On poly-Cauchy numbers and polynomials

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The generating function of the Cauchy numbers of the first kind c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}\,.$$

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$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad \left(B_1 = \frac{1}{2}\right) .$$

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$$\frac{x}{1-e^{-x}}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad\left(B_1=\frac{1}{2}\right).$$

or

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad\left(B_1=-\frac{1}{2}\right).$$

Euler numbers

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Euler numbers E_n

$$\frac{2}{e^x - 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}\,.$$

Euler polynomials $E_n(z)$

$$\frac{2e^{xz}}{e^x - 1} = \sum_{n=0}^{\infty} E_n(z) \frac{x^n}{n!}\,.$$

Multiple Euler numbers $\mathcal{E}_n^{(r)}$

$$\left(\frac{2}{e^{x/2} + e^{-x/2}}\right)^r = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)} \frac{x^n}{n!}\,.$$

Similar numbers and polynomials

Similar numbers and polynomials

The Bernoulli polynomials of the order r (r is an integer) denoted by $\mathcal{B}_n^{(k)}(z)$ are defined by

$$\left(\frac{x}{e^x-1}\right)^r e^{xz}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(r)}(z)\frac{x^n}{n!}\,.$$

If $z=0$, $\mathcal{B}_n^{(k)}(z)=\mathcal{B}_n^{(k)}$ is the Bernoulli number of the order r.
If $z=0$ and $k=1$, $\mathcal{B}_n^{(1)}=\mathcal{B}_n$ is the ordinary Bernoulli number.

Nörlund's number $\mathcal{B}_n^{(n)}$ (1954, Nörlund)

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(n)} \frac{x^n}{n!}\,.$$

Nörlund's number $\mathcal{B}_n^{(n)}$ (1954, Nörlund)

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(n)} \frac{x^n}{n!}\,.$$

Bernoulli number of the second kind b_n

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Bernoulli number of the second kind b_n

$$\frac{x}{\ln(1+x)}=\sum_{n=0}^{\infty}b_n x^n\,.$$

In fact, $b_n=c_n/n!$, where c_n is the Cauchy number of the first kind.

Jordan's polynomial $\Psi_n(z)$ (1965, Jordan)

$$\frac{x(1+x)^z}{\ln(1+x)} = \sum_{n=0}^{\infty} \Psi_n(z) x^n$$

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Carlitz' polynomial $\beta_n^{(k)}(z)$ (1961, Carlitz)

$$\left(\frac{x}{\ln(1+x)}\right)^k (1+x)^z = \sum_{n=0}^{\infty} \beta_n^{(k)}(z) \frac{x^n}{n!} .$$

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Carlitz' polynomial $\beta_n^{(k)}(z)$ (1961, Carlitz)

$$\left(\frac{x}{\ln(1+x)}\right)^k (1+x)^z = \sum_{n=0}^{\infty} \beta_n^{(k)}(z) \frac{x^n}{n!} \,.$$

In fact, $\beta_n^{(1)}(z) = n! \Psi_n(z)$.

The Bernoulli polynomials of the second kind $b_n(z)$

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Degenerate Bernoulli polynomials $\beta_m(\lambda,z)$ (1956, Carlitz)

$$\frac{x(1+\lambda x)^{\mu z}}{(1+\lambda x)^\mu-1}=\sum_{n=0}^\infty \beta_m(\lambda,z)\frac{x^n}{n!}\quad(\lambda\mu=1)\,.$$

Korobov polynomials of the first kind K_n (2001, Korobov; 2003, Ustinov)

$$\frac{qx(1+x)^z}{(1+x)^q-1}=\sum_{n=0}^\infty K_n(z)\frac{x^n}{n!}\,.$$

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Korobov polynomilas of the second kind k_n (2003, Ustinov)

$$\frac{x(1+qx)^{z/q}}{(1+qx)^{1/q}-1}=\sum_{n=0}^\infty k_n(z)\frac{x^n}{n!}\,.$$

Harmonic numbers $H_n := \sum_{k=1}^{n} \frac{1}{k}$

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Generalized harmonic numbers
$H(n,r) := \sum_{1\le n_0+n_1+\cdots+n_r\le n}(n_0 n_1 \dots n_r)^{-1}$ **(**$n \ge 1$**,** $r \ge 0$**) (1997, Gertsch; 1997 Santmyer)**

$$\frac{(-1)^r (\ln(1-x))^{r+1}}{1-x} = \sum_{n=r+1}^{\infty} H(n,r) x^n\,.$$

The (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$

$$\frac{(\ln(1+x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} \left[{n \atop m}\right] \frac{x^n}{n!}.$$

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$$\frac{(\ln(1+x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} \left[{n \atop m}\right] \frac{x^n}{n!}.$$

The Stirling numbers of the second kind $\left\{{n \atop m}\right\}$

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{{n \atop m}\right\} \frac{x^n}{n!}.$$

The (unsigned) Stirling numbers of the first kind $\left[{n \atop m} \right]$ arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1)=\sum_{m=0}^{n}\left[{n \atop m} \right] x^m\,.$$

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The Stirling numbers of the second kind $\left\{{n \atop m}\right\}$ are determined by

$$\left\{{n \atop m}\right\}=\frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n\,.$$

There are many identities about the Bernoulli numbers. They are much related to the (unsigned) Stirling numbers of the first kind $\left[{n \atop m} \right]$ and the Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$. Some of them are

There are many identities about the Bernoulli numbers. They are much related to the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ and the Stirling numbers of the second kind $\left\{{n \atop m}\right\}$. Some of them are

$$\frac{1}{n!}\sum_{m=0}^{n}(-1)^m\left[{n+1 \atop m+1}\right]B_m=\frac{1}{m+1},$$

$$B_n=(-1)^n\sum_{m=0}^{n}\left\{{n \atop m}\right\}\frac{(-1)^m m!}{m+1}.$$

In 1997 M. Kaneko introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

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$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^\infty B_n^{(k)}\frac{x^n}{n!}\,,$$

where

$$\mathrm{Li}_k(z)=\sum_{m=1}^\infty\frac{z^m}{m^k}$$

is the k-th polylogarithm function. When $k=1$, $B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)}=1/2$.

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$$e^x \cdot \underbrace{\frac{1}{e^x-1}\int_0^x \frac{1}{e^x-1}\int_0^x \cdots \frac{1}{e^x-1}\int_0^x}_{k-1} \frac{x}{e^x-1}\underbrace{dxdx\ldots dx}_{k-1}$$

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$$= \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} .$$

An explicit formula for $B_n^{(k)}$ is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1) . \quad (1)$$

Let n and k be integers with $c \ge 0$ and $k \ge 1$. Define the poly-Cauchy numbers $c_n^{(k)}$ by the following.

$$c_n^{(k)} = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} \binom{x_1 x_2 \dots x_k}{n} dx_1 dx_2 \dots dx_k \,.$$

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The Cauchy numbers $c_n = c_n^{(1)}$ can be expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$:

$$c_n^{(1)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-1)^m}{m+1} \,.$$

The poly-Cauchy numbers $c_n^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind $\left[n \atop m \right]$. This is considered as an analogous identity to the identity (1).

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Theorem 1

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-1)^m}{(m+1)^k}\,.$$

By using Theorem 1, we get

$$
\begin{aligned}
c_0^{(k)} &= 1\,,\\
c_1^{(k)} &= \frac{1}{2^k}\,,\\
c_2^{(k)} &= -\frac{1}{2^k} + \frac{1}{3^k}\,,\\
c_3^{(k)} &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k}\,,\\
c_4^{(k)} &= -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k}\,,\\
c_5^{(k)} &= \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k}\,.
\end{aligned}
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Denote the polylogarithm factorial function $\mathrm{Lif}_k(z)$ by

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Theorem 2

The generating function of the poly-Cauchy numbers $c_n^{(k)}$ is given by the following:

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

The generating function of the Cauchy numbers of the first kind c_n is also given by

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The generating function of the poly-Cauchy numbers in Theorem 2 can be written in the form of iterated integrals.

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The generating function of the poly-Cauchy numbers in Theorem 2 can be written in the form of iterated integrals.

Corollary 1

For $k \ge 2$ we have

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)\ln(1+x)}\underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

Theorem 3

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} c_m^{(k)} = \frac{1}{(n+1)^k} .$$

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$$\sum_{m=0}^{n} \left\{ n \atop m \right\} c_m^{(k)} = \frac{1}{(n+1)^k}\,.$$

The Cauchy numbers of the second kind $\hat{c}_n$ is defined by

$$\begin{aligned}\hat{c}_n &= n! \int_0^1 \binom{-x}{n} dx \\ &= \int_0^1 (-x)(-x-1)\dots(-x-n+1)dx \\ &= (-1)^n \int_0^1 \langle x \rangle_n \, dx \,,\end{aligned}$$

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where $\langle x \rangle_n = x(x+1)\dots(x+n-1)$ $(n \geq 1)$ is the rising factorial with $\langle x \rangle_0 = 1$.

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where $\langle x \rangle_n = x(x+1)\dots(x+n-1)$ ($n \ge 1$) is the rising factorial with $\langle x \rangle_0 = 1$.
We call the Cauchy numbers c_n as the Cauchy numbers of the first kind, in order to distinguish with those of the second kind.

Similarly to the poly-Cauchy numbers of the first kind, we define the poly-Cauchy numbers of the second kind as follows.

$$\hat{c}_n^{(k)} = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 x_2 \dots x_k}{n} dx_1 dx_2 \dots dx_k \,.$$

The Cauchy numbers of the second kind $\hat{c}_n = \hat{c}_n^{(1)}$ is expressed in terms of the Stirling numbers of the first kind:

$$\hat{c}_n^{(1)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{1}{m+1}\,.$$

The Cauchy numbers of the second kind $\hat{c}_n = \hat{c}_n^{(1)}$ is expressed in terms of the Stirling numbers of the first kind:

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The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind.

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$$\hat{c}_n^{(1)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{m+1}\,.$$

The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can be also expressed in terms of the Stirling numbers of the first kind.

Theorem 4

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}\,.$$

By using Theorem 4, we get

$$\begin{aligned}
\hat{c}_0^{(k)} &= 1\,,\\
\hat{c}_1^{(k)} &= -\frac{1}{2^k}\,,\\
\hat{c}_2^{(k)} &= \frac{1}{2^k}+\frac{1}{3^k}\,,\\
\hat{c}_3^{(k)} &= -\frac{2}{2^k}-\frac{3}{3^k}-\frac{1}{4^k}\,,\\
\hat{c}_4^{(k)} &= \frac{6}{2^k}+\frac{11}{3^k}+\frac{6}{4^k}+\frac{1}{5^k}\,,\\
\hat{c}_5^{(k)} &= -\frac{24}{2^k}-\frac{50}{3^k}-\frac{35}{4^k}-\frac{10}{5^k}-\frac{1}{6^k}\,.
\end{aligned}$$

In similar manners to the results in Theorem 2, Corollary 1 and Theorem 3 about the poly-Cauchy numbers of the first kind $c_n^{(k)}$, we can obtain the following corresponding results about the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$.

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Theorem 5

The generating function of the poly-Cauchy numbers $\hat{c}_n^{(k)}$ is given by the following:

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}.$$

The generating function of the Cauchy numbers of the second kind $\hat{c}_n = \hat{c}_n^{(1)}$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n^{(1)} \frac{x^n}{n!}\,.$$

The generating function of the Cauchy numbers of the second kind $\hat{c}_n = \hat{c}_n^{(1)}$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \hat{c}_n^{(1)} \frac{x^n}{n!}\,.$$

The generating function of the poly-Cauchy numbers of the second kind can be also written in the form of iterated integrals by putting $z = -\ln(1+x)$ in

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \dots \frac{1}{z}\int_0^z}_{k-1} \frac{e^z - 1}{z} \underbrace{dz dz \dots dz}_{k-1}\,.$$

Corollary 2

For $k \geq 2$ we have

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)^2\ln(1+x)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}\,.$$

Corollary 2

For $k \ge 2$ we have

$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)^2\ln(1+x)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}\,.$$

Theorem 6

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} \hat{c}_m^{(k)} = \frac{(-1)^n}{(n+1)^k}\,.$$

There are some relations between the poly-Cauchy numbers of the first kind and those of the second kind.

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Theorem 7

For $n \ge 1$ we have

$$\boldsymbol{(-1)^n \frac{c_n^{(k)}}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{\hat{c}_m^{(k)}}{m!}\,,}$$

$$\boldsymbol{(-1)^n \frac{\hat{c}_n^{(k)}}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{c_m^{(k)}}{m!}\,.}$$

There are relations between poly-Bernoulli numbers and poly-Cauchy numbers.

Theorem 8

For $n \geq 1$ we have

$$\boldsymbol{B_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)} \,,}$$

$$\boldsymbol{c_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)} \,.}$$

Duality theorem

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It is known that the duality theorem holds for poly-Bernoulli numbers. Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad \text{for } n, k \geq 0\,.$$

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$$B_n^{(-k)} = B_k^{(-n)} \quad \text{for } n, k \geq 0\,.$$

It is due to the symmetric formula:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}\,.$$

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However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} = e^y(1+x)^{e^y},$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \hat{c}_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^y}{(1+x)^{e^y}}.$$

$$\mathrm{Lif}_{-r}(x) = e^x \sum_{j=0}^{r} \left\{ {r+1 \atop j+1} \right\} x^j \quad (r = 0, 1, 2, \dots)\,.$$

$$\mathrm{Lif}_{-r}(x) = e^x \sum_{j=0}^{r} \left\{ {r+1 \atop j+1} \right\} x^j \quad (r = 0, 1, 2, \dots)\,.$$

We have the record for the first some values r.

$$\begin{aligned}
\mathrm{Lif}_0(x) &= e^x,\\
\mathrm{Lif}_{-1}(x) &= (1+x)e^x,\\
\mathrm{Lif}_{-2}(x) &= (1+3x+x^2)e^x,\\
\mathrm{Lif}_{-3}(x) &= (1+7x+6x^2+x^3)e^x,\\
\mathrm{Lif}_{-4}(x) &= (1+15x+25x^2+10x^3+x^4)e^x,\\
\mathrm{Lif}_{-5}(x) &= (1+31x+90x^2+65x^3+15x^4+x^5)e^x.
\end{aligned}$$

For $k \geq 2$

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z)\,,$$

so

$$\mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}dt\,;$$

For $k \geq 2$

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z)\,,$$

so

$$\mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}dt\,;$$

on the other hand,

$$\frac{d}{dz}(z\mathrm{Lif}_k(z)) = \mathrm{Lif}_{k-1}(z)\,,$$

so

$$\mathrm{Lif}_k(z) = \frac{1}{z}\int_0^z \mathrm{Lif}_{k-1}(t)dt\,.$$

For $k \ge 2$

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z)\,,$$

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on the other hand,

$$\frac{d}{dz}(z\mathrm{Lif}_k(z)) = \mathrm{Lif}_{k-1}(z)\,,$$

so

$$\mathrm{Lif}_k(z) = \frac{1}{z}\int_0^z \mathrm{Lif}_{k-1}(t)dt\,.$$

In addition, $\mathrm{Li}_1(z) = -\ln(1-z)$ and $\mathrm{Lif}_1(z) = (e^z-1)/z$.

Poly-Cauchy polynomials of the first kind

Define the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} \binom{x_1 x_2 \dots x_k - z}{n} dx_1 dx_2 \dots dx_k \,.$$

Poly-Cauchy polynomials of the first kind

Define the **poly-Cauchy polynomials of the first kind** $c_n^{(k)}(z)$ by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} \binom{x_1 x_2 \dots x_k - z}{n} dx_1 dx_2 \dots dx_k \, .$$

The first several polynomials are

$$c_0^{(k)}(z) = 1,$$

$$c_1^{(k)}(z) = \frac{1}{2^k} - z,$$

$$c_2^{(k)}(z) = -\frac{1}{2^k} + \frac{1}{3^k} + \left(1 - \frac{2}{2^k}\right) z + z^2,$$

$$
\begin{aligned}
c_3^{(k)}(z) &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k} + \left(-2 + \frac{6}{2^k} - \frac{3}{3^k}\right) z \\
&\quad + \left(-3 + \frac{3}{2^k}\right) z^2 - z^3, \\
c_4^{(k)}(z) &= -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k} + \left(6 - \frac{22}{2^k} + \frac{18}{3^k} - \frac{4}{4^k}\right) z \\
&\quad + \left(11 - \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 + \left(6 - \frac{4}{2^k}\right) z^3 + z^4.
\end{aligned}
$$

$$\begin{aligned}c_3^{(k)}(z)&=\frac{2}{2^k}-\frac{3}{3^k}+\frac{1}{4^k}+\left(-2+\frac{6}{2^k}-\frac{3}{3^k}\right)z\\&\quad+\left(-3+\frac{3}{2^k}\right)z^2-z^3,\\c_4^{(k)}(z)&=-\frac{6}{2^k}+\frac{11}{3^k}-\frac{6}{4^k}+\frac{1}{5^k}+\left(6-\frac{22}{2^k}+\frac{18}{3^k}-\frac{4}{4^k}\right)z\\&\quad+\left(11-\frac{18}{2^k}+\frac{6}{3^k}\right)z^2+\left(6-\frac{4}{2^k}\right)z^3+z^4.\end{aligned}$$

If $k=1$, then $c_n^{(1)}(z)=c_n(z)$ are the Cauchy polynomials of the first kind.

$$
\begin{aligned}
c_3^{(k)}(z) &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k} + \left(-2 + \frac{6}{2^k} - \frac{3}{3^k}\right) z \\
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c_4^{(k)}(z) &= -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k} + \left(6 - \frac{22}{2^k} + \frac{18}{3^k} - \frac{4}{4^k}\right) z \\
&\quad + \left(11 - \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 + \left(6 - \frac{4}{2^k}\right) z^3 + z^4.
\end{aligned}
$$

If $k = 1$, then $c_n^{(1)}(z) = c_n(z)$ are the Cauchy polynomials of the first kind. If $z = 0$, then $c_n^{(k)}(0) = c_n^{(k)}$ are the poly-Cauchy numbers of the first kind.

Theorem 9

$$c_n^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

Theorem 9

$$c_n^{(k)}(z)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{(-z)^i}{(m-i+1)^k}\,.$$

Theorem 10

The generating function of the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ is given by

$$\frac{\mathrm{Lif}_k(\ln(1+x))}{(1+x)^z}=\sum_{n=0}^{\infty}c_n^{(k)}(z)\frac{x^n}{n!}\,.$$

Corollary 3

For $k \geq 2$ we have

$$\frac{1}{(1+x)^z} \times \underbrace{\frac{1}{\ln(1+x)} \int_0^x \frac{1}{(1+x)\ln(1+x)} \int_0^x \cdots \frac{1}{(1+x)\ln(1+x)} \int_0^x}_{k-1} \frac{x}{(1+x)\ln(1+x)} \underbrace{dx dx \ldots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} c_n^{(k)}(z) \frac{x^n}{n!}.$$

Corollary 3

For $k \ge 2$ we have

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$$\underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)\ln(1+x)} \underbrace{dx dx \dots dx}_{k-1}$$
$$= \sum_{n=0}^{\infty} c_n^{(k)}(z)\frac{x^n}{n!}\,.$$

Theorem 11

$$\sum_{m=0}^{n} \left\{ n \atop m \right\} c_m^{(k)}(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-z)^i}{(n-i+1)^k}\,.$$

Poly-Cauchy polynomials of the second kind

Define the poly-Cauchy polynomials of the second kind $\hat{c}_n^{(k)}(z)$ by

$$\hat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} \binom{-x_1 x_2 \dots x_k + z}{n} dx_1 dx_2 \dots dx_k \,.$$

Poly-Cauchy polynomials of the second kind

Define the **poly-Cauchy polynomials of the second kind** $\hat{c}_n^{(k)}(z)$ by

$$\hat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} \binom{-x_1 x_2 \dots x_k + z}{n} dx_1 dx_2 \dots dx_k \,.$$

The first several polynomials are

$$\hat{c}_0^{(k)}(z) = 1,$$
$$\hat{c}_1^{(k)}(z) = -\frac{1}{2^k} + z,$$
$$\hat{c}_2^{(k)}(z) = \frac{1}{2^k} + \frac{1}{3^k} - \left(1 + \frac{2}{2^k}\right) z + z^2,$$

$$
\begin{aligned}
\hat{c}_3^{(k)}(z) &= -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k} + \left(2 + \frac{6}{2^k} + \frac{3}{3^k}\right) z \\
&\quad - \left(3 + \frac{3}{2^k}\right) z^2 + z^3, \\
\hat{c}_4^{(k)}(z) &= \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} - \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right) z \\
&\quad + \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 - \left(6 + \frac{4}{2^k}\right) z^3 + z^4.
\end{aligned}
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$$
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&\quad - \left(3 + \frac{3}{2^k}\right) z^2 + z^3, \\
\hat{c}_4^{(k)}(z) &= \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} - \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right) z \\
&\quad + \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 - \left(6 + \frac{4}{2^k}\right) z^3 + z^4.
\end{aligned}
$$

If $k = 1$, then $\hat{c}_n^{(1)}(z) = \hat{c}_n(z)$ are the Cauchy polynomials of the second kind.

$$\begin{aligned}\hat{c}_3^{(k)}(z) &= -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k} + \left(2 + \frac{6}{2^k} + \frac{3}{3^k}\right) z\\ &\quad - \left(3 + \frac{3}{2^k}\right) z^2 + z^3,\\ \hat{c}_4^{(k)}(z) &= \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} - \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right) z\\ &\quad + \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 - \left(6 + \frac{4}{2^k}\right) z^3 + z^4.\end{aligned}$$

If $k = 1$, then $\hat{c}_n^{(1)}(z) = \hat{c}_n(z)$ are the Cauchy polynomials of the second kind. If $z = 0$, then $\hat{c}_n^{(k)}(0) = \hat{c}_n^{(k)}$ are the poly-Cauchy numbers of the second kind.

Theorem 12

$$\hat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k} .$$

Theorem 12

$$\hat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}.$$

Theorem 13

The generating function of the poly-Cauchy polynomials of the second kind $\hat{c}_n^{(k)}(z)$ is given by

$$(1+x)^z \mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)}(z) \frac{x^n}{n!}.$$

Corollary 4

For $k \ge 2$ we have

$$(1+x)^z \times \underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)^2\ln(1+x)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!}\,.$$

Corollary 4

For $k \geq 2$ we have

$$(1+x)^z \times \underbrace{\frac{1}{\ln(1+x)}\int_0^x \frac{1}{(1+x)\ln(1+x)}\int_0^x \cdots \frac{1}{(1+x)\ln(1+x)}\int_0^x}_{k-1} \frac{x}{(1+x)^2\ln(1+x)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{x^n}{n!} .$$

Theorem 14

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} \hat{c}_m^{(k)}(z) = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \frac{(-z)^i}{(n-i+1)^k} .$$

Relations between two kinds of poly-Cauchy polynomials

Theorem 15

For $n \ge 1$ we have

$$(-1)^n \frac{c_n^{(k)}(z)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\hat{c}_m^{(k)}(z)}{m!}\,,$$

$$(-1)^n \frac{\hat{c}_n^{(k)}(z)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{c_m^{(k)}(z)}{m!}\,.$$

Appell sequences

It is known that for an integer k and a positive integer n, the poly-Bernoulli polynomials satisfy

$$\frac{d}{dz}B_n^{(k)}(z) = nB_{n-1}^{(k)}(z)\,.$$

Appell sequences

It is known that for an integer k and a positive integer n, the poly-Bernoulli polynomials satisfy

$$\frac{d}{dz}B_n^{(k)}(z)=nB_{n-1}^{(k)}(z)\,.$$

The poly-Cauchy polynomials $c_n^{(k)}$ do not satisfy the similar identity.

Theorem 15

$$\frac{d}{dz}c_n^{(k)}(z) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)l!} c_l^{(k)}(z) \quad (n \geq 1)\,,$$

Theorem 15

$$\frac{d}{dz}c_n^{(k)}(z)=(-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)l!}c_l^{(k)}(z)\quad (n\ge 1)\,,$$

$$\frac{d}{dz}\hat{c}_n^{(k)}(z)=(-1)^{n-1} n!\sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)l!}\hat{c}_l^{(k)}(z)\quad (n\ge 1)\,.$$

Poly-Cauchy numbers with q parameter (by Mari Yokohama)

Let q be a real number with $q \neq 0$. Define the **poly-Cauchy numbers with q parameter** (of the first kind) $c_{n,q}^{(k)}$ by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - q) \dots (x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k \,.$$

Poly-Cauchy numbers with q parameter (by Mari Yokohama)

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$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - q)$$

$$\dots (x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k \,.$$

Hence, if $q = 1$, then $c_{n,1}^{(k)} = c_n^{(k)}$ are the poly-Cauchy numbers.

Poly-Cauchy numbers with q parameter $c_{n,q}^{(k)}$ can be expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$.

Theorem 16

For a real number $q \neq 0$

$$c_{n,q}^{(k)} = \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \geq 0,\, k \geq 1)\,.$$

Theorem 17

The generating function of the poly-Cauchy numbers with q parameter $c_{n,q}^{(k)}$ is given by

$$\mathrm{Lif}_k\left(\frac{\ln(1+qx)}{q}\right)=\sum_{n=0}^{\infty}c_{n,q}^{(k)}\frac{x^n}{n!}\quad (q\neq 0)\,.$$

The generating function of the poly-Cauchy numbers with q parameter in Theorem 17 can be also written in the form of iterated integrals as that of the poly-Cauchy numbers.

Corollary 5

For $k \geq 2$ we have

$$\underbrace{\frac{q}{\ln(1+qx)}\int_0^x \frac{q}{(1+qx)\ln(1+qx)}\int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)}\int_0^x}_{k-1} \frac{q((1+qx)^{1/q}-1)}{(1+qx)\ln(1+qx)}$$

$$\underbrace{dxdx\ldots dx}_{k-1}$$

$$=\sum_{n=0}^{\infty} c_{n,q}^{(k)}\frac{x^n}{n!}\,.$$

For $k=1$ we have

$$\frac{q((1+qx)^{1/q}-1)}{\ln(1+qx)}=\sum_{n=0}^{\infty} c_{n,q}\frac{x^n}{n!}\,.$$

Define the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ by

$$\hat{c}_{n,q}^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - q) \dots (-x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k \,.$$

Define the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ by

$$\hat{c}_{n,q}^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - q) \dots (-x_1 x_2 \dots x_k - (n-1)q) dx_1 dx_2 \dots dx_k \,.$$

Therefore, if $q = 1$, then $\hat{c}_{n,1}^{(k)} = \hat{c}_n^{(k)}$ are the poly-Cauchy numbers of the second kind.

Theorem 18

$$\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{q^{n-m}}{(m+1)^k}\,.$$

Theorem 18

$$\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] \frac{q^{n-m}}{(m+1)^k} \,.$$

Theorem 19

The generating function of the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ is given by

$$\mathrm{Lif}_k \left(-\frac{\ln(1+qx)}{q} \right) = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)} \frac{x^n}{n!} \,.$$

Corollary 6

For $k \geq 2$ we have

$$\underbrace{\frac{q}{\ln(1+qx)}\int_0^x \frac{q}{(1+qx)\ln(1+qx)}\int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)}\int_0^x}_{k-1} \frac{q(1-(1+qx)^{-1/q})}{(1+qx)\ln(1+qx)} \underbrace{dx dx \dots dx}_{k-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)} \frac{x^n}{n!}\,.$$

For $k = 1$ we have

$$\frac{q(1-(1+qx)^{-1/q})}{\ln(1+qx)} = \sum_{n=0}^{\infty} \hat{c}_{n,q} \frac{x^n}{n!}\,.$$

We shall consider integrals of the definition of poly-Cauchy numbers with q parameter in the range $[0, l]$, where l is a real number with $l \neq 0$ instead of the range $[0, 1]$.

We shall consider integrals of the definition of poly-Cauchy numbers with q parameter in the range $[0, l]$, where l is a real number with $l \neq 0$ instead of the range $[0, 1]$.
Define $c_{n,q}^{(k)}(l_1, l_2, \ldots, l_k)$, where $l_1, l_2, \ldots, l_k$ are nonzero real numbers, by

$$\begin{aligned} & c_{n,q}^{(k)}(l_1, l_2, \ldots, l_k) \\ & \quad = \int_0^{l_1} \int_0^{l_2} \cdots \int_0^{l_k} (x_1 x_2 \ldots x_k)(x_1 x_2 \ldots x_k - q) \\ & \qquad \ldots (x_1 x_2 \ldots x_k - (n-1)q) dx_1 dx_2 \ldots dx_k \,. \end{aligned}$$

Then $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$ can be also expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$.

Then $c_{n,q}^{(k)}(l_1, l_2, \ldots, l_k)$ can be also expressed in terms of the (unsigned) Stirling numbers of the first kind $\left[n \atop m \right]$.

Theorem 20

Let q be a real number with $q \neq 0$. Then for $n \geq 0$, $k \geq 1$ we have

$$c_{n,q}^{(k)}(l_1, l_2, \ldots, l_k) = \sum_{m=0}^{n} \left[n \atop m \right] \frac{(-q)^{n-m}(l_1 l_2 \ldots l_k)^{m+1}}{(m+1)^k}.$$

- **Is there any (combinatorial) interpretation of Poly-Cauchy numbers with negative index?**

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$$
\begin{aligned}
c_0^{(-k)} &= 1,\\
c_1^{(-k)} &= 2^k,\\
c_2^{(-k)} &= -2^k + 3^k,\\
c_3^{(-k)} &= 2 \cdot 2^k - 3 \cdot 3^k + 4^k,\\
c_4^{(-k)} &= -6 \cdot 2^k + 11 \cdot 3^k - 6 \cdot 4^k + 5^k.
\end{aligned}
$$

• Is there any (combinatorial) interpretation of Poly-Cauchy numbers with negative index?

$$
\begin{aligned}
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c_4^{(-k)} &= -6\cdot 2^k + 11\cdot 3^k - 6\cdot 4^k + 5^k.
\end{aligned}
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The number of $k \times n$ ($k, n \geq 1$) lonesome matrices is equal to $B_n^{(-k)}$.

• **Is there any (combinatorial) interpretation of Poly-Cauchy numbers with negative index?**

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The number of $k \times n$ $(k, n \geq 1)$ lonesome matrices is equal to $B_n^{(-k)}$.

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\} \quad (n, k \geq 0)\,.$$

It is easy to see

$$\boldsymbol{c_n^{(-1)} = (-1)^n (n-2)! \quad (n \geq 2)\,.}$$

It is easy to see

$$c_n^{(-1)} = (-1)^n (n-2)! \quad (n \ge 2)\,.$$

$$\{c_n^{(-2)}\}_{n\ge 0} = 1, 4, 5, -3, 4, -8, 20, -52, 72, 936, -17568, \dots$$
$$\{c_n^{(-3)}\}_{n\ge 0} = 1, 8, 19, -1, -10, 48, -234, 1302, -8328, 60672,$$
$$\{c_n^{(-4)}\}_{n\ge 0} = 1, 16, 65, 45, -116, 340, -1240, 5480, -28464,$$
$$\{c_n^{(-5)}\}_{n\ge 0} = 1, 32, 211, 359, -538, 984, -1866, 1110, 32640,$$

Sums of product of Cauchy numbers, including poly-Cauchy numbers

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- **Euler formula:**

$$\sum_{i=0}^{n}\binom{n}{i}B_i B_{n-i}=-nB_{n-1}-(n-1)B_n \quad (n\ge 1)\,.$$

Sums of product of Cauchy numbers, including poly-Cauchy numbers

- **Euler formula:**

$$\sum_{i=0}^{n}\binom{n}{i}B_iB_{n-i}=-nB_{n-1}-(n-1)B_n\quad(n\ge 1)\,.$$

$$\sum_{i=0}^{n}\binom{n}{i}c_i(c_{n-i}^{(k-1)}-c_{n-i}^{(k)})=n(n-1)c_{n-1}^{(k)}+nc_n^{(k)}\quad(n\ge 0)\,.$$

Denote $T_2^{(k)}(n) = \sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i}^{(k)}$.

Denote $T_2^{(k)}(n) = \sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i}^{(k)}$.

For $n \geq 0$ **and** $k \geq 1$ **we have**

$$T_2^{(0)}(n) = c_n^{(1)}(-1)\,,$$

$$T_2^{(k)}(n) = c_n^{(1)}(-1) - n \sum_{j=1}^{k} (c_n^{(j)} + (n-1)c_{n-1}^{(j)})\,,$$

$$T_2^{(-k)}(n) = c_n^{(1)}(-1) + n \sum_{j=0}^{k-1} (c_n^{(-j)} + (n-1)c_{n-1}^{(-j)})\,,$$

where $c_n^{(1)}(-1) = c_n + nc_{n-1}$.

Putting $k = 1$ in the second identity, we have

Putting $k = 1$ in the second identity, we have

$$\sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i} = -(n-1)c_n - n(n-2)c_{n-1} \quad (n \geq 0)\,.$$

Bernoulli polynomials

It is known that the Bernoulli polynomials $B_n(z)$ defined by the generating function

$$\frac{xe^{xz}}{e^x-1}=\sum_{n=0}^{\infty}B_n(z)\frac{x^n}{n!}$$

satisfy the identity

$$\sum_{l=0}^{n}\binom{n}{l}B_l(x)B_{n-l}(y)$$
$$=n(x+y-1)B_{n-1}(x+y)-(n-1)B_n(x+y)\quad(n\ge 0)$$

For $n \ge 0$

$$\sum_{i=0}^{n}\binom{n}{i}c_i(x)\bigl(c_{n-i}^{(k-1)}(y)-c_{n-i}^{(k)}(y)\bigr)$$
$$=n\bigl((x+y+n-1)c_{n-1}^{(k)}(x+y)+c_n^{(k)}(x+y)\bigr)\,.$$

Denote $T_2^{(k)}(n;x,y) = \sum_{i=0}^{n} \binom{n}{i} c_i(x) c_{n-i}^{(k)}(y)$**.**

Denote $T_2^{(k)}(n;x,y)=\sum_{i=0}^n\binom{n}{i}c_i(x)c_{n-i}^{(k)}(y)$.

For $n\ge 0$ and $k\ge 1$ we have

$$
\begin{aligned}
T_2^{(0)}(n;x,y)&=c_n^{(1)}(x+y-1)\,,\\
T_2^{(k)}(n;x,y)&=c_n^{(1)}(x+y-1)\\
&\qquad -n\sum_{j=1}^k\left(c_n^{(j)}(x+y)\right.\\
&\qquad\qquad \left.+(x+y+n-1)c_{n-1}^{(j)}(x+y)\right),
\end{aligned}
$$

$$\begin{aligned} T_2^{(-k)}(n;x,y) = c_n^{(1)}&(x+y-1) \\ &+ n\sum_{j=0}^{k-1}(c_n^{(-j)}(x+y) \\ &\quad + (x+y+n-1)c_{n-1}^{(-j)}(x+y))\,, \end{aligned}$$

where $c_n^{(1)}(x+y-1) = c_n(x+y) + nc_{n-1}(x+y)$.

Putting $k = 1$ in the second identity, we have

$$\sum_{l=0}^{n}\binom{n}{l}c_l(x)c_{n-l}(y)$$
$$= -n(x+y+n-2)c_{n-1}(x+y)-(n-1)c_n(x+y) \quad (n \geq 0)\,,$$