 1825$, as we had already guessed. In fact, the final case shows us that $t = 4$ would have sufficed, yielding with $l \in \mathscr{P}_4$ $c_l \in \mathscr{L}$, the two pairs 0, $c_l = 0$ and $1111)_7$, $c_l = 50$. The latter provides $66r_l = 7^4 \times 50 + 400 = 66 \times 1825$ as expected.

We conclude that convenient t may be inconveniently large.

Example 8. Take $g = 11$, $k = 40$ and $a = 0$. Here, $m = 1$, $D = 1$ and $11^2 \equiv 1 \pmod{40}$ so any even t is convenient. The elements of $\mathscr{L}$ less than 40 are $0, 1, 11, 12$.

In $\mathscr{P}_{11}$ one first finds l so that $c_l \in \mathscr{L}$. The pair providing the smallest positive r_l is $l = 1\ 01011\ 11111)_{11}$, $c_l = 11$, yielding $r_l = 7\ 91145\ 52723$. Thus, the smallest nontrivial element of $\mathscr{L}_0(11; 40)$ is $40r_l = 101\ 01011\ 11111)_{11}$. In this case, it is as if the smallest convenient t is inconveniently small. In fact, the only arithmetic required is $11^{2n} \equiv 1$, $11^{2n+1} \equiv 11 \pmod{40}$, and a look at $\mathscr{P}_{13}$ allows one to chance directly upon the sought for element of $\mathscr{L}_0$.

In concluding this section, we remark that our functional equations do not play an essential role in determining the smallest nontrivial element of a subset $\mathscr{L}_a(g; k)$. Indeed, for $h = 0, 1, \ldots$ set $b_h \equiv g^h \pmod{k}$, with $0 \leqslant b_h < k$ uniquely determining the b_h. The sequence $\mathscr{B} = (b_h)$ is, of course, periodic and one readily verifies that the sequence has preperiod of length m and period of length t, where $t > 0$ is minimal so that $g^t \equiv 1 \pmod{k'}$. In particular, if $(g, k) = 1$ then $\mathscr{B}$ is pure-periodic. In general, we may write:

$$\mathscr{B}(g; k) = \left\{ b_0, \ldots, b_{m-1}, \overline{b_m, \ldots, b_{m+t-1}} \right\}.$$

To find elements of, say, $\mathscr{L}_0$ we need only notice that $l \in \mathscr{L}_0$ implies $g^m \mid l$ so

$$l/g^m \equiv x_0 b_0 + \cdots + x_{t-1} b_{t-1} \equiv 0 \pmod{k'},$$

with nonnegative integers $x_0, \ldots, x_{t-1}$. Indeed, there is an evident correspondence between elements of $\mathscr{L}_0$ and such t-tuples $x_0, \ldots, x_{t-1}$. At the small cost of some extra notation, we may give a similar description of the elements of any $\mathscr{L}_a$, thereby obtaining an elementary proof of our Theorem.

We have made some brief suggestions as to how one might find, or, more usefully, verify that one has found the least nontrivial element of sets $\mathscr{L}_a(g; k)$. We recall that for $a = 0$ such sets are always infinite, and we denote by $\mathscr{M} = \mathscr{M}(g; k)$ the least positive multiple of k whose base g digits are 0 or 1. The arithmetic functions

$$k \mapsto \mathcal{M}(g; k)$$

seem quite complicated and it would be interesting to understand them more fully. To this end, we include a brief table listing $\mathcal{M}$ for $3 \leqslant g \leqslant 12$ and $1 \leqslant k \leqslant 100$. For compactness, elements of $\mathcal{M}$ are given in octal; thus the symbols in the body of the table are to be read as:

$$0:000 \quad 1:001 \quad 2:010 \quad 3:011 \quad 4:100 \quad 5:101 \quad 6:110 \quad 7:111,$$

thereby transforming the entries to their base g representation which, of course, employs only the digits 0 and 1.

$\mathcal{M}(g; k)$

k	g = 3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1
2	3	2	3	2	3	2	3	2	3	2
3	2	7	3	2	7	3	2	7	3	2
4	3	2	17	4	3	2	17	4	3	2
5	5	3	2	37	5	5	3	2	37	5
6	6	16	3	2	77	6	6	16	3	2
7	11	7	11	3	2	177	7	11	7	11
8	17	4	27	10	3	2	377	10	17	4
9	4	15	11	4	13	3	2	777	11	4
10	5	6	6	76	5	12	3	2	1777	12
11	37	37	37	15	23	27	37	3	2	3777
12	6	16	17	4	77	6	36	34	3	2
13	7	11	5	27	13	5	7	11	35	3
14	11	16	11	6	6	376	77	22	77	22
15	12	77	6	76	31	17	6	16	127	12
16	33	4	27	20	17	4	577	20	33	4
17	23	5	57	31	47	21	21	35	13	27
18	14	32	11	4	157	6	6	1776	11	4
19	43	47	67	53	7	11	35	31	7	11
20	17	6	36	174	17	12	17	4	1777	12
21	22	7	11	6	16	537	16	25	25	22
22	71	76	53	32	35	56	65	6	6	7776
23	45	13	45	15	53	51	31	65	47	35
24	36	34	377	10	77	6	776	70	17	4
25	61	33	4	75	5	137	33	4	67	67
26	77	22	5	56	71	12	77	22	35	6
27	10	15	33	10	13	11	4	1577	33	10
28	11	16	33	14	6	376	115	44	77	22
29	133	23	103	127	177	123	45	155	145	5
30	12	176	6	76	137	36	6	16	1777	12
31	13	37	7	11	47	37	71	73	115	53
32	47	10	65	40	33	4	737	40	47	10
33	76	51	135	32	23	41	76	77	6	7776
34	137	12	71	62	47	42	21	72	65	56
35	55	77	22	1777	12	477	77	22	155	55
36	14	32	33	4	157	6	36	3774	11	4
37	15	117	53	5	111	101	111	7	11	111
38	113	116	71	126	77	22	35	62	77	22
39	16	25	17	56	13	17	16	25	173	6
40	17	14	56	370	17	12	377	10	12577	24
41	21	41	27	107	155	47	5	37	161	71
42	22	16	11	6	176	1276	176	52	77	22
43	327	177	117	7	11	67	73	155	177	51
44	71	76	53	64	41	56	65	14	6	7776
45	24	173	22	174	31	17	6	1776	165	24
46	157	26	131	32	53	122	175	152	47	72
47	27	337	75	133	71	155	105	23	145	217
48	66	34	677	20	7777	14	1376	160	33	4
49	275	43	103	33	4	375	61	141	43	275
50	137	66	14	172	5	276	33	4	3677	156

$\mathscr{M}(g; k)$

k	$g =$ 3	4	5	6	7	8	9	10	11	12
51	46	137	71	62	103	63	42	43	207	56
52	77	22	17	134	71	12	7777	44	173	6
53	107	147	217	105	223	115	121	43	45	145
54	30	32	33	10	273	22	14	3376	33	10
55	207	47	76	73	137	175	113	6	76	13577
56	33	34	161	30	6	376	1337	110	113	44
57	106	111	71	126	7	11	72	31	25	22
58	353	46	115	256	207	246	157	332	145	12
59	35	557	65	373	43	267	43	337	237	53
60	36	176	36	174	473	36	36	34	1777	12
61	41	61	221	115	31	131	37	45	5	225
62	161	76	77	22	47	76	71	166	115	126
63	44	777	11	14	26	37777	16	1737	77	44
64	47	10	65	100	33	4	737	100	71	10
65	61	11	12	73	101	5	77	22	67	17
66	162	122	151	32	137	102	152	176	6	7776
67	225	433	127	107	513	227	263	153	153	145
68	355	12	71	144	47	42	63	164	207	56
69	112	13	157	32	237	173	62	205	47	72
70	55	176	22	3776	12	1176	77	22	11377	132
71	73	73	37	163	207	243	265	23	75	147
72	74	64	757	10	157	6	776	7770	33	4
73	101	111	213	225	23	7	11	21	47	255
74	123	236	53	12	333	202	333	16	11	222
75	142	237	14	172	137	151	66	34	165	156
76	113	116	71	254	77	22	35	144	77	22
77	135	205	103	41	46	337	43	11	16	13577
78	176	52	17	56	273	36	176	52	173	6
79	341	657	153	121	165	227	225	223	13	253
80	377	14	56	760	17	24	177777	20	12737	24
81	20	15	127	20	327	33	4	1775	207	20
82	21	102	27	216	161	116	5	76	161	162
83	463	307	75	435	471	203	107	53	27	471
84	22	16	33	14	176	1276	232	124	77	22
85	23	17	136	153	221	125	63	72	163	151
86	327	376	225	16	11	156	317	332	633	122
87	266	23	165	256	355	237	112	327	207	12
88	71	174	53	150	41	56	2177	30	36	17774
89	577	215	131	31	251	47	253	325	27	21
90	24	366	22	174	175	36	6	1776	3377	24
91	25	25	55	157	26	1467	7	11	421	11
92	157	26	131	64	237	122	225	324	47	72
93	26	2067	25	22	211	135	162	203	371	126
94	27	676	317	266	71	332	317	46	145	436
95	113	47	156	165	43	55	207	62	77777	55
96	116	70	773	40	15777	14	1676	340	47	10
97	325	147	35	101	117	145	35	341	241	73
98	275	106	275	66	14	772	365	302	257	572
99	174	51	671	64	301	41	76	777777	22	17774
100	151	66	74	364	17	276	33	4	3677	156

3. Remarks. The power series $L(X)$ and the nontrivial $L_a(X)$ that appear in the present note are transcendental functions with the unit circle as their natural boundary. Indeed, their only coefficients are 0 and 1, and each series has arbitrarily long sequences of zero coefficients (cf. Pólya and Szegö [3], Mahler [2]). Arithmetic properties of functions satisfying functional equations as in the present case were studied by the second author and have recently become the subject of further extensive investigation. In particular, if α is algebraic and $0 < |\alpha| < 1$, then $L_a(\alpha)$ is transcendental whenever $\mathscr{L}_a$ is nonempty; and $L(\alpha)$ is transcendental. Moreover, interest in these matters has been heightened by the realization that the class of functions, of which the present ones are examples, is the class of generating functions of sequences recognized by finite automata. For an informal introduction see FOLDS! [1], especially pp. 178ff.

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