Relative hyperbolicity and near-hyperbolicity

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A bit of background

- Hyperbolic groups are groups that act geometrically on δ-hyperbolic spaces.
- Suppose that G acts on a δ-hyperbolic space X.
- The boundary at infinity ∂X is a compact metrizable space.
- There is an induced action of G on ∂X by homeomorphisms.
- The existence and dynamics of such an action on a compact metric space K, actually characterises hyperbolic groups.
- Can also consider the induced equivalence relation on K.
  - Is it hyperfinite? Tame? Near-hyperbolic?

Relative hyperbolicity

Definition

Let G be a countable group acting a compact metric space K. The action is a *convergence action* if the induced action on the space of distinct triples is properly discontinuous.

Theorem (Tukia)

Suppose that a group G acts properly discontinuously on a proper δ-hyperbolic space X. Then the induced action on ∂X is a convergence action.

Theorem (Bowditch)

Suppose that a group G admits a convergence action on a compact metric space K. If the action on $[K]^3$ is cocompact, then G is hyperbolic. Then G is hyperbolic.

Definition

Let G be a countable group with a convergence action on compact metric space K.

1. An element $g \in G$ is *loxodromic* if it fixes exactly two points of K.
2. A subgroup $P \leqslant G$ is *parabolic* is if it is infinite and contains no loxodromic element. The fixed point of P is called a *parabolic* point.
3. A parabolic point $p \in K$ is called *bounded* if $\mathrm{Stab}_G(p)$ acts cocompactly on $K \setminus \{p\}$.
4. A point $\xi \in K$ is a *conical limit point* if there exists a sequence $\{g_i\} \subseteq G$ and distinct points α, $\beta \in K$ such that $g_i(\xi) \to \alpha$ and $g_i(\xi') \to \beta$ for all $\xi' \in K \setminus \{\xi\}$.

Relative hyperbolicity

Definition (Gromov, Farb, Bowditch,...)

A countable group G is *relatively hyperbolic* if it admits a properly discontinuous action on a proper δ-hyperbolic space X such that the induced convergence group action on ∂X is such that every point of ∂X is either a bounded parabolic point or a conical limit point.

We say that $(G, \mathcal{P})$ is relatively hyperbolic, where $\mathcal{P}$ is a set of representatives of the conjucacy classes of maximal parabolic subgroups.

Examples

- free groups
- geometrically finite Kleinian groups
- fundamental groups of complete non-compact finite volume Riemannian manifolds of pinched negative sectional curvature
- limit groups (Dahmani-Groves)

Consider $F_2 = \langle x, y \mid - \rangle$ acting on a regular 4-valent tree T (its Cayley graph).

Get an action on $K = \partial T$, the space of ends of T.

All elements of K are conical limit points:
Identifying K with the set of infinite reduced words over $\{x, x^{-1}, y, y^{-1}\}$
Given $\xi = (a_i) \in K$, let $g_i = (a_1 a_2 \dots a_i)^{-1}$, $\alpha = \lim g_i \xi$ and $\beta = \lim g_i$

Induced equivalence relation on K

The induced equivalence relation is *tail equivalence:*

$$(a_i) E_t (b_i) \iff \exists n \exists m \forall i (a_{n+i} = b_{m+i})$$

Theorem

1. *E_t is not tame.*
2. *E_t is hyperfinite. (Dougherty-Jackson-Kechris)*

Borel equivalence relations

Definition

Let X be a *standard Borel space,* that is, a set equipped with a σ-algebra that is Borel isomorphic to the σ-algebra of the Borel sets in a Polish space.

A *Borel equivalence relation* E on X is an equivalence relation which is Borel as a subset of X^2 (with the product Borel structure).

Let E be a Borel equivalence relation on a standard Borel space X.

1. E is *finite* if every equivalence class is finite.
2. E is called *hyperfinite* if $E = \cup_n E_n$, where $\{E_n\}$ is an increasing sequence of finite Borel equivalence relations.
3. E is called *tame* if there is a Borel map $f : X \to Y$ with Y a standard Borel space and $x_1 E x_2 \iff f(x_1) = f(x_2)$.

Near-hyperbolicity

Introduced by Adams 1996, Kechris-Hjorth 2005

Used to prove various results about Borel reducibility. For example, they produce an infinite family of countable groups $\{G_p\}$ each with a Borel, free action on a standard Borel space X_p so that $E_{G_p}^{X_p} \not\leqslant_B E_{G_q}^{X_q}$ if $p \neq q$.

Given a compact metric space K, denote by $\mathcal{M}(K)$ the compact metric space of all probability Borel measures on K. Denote by $\mathcal{M}_{\leqslant 2}(K)$ the subset of $\mathcal{M}(K)$ consisting of those measures having support of cardinality at most 2, that is,

$$\mathcal{M}_{\leqslant 2}(K) = \{\mu \in \mathcal{M}(K) \mid \exists\ a, b \in K \text{ such that } \mu(\{a, b\}) = 1\}.$$

Denote its complement in $\mathcal{M}(K)$ by $\mathcal{M}_3(K)$, that is,

$$\mathcal{M}_3(K) = \mathcal{M}(K) \setminus \mathcal{M}_{\leqslant 2}(K)$$

Definition

A countable group G is called *near-hyperbolic* if it admits a continuous action on a compact metric space K with the following properties:

1. The induced action on $\mathcal{M}_{\leqslant 2}(K)$ has amenable stabilizers and the induced equivalence relation on $\mathcal{M}_{\leqslant 2}(K)$ is hyperfinite.
2. The induced action of G on $\mathcal{M}_3(K)$ has finite stabilizers and the induced equivalence relation on $\mathcal{M}_3(K)$ is tame.

- Closed under taking subgroups.
- Amenable groups are near-hyperbolic.
- F_2 is near-hyperbolic.

Theorem (Kechris-Hjorth 2005)

Every hyperbolic group is near-hyperbolic.

Theorem

Let $(G, \mathcal{P})$ be relatively hyperbolic. If each element of $\mathcal{P}$ is amenable then G is near-hyperbolic.

F_2 is near-hyperbolic

Let $K = \partial F_2$

- For $a, b \in K$, $\mathrm{Stab}_{F_2}\{a, b\}$ is either trivial or $\mathbb{Z}$, so $F_2 \curvearrowright \mathcal{M}_{\leqslant 2}(K)$ has stabilizers that are either trivial or $\mathbb{Z}$

- $E_{F_2}^{\mathcal{M}_{\leqslant 2}(K)}$ is hyperfinite follows from Dougherty-Jackson-Kechris.
  - DJK $\implies$ $E_{F_2}^{[K]^2}$ is hyperfinite

- $E_{F_2}^{\mathcal{M}_3(K)}$ is tame:
  - 'median' map $\varphi : [K]^3 \to F_2$ is a Borel F_2-map
  - Define a Borel F_2-map $\theta : \mathcal{M}_3(K) \to \mathcal{F} = \{A \subseteq F_2 \mid |A| < \infty\}$ via

$$\theta(\mu) = \{g \in F_2 \mid \nu(\{g\}) \text{ is maximal } \} \quad \text{where} \quad \nu = \varphi_*(\mu^3 \mid_{[K]^3})$$

  - take 'centre of mass' of $A \in \mathcal{F}$ to get a Borel F_2-map $\mathcal{M}_3(K) \to F_2$
  - Follows that $E_{F_2}^{\mathcal{M}_3(K)}$ is tame with trivial stabilizers (Adams 1996)

Thanks for listening