(k, L)-complexes and s-arc transitive graphs

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(k, L)-complexes

Given an integer $k \geq 3$ and a graph L, a (k, L)-complex is a complex of vertices, edges and faces such that

- each face is a regular k-gon
- for each vertex v, the graph with
  - vertices: edges incident with v
  - edges: the faces incident with v
  - adjacency inherited from incidence.

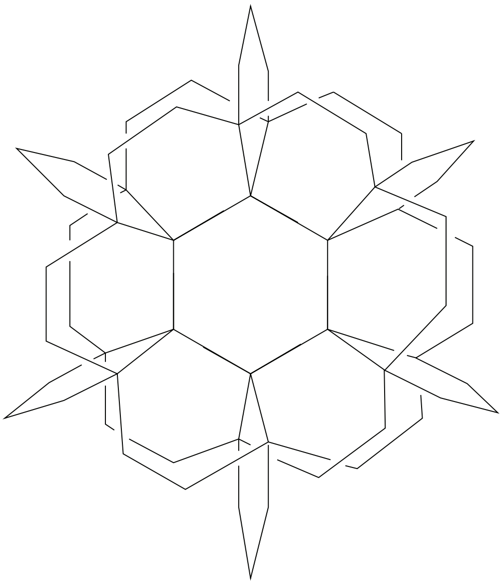
  is isomorphic to L.

The graph at v is called the link.

Examples

- Platonic solids
  - Tetrahedron is a $(3, C_3)$-complex
  - Cube is a $(4, C_3)$-complex
  - Icosahedron is a $(3, C_5)$-complex
- Tessellation of Euclidean plane by equilateral triangles is a $(3, C_6)$-complex.

Part of a $(6, K_4)$-complex



Existence

Let $\mathcal{X}(k, L)$ be the set of all simply connected (k, L)-complexes.

How large is $\mathcal{X}(k, L)$?

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$$\mathrm{girth}(L) \geq \frac{2k}{k-2}$$

Ballmann and Brin: For any pair (k, L) satisfying the Gromov link condition with L regular, there is a (k, L)-complex.

Existence II

If $k \geq 4$ is even then there is a (k, L)-complex (Davis complex).

If k is odd and $L = K_{m,n}$ with $m \neq n$ then there is no (k, L)-complex.

Uniqueness?

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Unique (k, L)-complex in following cases:

- (k, C_n) for $n \geq 3, 4, 6$ and $k \geq 6, 4, 3$: tiling of Euclidean/hyperbolic plane with regular k-gons with angles $\frac{2\pi}{n}$.
- $(4, K_{m,n})$: product of m-regular and n-regular tree.
- $(k, K_{m,m})$, $k > 4$: Bourdon's building

s-arc transitive graphs

An s-arc is an $(s+1)$-tuple $v_0, v_1, \ldots, v_s$ such that $v_i \sim v_{i+1}$ and $v_i \neq v_{i+2}$.

- locally s-arc transitive: $\mathrm{Aut}(\Gamma)_v$ is transitive on the set of s-arcs starting at v, for all vertices v.
- s-arc transitive: $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs.

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Świątkowski: If

- $k \geq 4$ and (k, L) satisfies Gromov link condition, and
- L has valency 3 and is 3-arc transitive

then $|\mathcal{X}(k, L)| = 1$.

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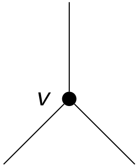
Star-transitive

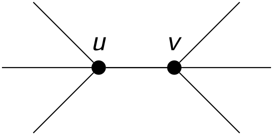
open star $\mathrm{st}(v)$:



We say that Γ is star-transitive if every isomorphism $\mathrm{st}(v_1) \to \mathrm{st}(v_2)$ lifts to an automorphism of Γ.

st(edge)-transitive

open edge-star, $\mathrm{st}(\{u, v\})$:



We say that Γ is st(edge)-transitive if every isomorphism $\mathrm{st}(\{u_1, v_1\}) \to \mathrm{st}(\{u_1, v_1\})$ lifts to an automorphism of Γ.

Uniqueness II

Lazarovich: If $k \geq 4$, (k, L) satisfies the Gromov Link condition and L is star-transitive and st(edge)-transitive then $|\mathcal{X}(k, L)| \leq 1$.

Already seen that when $k \geq 4$ is even then there is a (k, L).

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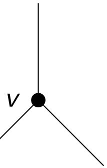
Which graphs are star-transitive and st(edge)-transitive?

Some observations:

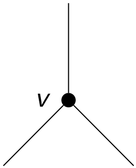
Suppose Γ is star-transitive and $G = \mathrm{Aut}(\Gamma)$:



Some observations:

Suppose Γ is star-transitive and $G = \mathrm{Aut}(\Gamma)$:

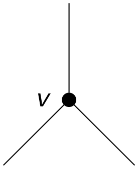
- If Γ is regular then Γ is vertex-transtiive.



Some observations:

Suppose Γ is star-transitive and $G = \mathrm{Aut}(\Gamma)$:

- If Γ is regular then Γ is vertex-transtiive.
- If $|\Gamma(v)| = k$ then $G_v^{\Gamma(v)} = S_k$ (locally fully symmetric).



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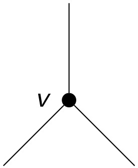
- If Γ is regular then Γ is vertex-transtiive.
- If $|\Gamma(v)| = k$ then $G_v^{\Gamma(v)} = S_k$ (locally fully symmetric).
- Γ is locally 2-arc-transitive.



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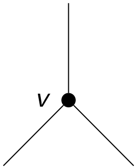

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- Γ is locally 2-arc-transitive.

Conversely, if Γ is locally fully symmetric and either

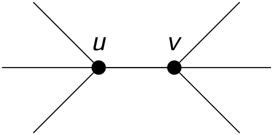
- Γ is vertex-transitive, or
- Γ has vertices of valency k and ℓ with $k \neq \ell$

then Γ is star-transitive.

Examples

- K_n is star-transitive: only connected star-transitive graph of girth 3.
- $K_{n,m}$
- Odd graphs O_k:
  - k-subsets of $2k+1$-set
  - adjacent if disjoint
  - vertex stabiliser $S_k \times S_{k+1}$

More observations:



Γ of minimal valency 3 is st(edge)-transitive if and only if it is edge-transitive and either:

- Γ is k-regular and for all edges

$$(G_{\{u,v\}})^{\Gamma(u)\cup\Gamma(v)} = S_{k-1} \operatorname{wr} S_2$$

- Γ is (k,ℓ)–biregular for $k \neq \ell$ and for all edges

$$(G_{\{u,v\}})^{\Gamma(u)\cup\Gamma(v)} = S_{k-1} \times S_{\ell-1}$$

Star-transitive and st(edge)-transitive

st(edge)-transitive and minimal valency at least three implies

- star-transitive
- 3-arc transitive.

Star-transitive and st(edge)-transitive

st(edge)-transitive and minimal valency at least three implies

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Examples

- K_3 (only st(edge)-transitive graph of girth 3).
- $K_{n,m}$ (only st(edge)-transitive graphs of girth 4)
- Point-line incidence graph of $\mathrm{PG}(2,2)$ and $\mathrm{PG}(2,3)$
- Point-line incidence graph of $W(3,2)$.

Cubic graphs

Tutte, Djokovič and Miller:

s	G_v	$G_{\{u,v\}}$
1	C_3	C_2
2	S_3	$C_2 \times C_2$ or C_4
3	$S_3 \times C_2$	D_8
4	S_4	D_{16} or QD_{16}
5	$S_4 \times C_2$	$(D_8 \times C_2) \rtimes C_2$

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So star-transitive if and only if $s \geq 2$ and st(edge)-transitive if and only if $s \geq 3$.

Valency 4 case

Possible vertex and edge stabilisers determined by Potočnik, Weiss.

Theorem

Star-transitive if and only if one of

- Γ *is 2-arc–transitive, and* $G_v = S_4$*;*
- Γ *is* 3*-arc-transitive, and* $G_v = S_4 \times S_3$ *or* $G_v = (A_4 \times C_3).2$*;*
- Γ *is* 4*-arc-transitive, and* $G_v = 3^2{:}\,\mathrm{GL}(2,3)$*;*
- Γ *is* 7*-arc-transitive, and* $G_v = [3^5]{:}\,\mathrm{GL}(2,3)$*.*

Parabolics of $\mathrm{PGL}(3,3)$ and $G_2(3)$.

Valency 4 case II

Theorem

Γ *is star-transitive and st(edge)-transitive if and only if one of the following is true.*

- Γ *is* 3*-arc-transitive, and* $G_v = S_4 \times S_3$*;*
- Γ *is* 4*-arc-transitive, and* $G_v = 3^2{:}\,\mathrm{GL}(2,3)$*;*
- Γ *is* 7*-arc-transitive, and* $G_v = [3^5]{:}\,\mathrm{GL}(2,3)$*.*

Arbitrary valency

Theorem

Γ a vertex-transitive graph. Then Γ is star-transitive and st(edge)-transitive if and only if one of the following holds:

- *Γ is 3-arc-transitive, and $G_v = S_r \times S_{r-1}$.*
- *Γ is cubic and 4-arc-transitive, and $G_v = S_4$ or $S_4 \times S_2$.*
- *Γ is of valency 4 and 4-arc-transitive, and $G_v = 3^2{:}\,\mathrm{GL}(2,3)$ or $[3^5]{:}\,\mathrm{GL}(2,3)$.*

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Vertex stabilisers of O_k, $\mathrm{PG}(2,2)$, $W(3,2)$, $\mathrm{PG}(2,3)$ or $G_2(3)$.

Vertex-intransitive examples

- $K_{m,n}$ for $m \neq n$
- Point-line incidence graph of GQ for $\mathrm{PSU}(4,2)$: valency $\{3,5\}$.
- k-subset, $(k-1)$-subsets of n-set with adjacency given by inclusion

$$G_v = S_k \times S_{n-k} \text{ and } G_w = S_{k-1} \times S_{n-k+1}$$

- Vertex-maximal clique graph of Hamming graph $H(k,n)$: valency $\{k,n\}$

$$G_v = S_{n-1} \operatorname{wr} S_k \text{ and } G_w = S_n \times (S_n \operatorname{wr} S_{k-1})$$

Vertex-intransitive case

Theorem

Γ a vertex-intransitive graph valency $\{\ell, r\}$. If Γ is star-transitive and st(edge)-transitive then one of the following holds:

- $G_v = S_r \times S_{\ell-1}$ *and* $G_w = S_\ell \times S_{r-1}$
- $$\begin{aligned} ((A_{r-1} \operatorname{wr} S_{\ell-1}) \times A_r).2 &\leqslant G_w \leqslant (S_{r-1} \operatorname{wr} S_{\ell-1}) \times S_r \\ A_{r-1} \operatorname{wr} S_\ell &\leqslant G_v \leqslant S_{r-1} \operatorname{wr} S_\ell \end{aligned}$$
- $r \leq 5$.