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Alf van der Poorten and Continued Fractions

Jeffrey Shallit
School of Computer Science, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
shallit@cs.uwaterloo.ca
http://www.cs.uwaterloo.ca/~shallit

Why Focus on Continued Fractions?

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- He obtained many interesting and novel results about them

VDP's Continued Fraction Themes

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- Use the "folding lemma" of Mendès France

Basics of Continued Fractions

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- rational numbers have terminating expansions;
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- $e^2 = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, \ldots]$

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"Though there had been earlier rumours of his [Apéry] claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions."

A continued fraction for $\zeta(3)$

In that paper:

$$\zeta(3) = \cfrac{6}{5 - \cfrac{1}{117 - \cfrac{64}{535 - \cfrac{729}{1436 - \cfrac{4096}{3105 - \cdots}}}}}$$

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$$-\frac{n^6}{34n^3 + 51n^2 + 27n + 5}.$$

A continued fraction for $\zeta(2)$

Also in that paper:

$$\zeta(2) = \frac{\pi^2}{6} = \cfrac{5}{3 + \cfrac{1}{25 + \cfrac{16}{69 + \cfrac{81}{135 + \cfrac{256}{223 + \cdots}}}}}$$

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Red herring or open problem?

Convergents

- define $p_{-2} = 0$; $p_{-1} = 1$; $q_{-2} = 1$; $q_{-1} = 0$, and

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

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- and so $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$, a theorem due to Galois in 1829.

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- Therefore
$$\frac{p_n}{q_n} = a_0 + \sum_{1 \le i \le n} \frac{(-1)^{n+1}}{q_{n-1} q_n}$$

Convergents

- now, letting $n \to \infty$, we see that if $\theta = [a_0, a_1, a_2, \ldots]$ is irrational, then
$$\theta = a_0 + \sum_{i \geq 1} \frac{(-1)^{n+1}}{q_{n-1} q_n}.$$

Continued Fractions for Algebraic Numbers

Theorem. (Bombieri & VDP, 1995) Let $\gamma > 1$ be the unique positive zero of the polynomial $f(X)$, and suppose

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Example. Let $f(X) = X^3 - 2$. Then

$$\sqrt[3]{2} = [1, a_1, a_2, \ldots]$$

with

$$a_{n+1} = \left\lfloor \frac{(-1)^{n+1}}{q_n} \frac{3p_n^2}{p_n^3 - 2q_n^3} - \frac{q_{n-1}}{q_n} \right\rfloor$$

for $n \geq 0$.

Some Unusual Continued Fraction Expansions

$$2\sum_{n\geq 0} 2^{-2^n} = [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, \ldots]$$

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(large partial quotients are close to powers of 2)

$$\begin{aligned}\sum_{n\geq 1} 10^{-n!} \quad = \quad & [0, 9, 11, 99, 1, 10, 9, 999999999999, 1, 8, 10, 1, 99, 11, \\ & 9, \overbrace{999999999999999 \cdots 9999999999999999999}^{72}, \ldots]\end{aligned}$$

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$$\sum_{i\geq 0} \frac{(-1)^i}{b_i} = [0, 1, 1, 1, 2^2, 3^2, 14^2, 129^2, 25298^2, \ldots]$$

where $b_0 = 1$, and $b_{n+1} = b_n^2 + b_n$ for $n \geq 0$. (Cahen's constant)

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Each of these can be viewed as *specializations* of continued fractions for elements of $\mathbb{Q}((X^{-1}))$: formal Laurent series.

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Let n be an integer. Any formal power series

$$f(X) = \sum_{-\infty < i \leq n} b_i X^i \in \mathbb{Q}((X^{-1}))$$

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where $a_i \in \mathbb{Q}[X]$ for $i \geq 0$ and $\deg a_i > 0$ for $i > 0$.

An Example

$$f(X) \quad := \quad \frac{1}{\sqrt{X^2-1}} = \sum_{k\geq 0} 2^{-2k}\binom{2k}{k} X^{-2k-1}$$

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Here are the first few convergents to the continued fraction for f:

n	0	1	2	3	4
a_n	0	X	$-2X$	$2X$	$-2X$
p_n	0	1	$-2X$	$-4X^2+1$	$8X^3-4X$
q_n	1	X	$-2X^2+1$	$-4X^3+3X$	$8X^4-8X^2+1$

An Example (continued)

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where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

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Lemma. Let

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Lemma. Let

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- let w^R denote the reversed word.

Then

$$\frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2} = [a_0,\ w,\ x - \frac{q_{n-1}}{q_n}] = [a_0,\ w,\ x,\ -w^R].$$

Proof of the Folding Lemma

Proof. We have

$$[a_0, w, x \quad - \quad q_{n-1}/q_n] \leftrightarrow \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} x - q_{n-1}/q_n & 1 \\ 1 & 0 \end{bmatrix}$$

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$$= \quad \begin{bmatrix} xp_n - (p_n q_{n-1} - p_{n-1} q_n)/q_n & p_n \\ xq_n & q_n \end{bmatrix}$$

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&= \begin{bmatrix} xp_n - (p_n q_{n-1} - p_{n-1} q_n)/q_n & p_n \\ xq_n & q_n \end{bmatrix} \\
&\leftrightarrow \frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2}.
\end{aligned}
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Application of the folding lemma

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In general, by the Folding Lemma, if

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In general, by the Folding Lemma, if

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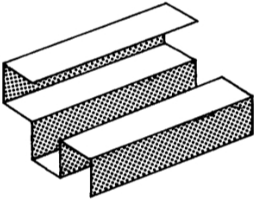
then

$$h_{i+1}(X) = [1,\ Y,\ -X,\ -Y^R].$$

Continued fractions and folding paper

So — as explained in VDP's 2nd most cited paper, entitled FOLDS! — the terms of this continued fraction are given by folding a piece of paper repeatedly, then unfolding and reading the sequence of folds as "up" or "down".



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a contradiction, since we cannot have
$p = a^2c + 2aa' = a(ac + 2a')$.

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So the palindrome must be of even length, say,

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expressing p as the sum of two squares.

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expressing p as the sum of two squares.

Taking the determinant, we get $p(b^2 + b'^2) - x^2 = 1$, or $x^2 \equiv -1 \pmod{p}$ — so we get a square root of $-1 \pmod{p}$, for free!

Specialization

The expansion

$$\begin{aligned} X \sum_{0 \le i \le n} X^{-2^i} &= [1, X, -X, -X, -X, X, X, -X, -X, X, -X, -X, \ldots \\ &= [a_0, a_1, a_2, \ldots] \end{aligned}$$

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is atypical in several respects:

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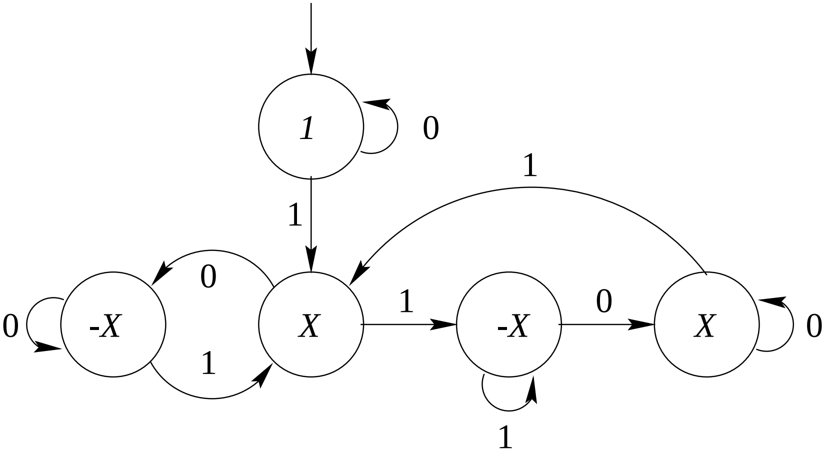
- the partial quotients are given by a finite automaton

A MSD-first automaton for the partial quotients



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Now "specialize", setting $X = 2$. We get

$$2\sum_{i\geq 0} 2^{-2^i} = [1, 2, -2, -2, -2, 2, 2, -2, -2, 2, \cdots],$$

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This can be done with the following

Lemma.

$$\begin{aligned}
[a, -b, c] &= [a-1,\ 1,\ b-2,\ 1,\ c-1]; \\
[a, 0, b] &= [a+b].
\end{aligned}$$

Application of the Folding Lemma

Thus

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&\cdots
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Let $a_0 = 1$, $a_i = \pm 1$ for $i \geq 1$.

Then the number

$$2 \sum_{i \geq 0} a_i 2^{-2^i}$$

is transcendental and its continued fraction expansion consists solely of 1's and 2's.

Liouville's Transcendental Number

Let $f_n(X) = \sum_{1 \leq k \leq n} X^{-k!}$, and define $f(X) = f_\infty(X)$.

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\begin{aligned}
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f_4(X) &= [0,\ X-1,\ X+1,\ X^2,\ -X-1,\ -X+1,\ -X^{12}, \\
&\qquad X-1,\ X+1,\ -X^2,\ -X-1,\ -X+1] \\
&\ \ \vdots
\end{aligned}
$$

Liouville's Transcendental Number

Hence, we get

$$\begin{aligned} f(X) &= [0,\ X-1,\ X+1,\ X^2,\ -X-1,\ -X+1,\ -X^{12}, \\ &\quad X-1,\ X+1,\ -X^2,\ -X-1,\ -X+1,\ -X^{72},\ldots] \end{aligned}$$

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where the large partial quotients are

$$n! - 2(n-1)! = (n-2)(n-1)!.$$

An Unusual Continued Fraction

$$
\begin{aligned}
2^{-1} + 2^{-2} \quad &+ \quad 2^{-3} + 2^{-5} + 2^{-8} + 2^{-13} + \cdots + 2^{-F_n} + \cdots \\
&= \quad [0, 1, 10, 6, 1, 6, 2, 14, 4, 124, 2, 1, 2, 2039, \\
& \qquad 1, 9, 1, 1, 1, 262111, 2, 8, 1, 1, 1, 3, 1, 536870655, \\
& \qquad 4, 16, 3, 1, 3, 7, 1, 140737488347135, \ldots].
\end{aligned}
$$

A Fibonacci Power Series

Similarly

$$\begin{aligned}
X^{-1} &+ X^{-2} + X^{-3} + X^{-5} + X^{-8} + \cdots + X^{-F_n} + \cdots + \\
&= [0,\ X-1,\ X^2+2X+2,\ X^3-X^2+2X-1, \\
&\quad -X^3+X-1,\ -X,\ -X^4+X,\ -X^2, \\
&\quad -X^7+X^2,\ -X-1,\ X^2-X+1,\ X^{11}-X^3, \\
&\quad -X^3-X,\ -X,\ X,\ X^{18}-X^5,\ -X,\ X^3+1,\ X, \\
&\quad -X,\ -X-1,\ -X+1,\ -X^{29}+X^8,\ X-1,\ldots]
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What's going on here?

A Fibonacci Power Series

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$$\begin{aligned} X^{-1} &+ X^{-2}+X^{-3}+X^{-5}+X^{-8}+\cdots+X^{-F_n}+\cdots+ \\ &= [0,\ X-1,\ X^2+2X+2,\ X^3-X^2+2X-1, \\ &\quad -X^3+X-1,\ -X,\ -X^4+X,\ -X^2, \\ &\quad -X^7+X^2,\ -X-1,\ X^2-X+1,\ X^{11}-X^3, \\ &\quad -X^3-X,\ -X,\ X,\ X^{18}-X^5,\ -X,\ X^3+1,\ X, \\ &\quad -X,\ -X-1,\ -X+1,\ -X^{29}+X^8,\ X-1,\ldots] \end{aligned}$$

What's going on here?

"We remark that to our surprise, and horror, continued fraction expansion of formal power series appears to adhere to the cult of Fibonacci." – VDP (1998)

Theorem. (VDP & JOS, 1993). Let (F_n) be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

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$$s_n = X^{-1} + X^{-2} + X^{-3} + X^{-5} + \cdots + X^{-F_n} = [0, f_n].$$

Then for $n \geq 11$ we have s_{n+1}

$$= \quad [0, f_n, 0, -f_{n-4}, -X^{L_{n-4}}, f^R_{n-4}, 0, -f_{n-3}, X^{F_{n-4}}, f^R_{n-3}]$$

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$$g_n = f_{n-1}, 0, -f_{n-5}, -X^{L_{n-5}}, f_{n-5}^R, 0, -f_{n-4}.$$

The Fibonacci power series

Corollary. A polynomial p is a partial quotient in the expansion of the Fibonacci power series if and only if p or $-p$ occurs in the following list:

$$
\begin{aligned}
&X+1;\\
&X^2 \pm X + 1;\\
&X^2 + 2X + 2;\\
&X^3 + 1;\\
&X^3 + X;\\
&X^3 - X + 1;\\
&X^3 - X^2 + 2X - 1;\\
&X^{F_n};\\
&X^{L_{n+2}};\\
&X^{L_{n+1}} - X^{F_n}
\end{aligned}
$$

for $n \geq 1$

The Fibonacci power series

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in the continued fraction expansion of

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-8} + \cdots$$

differ by 1 from the numbers

$$2^{L_{h+1}} - 2^{F_h}$$

for $h \geq 4$.

An open question

Let $f(X) = \sum_{n \geq 0} a_n X^{-n} \in \mathbb{Q}[[X^{-1}]]$, where $a_n \in \{0, 1\}$.

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Now let's go back to $h(X) = X \sum_{i \geq 0} X^{-2^i}$ ...

Convergents to $h(X)$

Here is a table of the first few convergents to $h(X)$:

n	a_n	$p_n(X)$	$q_n(X)$
0	1	1	1
1	X	$X+1$	X
2	$-X$	$-X^2-X+1$	$-X^2+1$
3	$-X$	X^3+X^2+1	X^3
4	$-X$	$-X^4-X^3-X^2-2X+1$	$-X^4-X^2+1$
5	X	$-X^5-X^4-X^2+X+1$	$-X^5+X$
6	X	$-X^6-X^5-X^4-2X^3$ $-X+1$	$-X^6-X^4+1$
7	$-X$	$X^7+X^6+X^4+1$	X^7
8	$-X$	$-X^8-X^7-X^6-2X^5$ $-X^4-2X^3-2X+1$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^8-X^6-X^5$ $-X^4-2X^2+X+1$	$-X^9-X^5+X$

Denominators of the Convergents

n	$a_n(X)$	$q_n(X)$
0	1	1
1	X	X
2	$-X$	$-X^2+1$
3	$-X$	X^3
4	$-X$	$-X^4-X^2+1$
5	X	$-X^5+X$
6	X	$-X^6-X^4+1$
7	$-X$	X^7
8	$-X$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^5+X$
10	$-X$	$X^{10}-X^8-X^4-X^2+1$
11	$-X$	$-X^{11}+X^3$
12	X	$-X^{12}+X^{10}-X^8-X^2+1$
13	X	$-X^{13}-X^9+X$
14	X	$-X^{14}-X^{12}-X^8+1$

Theorem. (Allouche, Lubiw, Mendès France, VDP, JOS)
All of the coefficients of the denominators of the convergents to $X \sum_{i \geq 0} X^{-2^i}$ lie in $\{0, \pm 1\}$.

Proof. (Sketch)

- The low-order terms of $q_{2^k+n-1}(X)$ (i.e., those of degree $< 2^k$) are exactly the same as those of $q_{2^k-n-1}(X)$;
- The high-order terms of $q_{2^k+n-1}(X)$ are, up to a change of signs of individual terms, equal to $X^{2^k} q_{n-1}(X)$;

A Converse: The Continuant Tree

We can obtain a converse to the preceding theorem.

Define an infinite labeled binary tree T with root r and node n labeled $L(n)$, as follows:

- $L(r) = 1$;
- $L(\mathrm{left}(n)) = XL(n) + L(\mathrm{parent}(n))$;
- $L(\mathrm{right}(n)) = -XL(n) + L(\mathrm{parent}(n))$.

The paths in this tree consist of the consecutive denominators of the convergents to the continued fraction

$$[1, \pm X, \pm X, \pm X, \ldots].$$

The convergents

Theorem.
If a path in T consists entirely of polynomials with coefficients in $\{0, \pm 1\}$, then it is the sequence of denominators of convergents to a formal power series of the form

$$X \sum_{i \geq 0} \pm X^{-2^i}.$$

The convergents for $h(X)$

Theorem. (A, L, MF, vdP, S)
Let

$$\begin{aligned} h(X) &= X \sum_{i \geq 0} X^{-2^i} \\ &= [a_0, a_1, a_2, \ldots] \end{aligned}$$

and set $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$. Then

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(a) $q_{2n+1}(X) = X q_n(X^2)$;

The convergents for $h(X)$

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$$h(X) = X \sum_{i \geq 0} X^{-2^i} = [a_0, a_1, a_2, \ldots]$$

and set $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$. Then

(a) $q_{2n+1}(X) = Xq_n(X^2)$;

(b) $q_{2n}(X) = (-1)^n(q_n(X^2) - q_{n-1}(X^2))$;

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The convergents for $h(X)$

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(b) $q_{2n}(X) = (-1)^n(q_n(X^2) - q_{n-1}(X^2))$;

(c) The polynomial $q_{2n+1}(X)$ is odd;

(d) The polynomial $q_{2n}(X)$ is even.

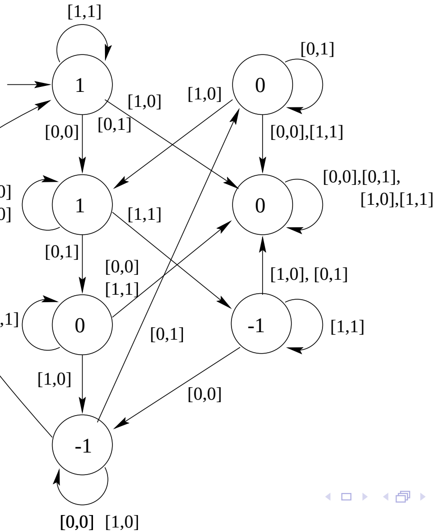
The Coefficient Table is Automatic

Theorem. (Allouche, Lubiw, Mendès France, VDP, JOS)
Define $c_{m,n} = [X^n]q_m(X)$, the coefficient of the X^n term in the polynomial $q_m(X)$. Then the double sequence (table) $(c_{m,n})_{m,n\geq 0}$ is automatic.

Here is what a small portion of this infinite table looks like:

$m \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	0	0	0	0
2	1	0	-1	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	1	0	-1	0	-1	0	0	0	0	0
5	0	1	0	0	0	-1	0	0	0	0
6	1	0	0	0	-1	0	-1	0	0	0
7	0	0	0	0	0	0	0	1	0	0
8	1	0	0	0	-1	0	-1	0	-1	0
9	0	1	0	0	0	-1	0	0	0	-1

A LSD-First Automaton for the Coefficients of the Denominators of the Convergents



More General Results

We can also consider the formal power series

$$g_\epsilon(X) := \sum_{i \geq 0} \epsilon_i X^{-2^i} \qquad h_\epsilon(X) := X g_\epsilon(X)$$

where $\epsilon_i = \pm 1$.

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Nearly everything we proved for $h(X)$ also holds for these series. In particular

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where $\epsilon_i = \pm 1$.

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An Open Question

Let the Fibonacci numbers be defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. VDP and JOS considered the continued fraction for the formal power series

$$r(X) = \sum_{i \geq 2} X^{-F_i} = X^{-1} + X^{-2} + X^{-3} + X^{-5} + X^{-8} + \cdots .$$

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Can this be proved?

VDP's Hadamard Quotient Theorem

Theorem. Let K be a field of characteristic 0. Suppose $\sum_{n\geq 0} b_n X^n$ and $\sum_{n\geq 0} c_n X^n$ in $K[[X]]$ are the expansions of rational functions with $c_n \neq 0$ for all $n \geq n_0$. If the quotients b_n/c_n all belong to a finitely generated ring over $\mathbb{Z}$, then $\sum_{n\geq n_0} \frac{b_n}{c_n} X^n$ is the expansion of a rational function.

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Full details written down by Robert Rumely (68 pages!) in 1986–87.

Application of the Hadamard Quotient Theorem

Theorem. (H. W. Lenstra, Jr., and JOS)
Let θ be an irrational real number with simple continued fraction expansion $\theta = [a_0, a_1, \ldots]$ and convergents p_n/q_n for $n \geq 0$. Then the following four conditions are equivalent:

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(b) $(q_n)_{n\geq 0}$ satisfies a linear recurrence with constant coefficients;
(c) $(a_n)_{n\geq 0}$ is ultimately periodic;
(d) θ is a quadratic irrational.

Since then, a better proof was found by Andrew Granville that does not depend on HQT, and generalizations by Bézivin.

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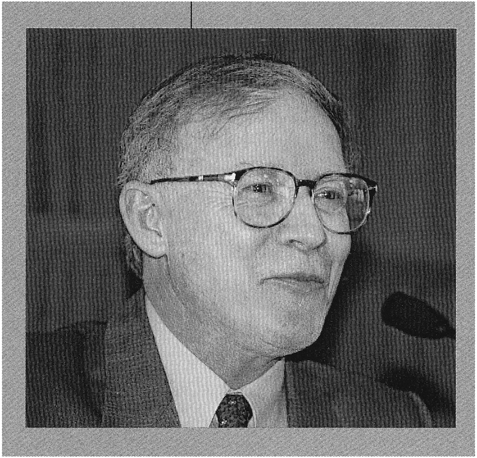
(about refereeing) "Happily, here there is no tradition that it is wrong to be scathing when that is appropriate."

Farewell VDP



For Further Reading

1. J.-P. Allouche, A. Lubiw, M. Mendès France, A. J. van der Poorten, and J. Shallit, Convergents of folded continued fractions, *Acta Arithmetica* **77** (1996), 77–96.
2. M. Mendès France, A. J. van der Poorten, and J. Shallit, On lacunary formal power series and their continued fraction exapnsion, in *Number Theory in Progress*, Walter de Gruyter, 1999, pp. 321–326.
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4. A. J. van der Poorten and J. Shallit, Folded continued fractions, *J. Number Theory* **40** (1992), 237–250.
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