Geometric Satake, Springer correspondence, and small representations

Anthony Henderson

University of Sydney

September 2012

Let G be a connected reductive algebraic group over $\mathbb{C}$ with maximal torus T and Weyl group $W = N_G(T)/T$. Define

$$\Phi : \mathrm{Rep}(G) \to \mathrm{Rep}(W) : V \mapsto V^T \otimes \varepsilon.$$

For which irreps V of G can we describe $\Phi(V)$?

Example

If $G = GL_n$, then W is the symmetric group S_n.

- For $n = 2$, one can easily calculate that

$$\Phi(V) = \begin{cases} \varepsilon^{m+1}, & \text{if } V = S^{2m}(\mathbb{C}^2) \otimes \det^{-m}, \\ 0, & \text{otherwise (centre acts nontrivially).} \end{cases}$$

- If λ is a partition of n, then $\Phi(V(\lambda_1 - 1, \cdots, \lambda_n - 1))$ is an irreducible representation of S_n, and all such arise in this way.
- There is no general formula for the decomposition of $\Phi(V)$.

Suppose $E \in \mathrm{Rep}(G)$ has an action of W that commutes with the action of G. Then we have another functor

$$\mathrm{Hom}_G(E, -) : \mathrm{Rep}(G) \to \mathrm{Rep}(W).$$

Example

When $G = GL_n$, let $E = (\mathbb{C}^n)^{\otimes n} \otimes \det^{-1}$, with S_n permuting the $\mathbb{C}^n$ factors. The weights $(a_1, \cdots, a_n)$ of E all satisfy

$$a_1 + \cdots + a_n = 0, \ \ a_i \geq -1.$$

On the subcategory of representations with weights of this kind,

$$\Phi \cong \mathrm{Hom}_{GL_n}(E, -).$$

However, there is no analogous E for general G.

Philosophically, representation theory associated with G should be related to geometry associated with the Langlands dual group $\check{G}$.

Theorem (Lusztig, Ginzburg, Mirković–Vilonen)

There is an equivalence ("geometric Satake")

$$\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Perv}(\mathrm{Gr}) : V \mapsto \mathrm{Sat}(V)$$

where $\mathrm{Gr} = \check{G}(\mathbb{C}[t, t^{-1}])/\check{G}(\mathbb{C}[t])$ *is the affine Grassmannian and the perverse sheaves are for the stratification into* $\check{G}(\mathbb{C}[t])$*-orbits.*

The dimension of V^T is encoded in $j^*\mathrm{Sat}(V)$, where $j : K \to \mathrm{Gr}$ is the open embedding defined by

$$K = \ker(\check{G}(\mathbb{C}[t^{-1}]) \to \check{G}), \quad j(k) = k\check{G}(\mathbb{C}[t]).$$

But there is no W-action on $j^*\mathrm{Sat}(V)$.

There *is* a perverse sheaf with a W-action, but not on Gr or K:

$$\mathrm{Spr} = \mu_* \mathbb{C} \in \mathrm{Perv}(\mathcal{N}),$$

where μ is the resolution of the nilpotent cone $\mathcal{N} \subset \mathrm{Lie}(\check{G})$. This gives rise to the Springer correspondence

$$\mathrm{Hom}_{\mathrm{Perv}(\mathcal{N})}(\mathrm{Spr}, -) : \mathrm{Perv}(\mathcal{N}) \to \mathrm{Rep}(W).$$

Moreover there is an obvious map

$$\pi : K \to \mathrm{Lie}(\check{G}) : 1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m} \mapsto x_1.$$

Questions

1. For which V is $\pi(\mathrm{supp}(j^* \mathrm{Sat}(V)))$ contained in $\mathcal{N}$?
2. For which such V is $\pi_* j^* \mathrm{Sat}(V)$ perverse?

Example

If $G = GL_2$ then $\check{G} = GL_2$ and $\mathrm{Lie}(\check{G}) = \mathrm{Mat}_2$. We have

$$\mathrm{supp}(j^*\mathrm{Sat}(S^{2m}(\mathbb{C}^2) \otimes \det^{-m})) = \{1 + x_1 t^{-1} + \cdots + x_m t^{-m} \mid (1 + x_1 t^{-1} + \cdots + x_m t^{-m})(1 + y_1 t^{-1} + \cdots + y_m t^{-m}) = 1 \text{ for some } y_1, \cdots, y_m\}.$$

This condition forces x_1 to be nilpotent if $m = 1$ but not if $m \geq 2$.

Theorem (Achar–H. arXiv:1108.4999)

For $V \in \mathrm{Rep}(G)$ with trivial action of the centre,

$$\pi(\mathrm{supp}(j^*\mathrm{Sat}(V))) \subset \mathcal{N} \iff V \text{ is small},$$

i.e. the convex hull of its weights doesn't include twice a root. If so, $\pi : \mathrm{supp}(j^*\mathrm{Sat}(V)) \to \mathcal{N}$ is finite so $\pi_* j^*\mathrm{Sat}(V) \in \mathrm{Perv}(\mathcal{N})$.

THE UNIVERSITY OF SYDNEY

Theorem (Achar–H.–Riche arXiv:1205.5089)

On the subcategory of small representations,

$$\Phi \cong \mathrm{Hom}_{\mathrm{Perv}(\mathcal{N})}(\mathrm{Spr}, \pi_* j^* \mathrm{Sat}(-)).$$

Moreover, this holds when G is defined over an arbitrary field k, using complex $\check{G}, K, \mathcal{N}$ and sheaves with coefficients in k.

Idea of proof: show that all functors commute (up to isomorphism) with suitably-defined restrictions to a Levi subgroup L of G. Taking great care to check the compatibility of isomorphisms, one is then reduced to showing the result for L of semisimple rank 1, since W is generated by the Weyl groups of such L. This amounts to the GL_2 case, which can be checked directly.