Ergodic Plünnecke inequalities

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First connection of additive combinatorics and ergodic theory

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Furstenberg Correspondence principle

Let $A \subset \mathbb{Z}$ be a set with $d^*(A) > 0$. Then there exists an ergodic $\mathbb{Z}$-system $(X, \mathbb{B}, \mu, T)$ and a set $\tilde{A} \in \Sigma$ such that for every $k \geq 1$, every $n_1, \ldots, n_k \in \mathbb{Z}$ we have

$$d^*((A - n_1) \cap \ldots \cap (A - n_k)) \geq \mu(T^{-n_1}\tilde{A} \cap \ldots \cap T^{-n_k}\tilde{A}).$$

Setting

Γ countable abelian group

A, B sets in Γ

$A + B = \{a + b \,|\, a \in A, b \in B\}$

Folner sequences

A sequence of **finite** sets $F_n \subset \Gamma$ is Folner if for every $\gamma \in \Gamma$ we have

$$\frac{|(\gamma + F_n) \cap F_n|}{|F_n|} \to 1 \text{ as } n \to \infty.$$

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Upper Banach density of $A\subset\Gamma$:

$$d^*(A)=\sup_{(F_n)\text{ Folner}} \limsup \frac{|A\cap F_n|}{|F_n|}.$$

Examples

$d^*(2\mathbb{Z}) = \frac{1}{2}$, $d^*(\cup_n [n!, n! + n] \cap \mathbb{Z}) = 1$, $d^*(\square) = 0$.

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Furstenberg correspondence principle for sumsets

$A \subset \Gamma$ with $d^*(A) > 0$. $\exists$ an ergodic Γ-system (X, Σ, μ, T) and $A \in \Sigma$ s.t. $\forall B \subset \Gamma$:

$$d^*(A+B) \geq \mu(\cup_{\gamma \in B} T_\gamma A),$$

$$d^*(A) = \mu(A).$$

Quasi-ergodic sets

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2) (Boshernitzan, Kolesnik, Quas, Wierdl) $\Rightarrow$

$$B = \{\lfloor n^\alpha \rfloor | n \in \mathbb{N}\}, \alpha \notin \mathbb{Q}$$

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***k* quasi-ergodic set**

Let $k \geq 1$. A set $B \subset \Gamma$ is k **quasi-ergodic** if the set kB is quasi-ergodic.

Examples

1) Vinogradov theorem $\Rightarrow$ Primes is 4-quasi-ergodic.
2) Laplace theorem $\Rightarrow$ $\Box$ is 4-quasi-ergodic.

Main results

Björklund, F.

$B \subset \Gamma$ is k-quasi-ergodic. Then for every ergodic Γ-system (X, Σ, μ, T) and every $A \in \Sigma$ we have

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$$d^*(A+B) \geq \mu(\cup_{\gamma \in B} T_\gamma A) \geq \mu(A)^{1-\frac{1}{k}} = d^*(A)^{1-\frac{1}{k}}.$$

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Renling Jin

$A, B \subset \mathbb{N}$, $d^*(kB) = 1$, then

$$d^*(A+B) \geq (d^*(A))^{1-\frac{1}{k}}.$$

Magnification ratios

For $A, B \subset \Gamma$ finite sets:

$$\mu_k = \inf_{\emptyset \neq A' \subset A} \frac{|A' + kB|}{|A'|}, \; k \geq 1,$$

for $\delta > 0$

$$\mu_{k,\delta} = \inf_{A' \subset A, \, |A'| \geq \delta |A|} \frac{|A' + kB|}{|A'|}, \; k \geq 1$$

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Corollary (Plünnecke)

There exist $c_{k,\delta} > 0$, $c_{k,\delta} \to 1$ as $\delta \to 0$ such that

$$\mu_1 \geq c_{2,\delta} \mu_2^{\frac{1}{2}} \geq c_{3,\delta} \mu_3^{\frac{1}{3}} \geq \ldots \geq c_{k,\delta} \mu_k^{\frac{1}{k}}$$

Ergodic Theory

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If $B \subset \Gamma$ is finite then $\mu_k^{\frac{1}{k}}$ is a decreasing sequence.

Assume $\mu_k^{\frac{1}{k}}$ is $\downarrow$ for any $B \subset \Gamma$.

Application of Ergodic Plünnecke inequalities

Assume $\mu_k^{\frac{1}{k}}$ is $\downarrow$ for any $B \subset \Gamma$.

$B \subset \Gamma$ s.t. kB is quasi-ergodic: $\mu(\cup_{\gamma \in kB} T_\gamma A) = 1$, $\forall A : \mu(A) > 0$

$$\mu_1^k \geq \mu_k = \inf_{A' \subset A, \mu(A') > 0} \frac{\mu(\cup_{\gamma \in kB} T_\gamma A')}{\mu(A')} = \frac{1}{\mu(A)}$$

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Sketch of proofs for $\Gamma = \mathbb{Z}$

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- Periodic case ($\mu(Fix) = 1$) follows from Plünnecke inequalities.
- Aperiodic case ($\mu(Fix) = 0$):
  - Prove inequalities for a special aperiodic ergodic $\mathbb{Z}$-system
  - Use conjugacy lemma of Halmos to prove inequalities for a general aperiodic $\mathbb{Z}$-system.

Special aperiodic ergodic $\mathbb{Z}$-system

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Periodic partitions of $(\mathbb{Z}_2, \mathbb{B}, m_{\mathbb{Z}_2}, T)$

For every n, there exists a partition $P_n = \{C_0, TC_0, \ldots, T^{2^n-1}C_0\}$ of $\mathbb{Z}_2$.

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Use the approximation by elements of partition P_n and Plünnecke inequalities in $\mathbb{Z}$ to prove that for the system $(\mathbb{Z}_2, \mathbb{B}, m_{\mathbb{Z}_2}, T)$ and a finite $B \subset \mathbb{Z}$ ergodic Plünnecke inequalities hold true.

(X, Σ, μ) is a standard measure space.

$$APER = \{S : X \to X, S\mu = \mu, S \text{ is aperiodic}\}.$$

Then for any $T \in APER$, the conjugacy class of T

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Halmos conjugacy lemma

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$\forall S \in APER$, $\forall \varepsilon > 0$, $\exists \sigma \in Aut(X, \mu)$ $\forall A \in \Sigma$:

$$\mu(S(A) \triangle \sigma T \sigma^{-1}(A)) < \varepsilon.$$

Theorem 2

$B \subset \mathbb{Z}$ s.t. B is quasi-ergodic, $\delta > 0$. Then

$$\sup_{B' \subset B, |B'| < \infty} \inf_{A' \subset A, \mu(A') \geq \delta \mu(A)} \frac{\mu(\cup_{n \in B'} T^n A')}{\mu(A')} \geq \frac{1}{\mu(A)}.$$

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Let $\delta > 0$. Then

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Proof

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Aperiodic case: Uses Rokhlin's lemma and poitnwise ergodic theorem.

Thank you!!!