ACTA ARITHMETICA
LIII (1989)

The representation of squares to the base 3

by

KURT MAHLER

In memory of V. G. Sprindžuk

1. Let N be the set of all positive integers. Each $x \in N$ has a unique representation to the base 3 of the form

(1) $$x = \sum_{h=0}^{n} d_h \cdot 3^h,$$

or in abbreviated form

(2) $$x = d_n d_{n-1} \ldots d_1 d_0(3).$$

Here the *digits* d_h have the allowed values 0, 1, or 2, and $n \geqslant 0$ is the largest suffix h such that $d_h > 0$. Those of the digits d_h of x distinct from zero will be called the *essential digits* of x.

Define three subsets I, J, and K of N, as follows.

I consists of all $x \in N$ with digits 0 or 1.

J consists of all $x \in N$ with digits 0 or 2.

K consists of all $x \in N$ with at least one digit 1, at least one digit 2, and with any number $\geqslant 0$ of digits 0.

The three sets I, J, and K evidently are disjoint in pairs, and their union is equal to N. From now on, for $x \in N$, $f(x)$ denotes that set I, J, or K, which contains x. This function $f(x)$ evidently is well defined.

One finds that for the first 100 integers $x = 1, 2, \ldots, 100$ the value of $f(x)$ is 23 times equal to I, 15 times equal to J, and 62 times equal to K. More generally, as x runs from 1 to a large integer X, both equations $f(x) = I$ and $f(x) = J$ hold only $o(X)$ times, while the number of solutions of $f(x) = K$ is asymptotic to X.

2. Two integers x and y in N are called *related* if their quotient x/y is an integral power of 3. Such related integers have the same sets of essential digits, but need not agree in the sets of their digits 0. Related integers x and y satisfy the equations

(3) $$f(x) = f(y) \quad \text{and} \quad f(x^2) = f(y^2).$$

The integer $x \in N$ is said to be *reduced* if its lowest digit d_0 does not vanish. Since

(4) $$x \equiv d_0 \pmod 3,$$

this is the case if and only if $(x, 3) = 1$, i.e., x and 3 are relatively prime.

It is easily seen that to every integer $y \in N$ there exists a related integer x which is reduced. In order to study the values $\{f(x) | \; x \in N\}$ it suffices therefore to consider reduced integers.

3. The equation

(5) $$x' = 2x$$

establishes a reversible 1-to-1 correspondence between the elements x of I and the elements x' of J. If x' has the representation

$$x' = \sum_{h=0}^{n} d'_h 3^h = d'_n d'_{n-1} \ldots d'_1 d'_0(3)$$

to the base 3, then the digits d'_h of x' are connected with the digits d_h of x by the formulae

(6) $$d'_h = 2d_h \quad \text{for} \quad h = 0, 1, \ldots, n.$$

Thus the essential digits of x are equal to 1 and those of x' are equal to 2. The elements of J are even integers, while those of I may be even or odd.

4. The following problem will be studied.

Problem. *If $x \in N$, and $f(x)$ is given, find $f(x^2)$.*

A first almost obvious result is as follows.

Theorem 1. *If $x \in N$, then always*

$$f(x^2) \neq J.$$

Proof. If the assertion is false, then there exists an element y of N such that $f(y^2) = J$. There also exists a second element x of N such that

(i) x and y are related, and

(ii) x is reduced.

Here by (i) and (3),

$$f(x^2) = f(y^2) = J.$$

Further, by (ii), the lowest digit d_0 of x does not vanish, and therefore the lowest digit, $d_0^{(2)}$ say, of x^2, also is distinct from 0. Since $f(x^2) = J$, $d_0^{(2)}$ necessarily is equal to 2, and so it follows that

$$x^2 \equiv d_0^{(2)} = 2 \pmod 3.$$

However, 2 is a quadratic non-residue of 3, giving a contradiction.

One can easily find examples of elements x of N such that all but one of the essential digits of x^2 are equal to 2, while to lowest digit $d_0^{(2)}$ is equal to 1.

5. Let us next consider simultaneously any pair x, x' of integers in N such that

$$x' = 2x, \quad f(x) = I, \quad \text{and} \quad f(x') = J.$$

Let again d_n run over the digits of x and d'_h over the digits of x', so that $d'_h = 2d_h$ for $h = 0, 1, \ldots, n$.

Assume, say, that x has $r \geqslant 1$ essential digits, and that these essential digits correspond to the suffixes

$$h = j_1, j_2, \ldots, j_r$$

where without loss of generality

$$j_1 > j_2 > \ldots > j_r \geqslant 0.$$

Then by the hypothesis,

$$d_h = 1 \quad \text{and} \quad d'_h = 2 \quad \text{for} \quad h = j_1, j_2, \ldots, j_r,$$

while $d_h = d'_h = 0$ for all other suffixes h. It follows that, by the representations to the base 3 of x and x', these integers can also be written as

(7) $$x = 3^{j_1} + 3^{j_2} + \ldots + 3^{j_r} \quad \text{and} \quad x' = 2(3^{j_1} + 3^{j_2} + \ldots + 3^{j_r}),$$

thus as 1 times or 2 times a sum of r distinct integral powers of 3. This property will now enable us to evaluate $f(x^2)$ and $f(x'^2)$, where the result depends however on r.

In the lowest case $r = 1$, (7) states that

$$x = 3^{j_1} \quad \text{and} \quad x' = 2 \times 3^{j_1},$$

hence

$$x^2 = 3^{2j_1} \quad \text{and} \quad x'^2 = 4x^2 = 3^{2j_1+1} + 3^{2j_1}.$$

This means that the representations of x^2 and x'^2 have 1 or 2 essential digits, respectively, where these digits are in both cases equal to 1 and where all other digits are 0. Hence

(8) $$f(x^2) = f(x'^2) = I \quad \text{if} \quad r = 1.$$

Next let $r = 2$, so that

$$x = 3^{j_1} + 3^{j_2} \quad \text{and} \quad x' = 2(3^{j_1} + 3^{j_2}),$$

where

$$j_1 > j_2 \geqslant 0.$$

On squaring x and x',

$$x^2 = 3^{2j_1} + 2 \times 3^{j_1+j_2} + 3^{2j_2}, \quad x'^2 = 4x^2 = 4(3^{2j_1} + 2 \times 3^{j_1+j_2} + 3^{2j_2}).$$

In the first equation,

$$2j_1 > j_1 + j_2 > 2j_2 \geqslant 0,$$

so that this representation of x^2 is already its representation to the base 3. We see that x^2 has three essential digits, and of these two are equal to 1 and one is equal to 2, showing that

$$f(x^2) = K.$$

In the formula for x'^2 we split the factor 4 into $3+1$ and find that

$$x'^2 = (3^{2j_1+1} + 3^{2j_1}) + 2(3^{j_1+j_2+1} + 3^{j_1+j_2}) + (3^{2j_2+1} + 3^{2j_2}).$$

If now $j_1 \geqslant j_2 + 2$, then

$$2j_1 + 1 > 2j_1 > j_1 + j_2 + 1 > j_1 + j_2 > 2j_2 + 1 > 2j_2 \geqslant 0,$$

so that by the last formula x'^2 has four essential digits 1, and two essential digits 2, whence

$$f(x'^2) = K.$$

If, however, $j_1 = j_2 + 1$, the formula for x'^2 can be simplified to

$$x'^2 = 2 \times 3^{2j_2+3} + 3^{2j_2+2} + 3^{2j_2}$$

and shows that x'^2 has one essential digit 2 and two essential digits 1, so that also in this case

$$f(x'^2) = K.$$

Hence

(9) $$f(x^2) = f(x'^2) = K \quad \text{if} \quad r = 2.$$

Finally assume that $r \geqslant 3$. Then x can be split into the sum

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \sum_{h=1}^{r-2} 3^{j_h} \quad \text{and} \quad x_2 = \sum_{h=r-1}^{r} 3^{j_h},$$

while

$$x^2 = x_1^2 + 2x_1x_2 + x_2^2 \quad \text{and} \quad x'^2 = 4x^2 = 4(x_1^2 + 2x_1x_2 + x_2^2).$$

Here

$$x_1^2 = \sum_{h=1}^{r-2}\sum_{k=1}^{r-2} 3^{j_h+j_k}, \quad x_1x_2 = \sum_{h=1}^{r-2}\sum_{k=r-1}^{r} 3^{j_h+j_k}, \quad x_2^2 = 3^{2j_{r-1}} + 2 \times 3^{j_{r-1}+j_r} + 3^{2j_r}.$$

In the double sums for x_1^2 and x_1x_2 all the exponents $j_h + j_k$ are at least equal to $j_{r-2} + j_r$ and therefore are greater than the exponents $j_{r-1} + j_r$ and $2j_r$ in the second and third term of x_2^2 because by hypothesis

$$j_1 > j_2 > \ldots > j_r \geqslant 0.$$

It follows that in the representation of x^2 to the base 3 there is at least one essential digit 2 and at least one essential digit 1, whence

$$f(x^2) = K.$$

Finally x'^2 can be written in the form

(10) $$x'^2 = 4x^2 = (3+1)x_1^2 + 2(3+1)x_1x_2 + (3+1)x_2^2$$

where

$$(3+1)x_1^2 = \sum_{h=1}^{r-2}\sum_{k=1}^{r-2} (3^{j_h+j_k+1} + 3^{j_h+j_k}),$$

$$2(3+1)x_1x_2 = 2\sum_{h=1}^{r-2}\sum_{k=r-1}^{r} (3^{j_h+j_k+1} + 3^{j_h+j_k}),$$

and

$$(3+1)x_2^2 = (3^{2j_{r-1}+1} + 3^{2j_{r-1}}) + 2(3^{j_{r-1}+j_r+1} + 3^{j_{r-1}+j_r}) + (3^{2j_r+1} + 3^{2j_r}).$$

Again the exponents of 3 in the terms $4x_1^2$ and $8x_1x_2$ are not less than $j_{r-2} + j_r$ and hence are greater than the exponents $j_{r-1} + j_r$ and $2j_r$ of 3 in $4x_2^2$. The terms $2 \times 3^{j_{r-1}+j_r}$ and 3^{2j_r} of $4x_2^2$ are therefore not affected by the other terms in the expression (10) for x'^2. Hence at least one essential digit in the representation of x'^2 to the base 3 is equal to 2 and one is equal to 1. This concluded the proof of

(11) $$f(x^2) = f(x'^2) = K \quad \text{if} \quad r \geqslant 3.$$

On combining the partial results (8), (9), and (11), the following theorem is obtained.

THEOREM 2. *Let x and $x' = 2x$ be two elements of N such that*

$$f(x) = I \quad \textit{and} \quad f(x') = J.$$

If x is an integral power of 3 and hence x' is twice such a power, then

$$f(x^2) = f(x'^2) = I.$$

Otherwise

$$f(x^2) = f(x'^2) = K.$$

6. There remains the evaluation of $f(x^2)$ when $x \in N$ satisfies

$$f(x) = K.$$

By Theorem 1 either $f(x^2) = I$ or $f(x^2) = K$. We can show the following result.

THEOREM 3. *The set K contains infinitely many integers x prime to 3 such that*

$$f(x^2) = I,$$

and it also contains infinitely many such integers such that

$$f(x^2) = K.$$

Here the restriction $(x, 3) = 1$ has been imposed since without it the assertion of the theorem is obvious.

A small table of the development of x^2 to the base 3 shows that for $x = 1, 2, \ldots, 60$ the equation

$$f(x^2) = I$$

holds for the 16 integers,

$$x = 1, 2, 3, 6, 9, 11, 16, 18, 19, 27, 29, 33, 48, 54, 55, 57.$$

However the condition $(x, 3) = 1$ excludes the 9 integers

$$3, 6, 9, 18, 27, 33, 48, 54, 57,$$

and leaves only the seven reduced integers

$$1, 2, 11, 16, 19, 29, 55.$$

Table 1

x	x^2	$x^2(3)$	x	x^2	$x^2(3)$
1	1	1	31	961	1022121
2	4	11	32	1024	1101221
3	9	100	33	1089	1111100
4	16	121	34	1156	1120211
5	25	221	35	1225	1200101
6	36	1100	36	1296	1210000
7	49	1211	37	1369	1212201
8	64	2101	38	1444	1222111
9	81	10000	39	1521	2002100
10	100	10201	40	1600	2012021
11	121	11111	41	1681	2022021
12	144	12100	42	1764	2102100
13	169	20021	43	1849	2112111
14	196	21021	44	1936	2122201
15	225	22100	45	2025	2210000
16	256	100111	46	2116	2220101
17	289	101201	47	2209	10000211
18	324	110000	48	2304	10011100
19	361	111101	49	2401	10021221
20	400	112211	50	2500	10102121
21	441	121100	51	2601	10120100
22	484	122221	52	2704	10201011
23	529	201121	53	2809	10212001
24	576	210100	54	2916	11000000
25	625	212011	55	3025	11011001
26	676	221001	56	3136	11022011
27	729	1000000	57	3249	11110100
28	784	1002001	58	3364	11121121
29	841	1011011	59	3481	11202220
30	900	1020100	60	3600	11221100

In a second table, I have tabulated reduced integral solutions up to $x = 6563$ of $f(x^2) = I$; however, this table is probably not complete.

Table 2. $f(x^2) = I$

x	$x(3)$	x^2	$x^2(3)$
1	1	1	1
2	2	4	11
11	102	121	11111
16	121	256	100111
19	201	361	111101
29	1002	841	1011011
55	2001	3025	11011001
83	10002	6889	100110011
143	12022	20449	1001001101
163	20001	26569	1100110001
245	100002	60025	10001100011
262	100201	68644	10111011101
421	120121	177241	100000010111
451	121201	203401	101100000101
487	200001	237169	110001100001
731	1000002	534361	1000011000011
889	1012221	790321	1111011010011
1331	1211022	1771561	10100000010101
1459	2000001	2128681	11000011000001
1487	2001002	2211169	11011100011011
2189	10000002	4791721	100000110000011
2242	10002001	5026564	100110101011001
2323	10012001	5396329	101011011101001
2537	10110222	6436369	110010000001001
2644	10121221	6990736	111011011111011
2662	10122121	7086244	111100000111111
3788	12012022	14348944	1000000000001101
4375	20000001	19140625	1100000110000001
6563	100000002	43072969	10000001100000011

It is easy to verify that the equation $f(x^2) = I$ has infinitely many reduced solutions. Consider all positive integers

$$x = 3^n + 2 \quad \text{where} \quad n = 2, 3, 4, 5, \ldots$$

Their squares are

$$x^2 = 3^{2n} + 3^{n+1} + 3^n + 3^1 + 1$$

and obviously lie in I and are prime to 3.

On the other hand, the squares of the integers

$$x = 3^n + 1 \quad \text{where} \quad n = 1, 2, 3, \ldots$$

are

$$x^2 = 3^{2n} + 2 \times 3^n + 1;$$

these lie in K and also are prime to 3.

For $n = 2, 3, 4, \ldots,$

$$(2\times 3^n+1)^2 = 3^{2n+1}+3^{2n}+3^{n+1}+3^n+1 \in I,$$

and

$$(3^{2n+1}+2\times 3^n+1)^2 = 3^{4n+2}+3^{3n+2}+3^{3n+1}+3^{2n+2}+3^{2n}+3^{n+1}+3^n+1 \in I.$$

It seems probable that there exist infinitely many similar identities of this type giving solutions $x \in N$ of

$$f(x) = I, \quad (x, 3) = 1.$$

Canberra 1/1/88

Final remark. The problem discussed in this note can be generalised. Let $g \geqslant 2$ and $x \geqslant 1$ be integers. Then x can be represented to the general base g in the form

$$x = \sum_{h=0}^{n} d_h g^h, \quad \text{or say} \quad = d_n d_{n-1} \ldots d_1 d_0(g)$$

where now the digits d_h belong to the set $\{0, 1, 2, \ldots, g-1\}$. We can now again ask whether there exist infinitely many squares $x = y^2$ where y is a positive integer such that the following properties are satisfied,

(a) x is prime to g.

(b) All digits of x to the base g are either 0 or 1.

This problem is trivial if $g = 2$ and has been solved for $g = 3$ in this note affirmatively.

The trivial identity

$$(\tfrac{1}{2}\times 4^n+1)^2 = 4^{2n-1}+4^n+1 \quad (n = 2, 3, 4, \ldots)$$

shows that the problem has also for $g = 4$ a positive answer, with the special solutions

$$9^2 = 1101(4), \quad 33^2 = 101001(4), \quad 129^2 = 10010001(4),$$
$$513^2 = 1000100001(4), \quad \text{etc.}$$

I have not succeeded in solving the problem for any base $g \geqslant 5$, and I found only the one solution

$$20^2 = 1111(7)$$

for the special case $g = 7$. It would have some interest to study the case of a general base $g \geqslant 5$ and in particular the case $g = 10$.

Canberra, 28/1/1988

Received on 9. 2. 1988 (1785)