2–graphs and Textile Systems

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[1] with extra thanks to Aidan Sims for his brilliant pictures!

What is a 2–graph?

Definition 1 (Kumjian-P)

A *2-graph* (Λ, d) consists of a countable small category Λ with a functor $d : \Lambda \to \mathbb{N}^2$ satisfying the factorization property : for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^2$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$.

For $m \in \mathbb{N}^2$, we let $\Lambda^m = d^{-1}(m)$. For $m, n \in \mathbb{N}^2$, we say that $m \le n$ if $m_i \le n_i$ for each i.

Notation 2 (Raeburn–Sims–Yeend)

For $0 \le m \le n \le d(\lambda)$, by the factorization property we have $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, d(\lambda))$ where $d(\lambda(0, m)) = m$, $d(\lambda(m, n)) = n - m$ and $d(\lambda(n, d(\lambda))) = d(\lambda) - n$.

This talk concerns two equivalent ways of studying 2–graphs using directed graphs.

Directed graphs

A *directed graph* E consists of a set E^0 of vertices, a set E^1 of edges and maps $r, s : E^1 \to E^1$ giving the direction of each edge. Let E^n denote the paths of length n in E, and E^* denote the collection of finite paths in E.

A *graph morphism* $\phi : F \to E$ is a pair $\phi = (\phi^0, \phi^1)$ of maps $\phi^i : F^i \to E^i$ for $i = 0, 1$ such that for all $f \in F^1$

$$s(\phi^1(f)) = \phi^0(s(f)), \quad r(\phi^1(f)) = \phi^0(r(f)).$$

A graph morphism $\phi : F \to E$ has [unique] r-path (resp. s–path) lifting if for every $v \in \phi^0(F^0)$, $e \in \phi^1(F^1)$ with $r(e) = v$ (resp. $s(e) = v$) and $w \in F^0$ with $\phi^0(w) = v$ there is a [unique] $f \in \phi^1(F^1)$ with $r(f) = w$ (resp. $s(f) = w$) such that $\phi^1(f) = e$.

Coloured graphs

Let $\mathbb{F}_2$ be the free semigroup on 2-generators $\{b, r\}$.

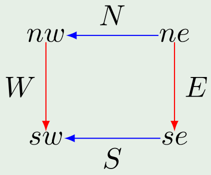
Definition 3 (Hazelwood, Raburn, Sims, Webster)

A 2-*coloured graph* (E, c) is a directed graph E together with a map $c : E^1 \to \{r, b\}$, which we can extend to a functor $c : E^* \to \mathbb{F}_2^+$.

Example 4

With $c_2(N) = b = c_2(S)$ and $c_2(E) = r = c_2(W)$ we have

$(E_2, c_2) :=$



Coloured graphs II

A 2-*coloured-graph morphism* from (E, c_E) to (F, c_F) is a graph morphism ψ from E to F such that $c_E(e) = c_F(\psi^1(e))$ for every $e \in E^1$.

Given a 2-coloured graph (E, c), a *square* in E is a coloured-graph morphism $\phi : (E_2, c_2) \to (E, c)$, and we identify a square ϕ with its image in E.

For a 2-coloured graph (E, c), a *complete collection of squares* is a collection $\mathcal{C}$ of squares in E such that for each $ef \in E^2$ with $c(e) \neq c(f)$ there exists a unique $\phi \in \mathcal{C}$ such that $ef = \phi(S)\phi(E)$ if $c(ef) = br$ and $ef = \phi(W)\phi(N)$ if $c(ef) = rb$.

2–coloured graphs and 2–coloured graphs

Let Λ be a 2–graph, then we may define a 2-coloured graph (E_Λ, c_Λ) and a complete collection of squares $\mathcal{C}_\Lambda$ in E_Λ as follows:

Let E_Λ be the directed graph with $E_\Lambda^0 = \Lambda^0$ and $E_\Lambda^1 = \Lambda^{(1,0)} \cup \Lambda^{(0,1)}$ with range and source maps inherited from Λ. The graph E_Λ is usually referred to as the 1-*skeleton* of Λ. Define $c_\Lambda : E_\Lambda^1 \to \mathbb{F}_2$ by $c_\Lambda(e) = b$ if $d(e) = (1,0)$ and $c_\Lambda(e) = r$ if $d(e) = (0,1)$.

For $\lambda \in \Lambda^{(1,1)}$ define a square $\phi_\lambda : (E_2, c) \to (E_\Lambda, c_\Lambda)$ by

$$W \mapsto \lambda\big((0,0),(0,1)\big) \in \Lambda^{(0,1)},\ N \mapsto \lambda\big((0,1),(1,1)\big) \in \Lambda^{(1,0)},$$
$$S \mapsto \lambda\big((0,0),(1,0)\big) \in \Lambda^{(1,0)},\ E \mapsto \lambda\big((1,0),(1,1)\big) \in \Lambda^{(0,1)}.$$

One checks that $\mathcal{C}_\Lambda = \{\phi_\lambda : \lambda \in \Lambda^{(1,1)}\}$ is a complete collection of squares in E_Λ

2–coloured graphs and 2–coloured graphs II

The converse to this result is given by.

Theorem 5 (Hazelwood, Raeburn, Sims, Webster)

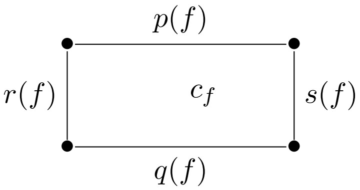
Let (E, c) be a 2-colored graph, and $\mathcal{C}$ be a complete collection of squares in E. Then there is a unique 2-graph $\Lambda = \Lambda(E, c)$ and a 2-coloured graph isomorphism from (E, c) to (E_Λ, c_Λ) taking $\mathcal{C}$ to $\mathcal{C}_\Lambda$.

Hence 2–graphs and 2–coloured graphs with a complete collections of squares are in bijective correspondence.

Textile systems

A *textile system*[2] is a quadruple $T = (F, E, p, q)$, where $F = (F^0, F^1, r_F, s_F)$, $E = (E^0, E^1, r_E, s_E)$ are two directed graphs, and $p, q : F \to E$ are two graph morphisms such that the map $C : F^1 \to E^1 \times E^1 \times F^0 \times F^0$ given by $f \mapsto (p(f), q(f), r_F(f), s_F(f))$ is injective.

Intuitively, the textile condition implies that for every $f \in F^1$ there is a unique "tile" c_f as shown:



(1)

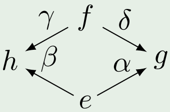
[2]adapted from Nasu

Examples

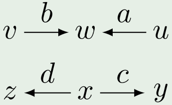
Example 6

Let

$F :=$



$E :=$



Define $p, q : F \to E$ by

$$p(\alpha) = a, p(\beta) = a, p(\gamma) = b, p(\delta) = b, \text{ and}$$
$$q(\alpha) = c, q(\beta) = d, q(\gamma) = d, q(\delta) = c.$$

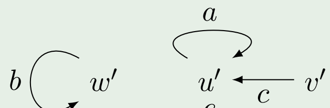
Then $T_1 = (F, E, p, q)$ is a textile system.

Examples II

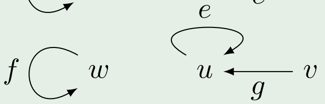
Example 7

Let

$F :=$



$E :=$



and define $p, q : F \to E$ by

$$p(u') = u, p(v') = v, p(w') = w \text{ and } p(a) = e, p(c) = g, p(b) = f$$
$$q(u') = u, q(v') = v, q(w') = u \text{ and } q(a) = e, q(c) = g, q(b) = e.$$

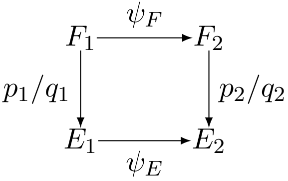
Then $T_2 = (F, E, p, q)$ is a textile system.

Equivalence

Let $T_1 = (F_1, E_1, p_1, q_1)$, $T_2 = (F_2, E_2, p_2, q_2)$ be two textile systems. Then T_1, T_2 are *equivalent* if there are graph isomorphisms $\psi_F : F_1 \to F_2$, $\psi_E : E_1 \to E_2$ such that for all $f \in F^1$ we have

$$p_2(\psi_F^1(f)) = \psi_E^1(p_1(f)), \quad q_2(\psi_F^1(f)) = \psi_E^1(q_1(f)).$$

That is, the squares



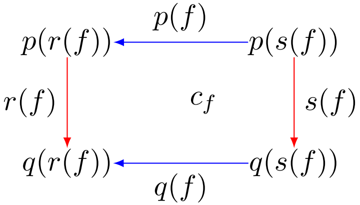
commute.

Textile ststems and coloured graphs

Given a textile system $T = (F, E, p, q)$, we may define 2-colored graph (G_T, c_T) as follows. Let $G_T^0 = E^0$, $G_T^1 = E^1 \sqcup F^0$, and

$$r(e) = r_E(e),\ s(e) = s_E(e),\ c_T(e) = b \text{ for } e \in E^1,$$
$$r(v) = q(v),\ s(v) = p(v),\ c_T(v) = r \text{ for } v \in F^0.$$

Since the map C is injective, each $f \in F^1$ uniquely determines a square $c_f : (E_2, c_2) \to (G_T, c_T)$ with image in G_T given by:



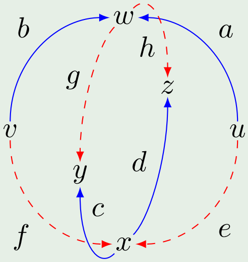
(2)

If p has unique r–path lifting and q has unique s–path lifting then $\mathcal{C}_T = \{c_f : f \in F^1\}$ is a complete collection of squares in G_T.

Back to examples

Example 8

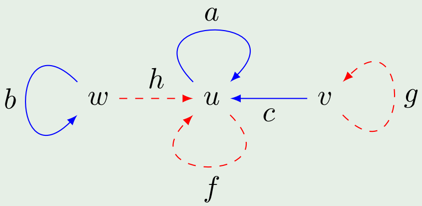
Recall the textile system T_1 described in Example 6 then (G_{T_1}, c_{T_1}) is the 2-coloured graph



Back to examples II

Example 9

Recall the textile system T_2 described in Example 7 then (G_{T_2}, c_{T_2}) is the 2-coloured graph



Textile ststems and coloured graphs II

Given a 2–coloured graph (G, c) with a complete collection of squares $\mathcal{C} = \{\phi_i : i \in I\}$ we may define a textile ststem $T_{G,c} = (F, E, p, q)$ as follows:

Set E to be the graph with $E^0 = G^0$, $E^1 = c^{-1}(b)$ and r_E, s_E inherited from G.

Set F to be the graph with $F^0 = c^{-1}(r)$, $F^1 = \mathcal{C}$, and for $\phi_i \in \mathcal{C}$ we put $r_F(\phi_i) = \phi_i(W)$ and $s_F(\phi_i) = \phi_i(E)$.

Finally, we define $p, q : F \to E$ by $p^0(w) = r(w)$, $p^1(\phi_i) = \phi_i(N)$; $q^0(w) = s(w)$, $q^1(\phi_i) = \phi_i(S)$.

Since $\mathcal{C}$ is a complete collection of squares it follows that p has unique r–path lifting and q has unique s–path lifting.

Textile systems and coloured graphs III

Theorem 10

(1) *Let $T = (F, E, p, q)$ be a textile system such that p has unique r–path lifting and q has unique s–path lifting. Then T_{G,c_T} is equivalent to T.*

(2) *Let (G, c) be a 2–coloured graph with a complete collection of squares $\mathcal{C}$. Then there is a 2–coloured graph isomorphism from (G, c) to $(G_{T_{G,c}}, c_{T_{G,c}})$ which takes $\mathcal{C}$ to $\mathcal{C}_{T_{G,c}}$.*

THANK YOU!