Calculating with string diagrams

Ross Street
Macquarie University

Workshop on Diagrammatic Reasoning in Higher Education
University of Newcastle

Reasons for choice of this topic

- A conviction that string diagrams can be understood better than algebraic equations by most students
- My experience with postgraduate students and undergraduate vacation scholars using strings
- As seen by the general public, knot theory for mathematics seems a bit like astronomy for physics
- A belief that string diagrams are widely applicable and powerful in communicating and in discovery
- That this is "advanced mathematics from an elementary viewpoint" (to quote Ronnie Brown's twist on Felix Klein)

Intentions

- Moving from linear algebra, we will look at braided monoidal categories (bmc) and explain the string diagrams for which bmc provide the environment.
- Familiar operations from vector calculus will be transported to bmc where the properties can be expressed in terms of equalities between string diagrams.
- Geometrically appealing arguments will be used to prove the scarcity of multiplications on Euclidean space, a theorem of a type originally proved using higher powered methods.

Arrows and categories

- Already introduced in undergraduate teaching is the notation $f\colon X \to A$ for a function taking each element x in the set X to an element $f(x)$ of the set A.

- In the situation $X \xrightarrow{f} A \xrightarrow{g} K$ we can follow f by g and obtain a new function, called the composite of f and g, denoted by $g \circ f\colon X \to K$.

- There is an identity function $1_X\colon X \to X$ for every set X: $1_X(x) = x$.

- If we now ignore the fact that X, A, K are sets (just call them vertices or objects) and that f, g are functions (just call them edges or morphisms) we are looking at a big directed graph.

- If we admit the existence of a composition operation $\circ$ which is associative and has identities 1_X, we are looking at a category.

Euclidean space

- The set of real numbers is denoted by $\mathbb{R}$.
- A vector of length n is a list $x = (x_1, \ldots, x_n)$ of real numbers. The set of these vectors is n-dimensional Euclidean space, denoted $\mathbb{R}^n$.
- Algebra is about operations on sets. We can add vectors x and y entry by entry to give a new vector $x + y$. We can scalar multiply a real number r by a vector x to obtain a vector rx.
- For example, $\mathbb{R}^3$ is ordinary 3-dimensional space. We have three particular unit vectors:

$$e^1 = (1,0,0),\ e^2 = (0,1,0),\ e^3 = (0,0,1)\ .$$

Every vector x in $\mathbb{R}^3$ is a unique linear combination $x = x_1 e^1 + x_2 e^2 + x_3 e^3$. Similarly in $\mathbb{R}^n$

Linear algebra

- A function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear when it preserves linear combinations: $f(x+y) = f(x) + f(y)$, $f(rx) = rf(x)$.
- Thus we have a category $\mathscr{E}$: objects are Euclidean spaces and morphisms are linear functions. We write $\mathscr{E}(V, W)$ for the set of morphisms from object V to object W.
- For this category $\mathscr{E}$, we can add the morphisms in $\mathscr{E}(V, W)$: define $f+g$ by $(f+g)(x) = f(x) + g(x)$. Composition distributes over this addition. Such a category is called additive.
- Notice that the only linear functions $f\colon \mathbb{R} \to \mathbb{R}$ are those given by multiplying by a fixed real number. So $\mathscr{E}(\mathbb{R}, \mathbb{R})$ can be identified with $\mathbb{R}$.

Multilinear algebra

- Categorical algebra is about operations on categories. The category $\mathscr{E}$ has such an operation called tensor product:

$$\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn} \ .$$

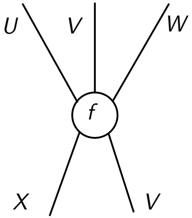
However, when thinking of the mn unit vectors of $\mathbb{R}^{mn}$ as being in the tensor product they are denoted by $e^i \otimes e^j$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Every element of $\mathbb{R}^m \otimes \mathbb{R}^n$ is a unique linear combination of these.

- Bilinear functions $U \times V \to W$ are in bijection with linear functions $U \otimes V \to W$.
- Note that $\mathbb{R}$ acts as unit for the tensor.
- For linear functions $f \colon \mathbb{R}^m \to \mathbb{R}^{m'}$ and $g \colon \mathbb{R}^n \to \mathbb{R}^{n'}$, we have a linear function $f \otimes g \colon \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^{m'} \otimes \mathbb{R}^{n'}$ defined by

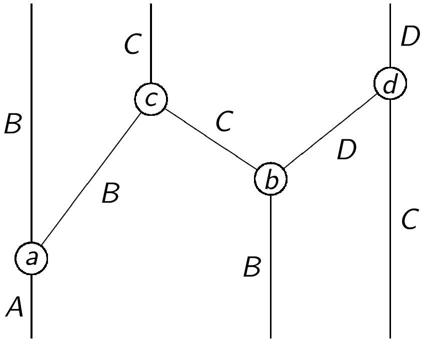
$$(f \otimes g)(e^i \otimes e^j) = f(e^i) \otimes g(e^j) \ .$$

Monoidal categories and their string diagrams

- A category $\mathscr{V}$ is monoidal when it is equipped with an operation called tensor product taking pairs of objects V, W to an object $V \otimes W$ and pairs of morphisms $f : V \to V'$, $g : W \to W'$ to a morphism $f \otimes g : V \otimes W \to V' \otimes W'$. There is also an object I acting as a unit for tensor. Composition and identity morphisms are respected in the expected way. An example is $\mathscr{V} = \mathscr{E}$ with $I = \mathbb{R}$.
- A morphism such as $f : U \otimes V \otimes W \to X \otimes V$ is depicted as



- Composition is performed vertically with splicing involved; tensor product is horizontal placement.

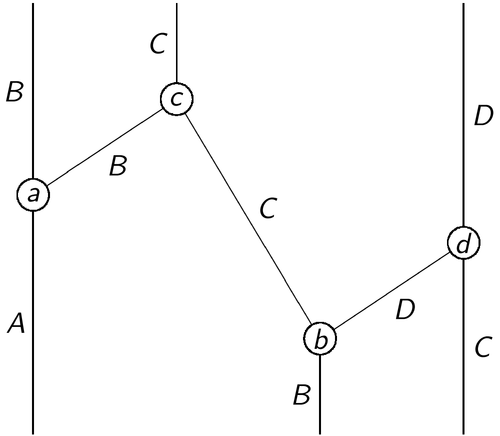


$$B \otimes B \xrightarrow{a} A\,, \quad C \otimes D \xrightarrow{b} B\,, \quad C \xrightarrow{c} B \otimes C\,, \quad D \xrightarrow{d} D \otimes C\,.$$

The *value* of the above diagram Γ is the composite

$$\begin{aligned} \mathrm{v}(\Gamma) = &(B \otimes C \otimes D \xrightarrow{1_B \otimes c \otimes d} B \otimes B \otimes C \otimes D \otimes C \\ &\xrightarrow{1 \otimes 1 \otimes b \otimes 1} B \otimes B \otimes B \otimes C \xrightarrow{a \otimes 1 \otimes 1} A \otimes B \otimes C)\,. \end{aligned}$$

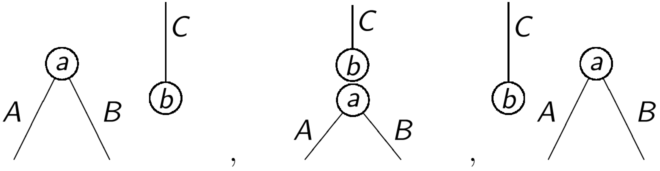
Here is a *deformation* of the previous Γ; the value is the same using monoidal category axioms.



$$\mathrm{v}(\Gamma) = (B \otimes C \otimes D \xrightarrow{1_B \otimes c \otimes 1} B \otimes B \otimes C \otimes D \xrightarrow{a \otimes 1 \otimes 1} A \otimes C \otimes D$$

$$\xrightarrow{1 \otimes 1 \otimes d} A \otimes C \otimes D \otimes C \xrightarrow{1 \otimes b \otimes 1} A \otimes B \otimes C).$$

The geometry handles units well: if $I \xrightarrow{a} A \otimes B$ and $C \xrightarrow{b} I$, then the following three string diagrams all have the same value.

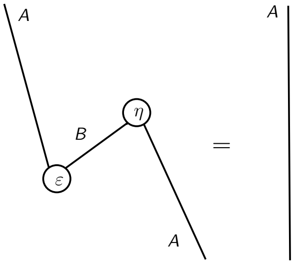


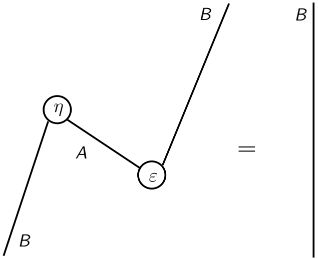
The straight lines can be curved while the nodes are really labelled points.
There is no bending back of the curves allowed: the diagrams are progressive.
These planar deformations are part of the geometry of monoidal categories.

Progressive graph on Mollymook Beach

Duals

A morphism $\varepsilon : A \otimes B \to I$ is a counit for an adjunction $A \dashv B$ when there exists a morphism $\eta : I \to B \otimes A$ satisfying the two equations:

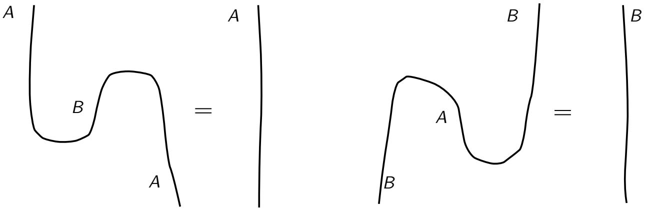




We call B a right dual for A.

Backtracking

When there is no ambiguity, we denote counits by cups $\cup$ and units by caps $\cap$. So the duality condition becomes the more geometrically "obvious" operation of pulling the ends of the strings as below.



The above are sometimes called the *snake equations*. The geometry of duality in monoidal categories allows backtracking in the plane.

Dot product, vector product and the quaternions

- For any x and y in $\mathbb{R}^n$, the dot product

$$x \bullet y = x_1 y_1 + \cdots + x_n y_n$$

defines a bilinear function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and so a linear function $\bullet\colon \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$.

- For any x and y in $\mathbb{R}^3$, the vector product

$$x \wedge y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

defines a bilinear function $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and so a linear function $\wedge\colon \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3$.

- The quaternions is the non-commutative ring $\mathbb{H} = \mathbb{R} \times \mathbb{R}^3 (\cong \mathbb{R}^4)$ with componentwise addition and associative multiplication defined by

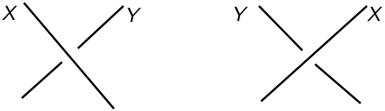
$$(r, x)(s, y) = (rs - x \bullet y, ry + sx + x \wedge y)$$

Braiding

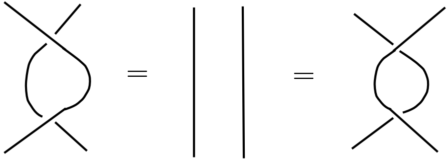
Now suppose the monoidal category is *braided*. Then we have isomorphisms

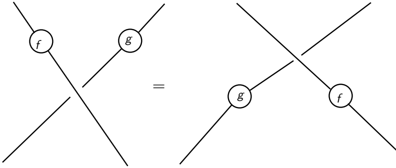
$$c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

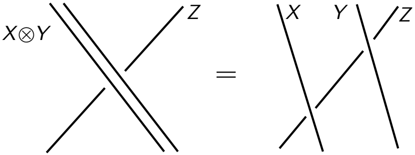
which we depict by a left-over-right crossing of strings in three dimensions; the inverse is a right-over-left crossing.

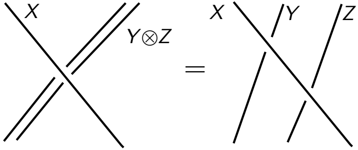


The braiding axioms reinforce the view that it behaves like a crossing.

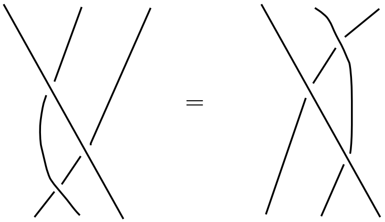








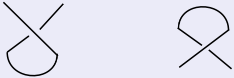
The following Reidemeister move or Yang-Baxter equation is a consequence.



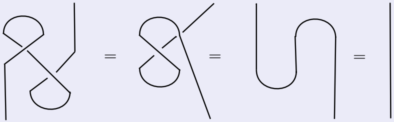
We will refer to these properties as the geometry of braiding.

Proposition

If $\mathscr{V}$ is braided and $A \dashv B$ with counit and unit depicted by $\cup$ and $\cap$ then $B \dashv A$ with counit and unit depicted by

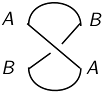


Proof.

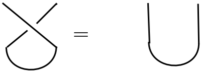


$\square$

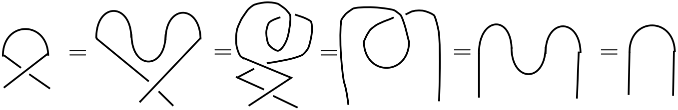
Objects with duals have dimension: if $A \dashv B$ then the dimension $d = d_A$ of A is the following element of the commutative ring $\mathscr{V}(I, I)$.

$d \quad =$ 

A self-duality $A \dashv A$ with counit $\cup$ is called symmetric when

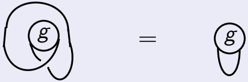


It follows that



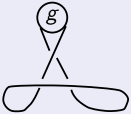
Proposition

If $A \dashv A$ is a symmetric self-duality and $g : I \to A{\otimes}A$ is a morphism then



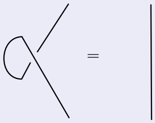
Proof.

Both sides are equal to:



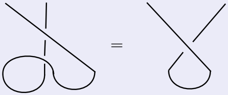
Proposition

If $A \dashv A$ is a symmetric self-duality then the following Reidemeister move holds



Proof.

By dragging the bottom strings to the right and up over the top string we see that the proposition is the same as



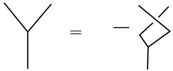
$\square$

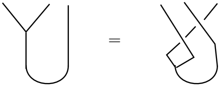
References

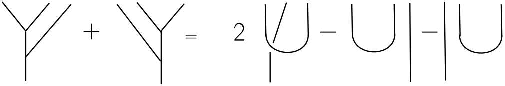
The remainder of this talk is built on the work of Rost and his students.

- Markus Rost, *On the dimension of a composition algebra*, Documenta Mathematica 1 (1996) 209–214.
- Dominik Boos, *Ein tensorkategorieller Zugang zum Satz von Hurwitz*, (Diplomarbeit ETH Zürich, March 1998) 42 pp.
- Susanne Maurer, *Vektorproduktalgebren*,(Diplomarbeit Universität Regensburg, April 1998) 39 pp.

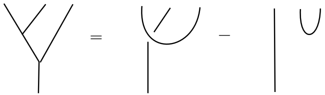
A vector product algebra (vpa) in a braided monoidal additive category $\mathscr{V}$ is an object V equipped with a symmetric self-duality $V \dashv V$ (depicted by a cup $\cup$) and a morphism $\wedge : V \otimes V \to V$ (depicted by a Y) such that the following three conditions hold.



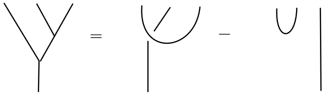




A vpa is associative when it satisfies



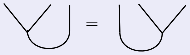
Using the first two axioms for a vpa, we see that associativity is equivalent to:



By adding these two expressions of associativity we obtain the third condition on a vpa. So the third vpa axiom is redundant in the definition of associative vpa.

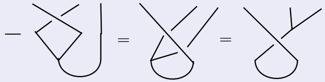
Proposition

The following is a consequence of the first two vpa axioms.



Proof.

Using those first two axioms for the first equality below then the geometry of braiding for the second, we have



However, the left-hand side is equal to the left-hand side of the equation in the proposition by the first vpa axiom while the right-hand sides are equal by symmetry of inner product $\cup$. $\square$

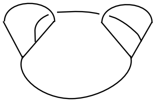
Theorem

For any associative vector product algebra V in any braided monoidal additive category $\mathscr{V}$, the dimension $d = d_V$ satisfies the equation

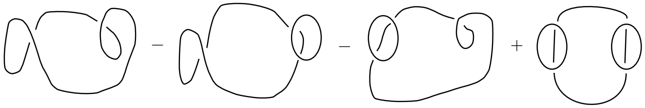
$$d(d-1)(d-3) = 0$$

in the endomorphism ring $\mathscr{V}(I, I)$ of the tensor unit I.

To prove this we perform two string calculations each beginning with the following element Ω of $\mathscr{V}(I, I)$.



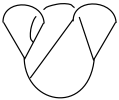
Using associativity twice, we obtain



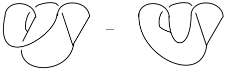
in which, using the first Reidemeister move and the geometry of braiding, each term reduced to a union of disjoint circles:

$$\Omega = d - dd - dd + ddd = d(d-1)^2 \ .$$

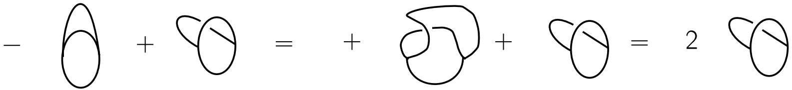
Return now to Ω and apply the last Proposition to obtain:

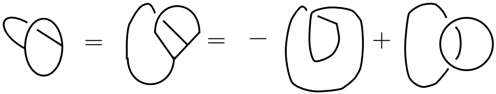


in which we see we can apply associativity to obtain:



In both terms we can apply the first vpa axiom.





$$\Omega = 2(-d + d^2)\ ,\ \text{yet from before}\ \ \Omega = d(d-1)^2$$

$$d(d-1)^2 = 2d(d-1)$$

$$0 = d(d-1)(d-1-2) = d(d-1)(d-3)$$

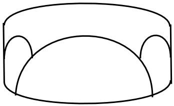
Theorem

For any vector product algebra V in any braided monoidal additive category $\mathscr{V}$ such that 2 can be cancelled in $\mathscr{V}(I, V)$ and $\mathscr{V}(I, I)$, the dimension $d = d_V$ satisfies the equation

$$d(d-1)(d-3)(d-7) = 0$$

in the endomorphism ring $\mathscr{V}(I, I)$ of the tensor unit I.

The proof involves performing two string calculations each beginning with the following element of $\mathscr{V}(I, I)$.



Thank You

☺