# Proving Quantum Indeterminism: Measurements of Value Indefinite Observables Are Unpredictable

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  - Schrödinger equation
  - cellular automata, non-deterministic Turing machines

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- "...randomness is not in the world, it is in the interface between our theoretical descriptions and 'reality' as accessed by measurement. Randomness is unpredictability with respect to the intended theory and measurement." (G. Longo)

# Eigenvalue-eigenstate principle

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Eigenvalue-eigenstate principle: *A system in a state $|\psi\rangle$ has a definite property of an observable A if and only if $|\psi\rangle$ is an eigenstate of A.*

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**Kochen-Specker Theorem.** *In $n \geq 3$ Hilbert space there is a finite set of (projection) observables $\mathcal{O}$ such that no value assignment function $v : \mathcal{O} \to \{0, 1\}$ can have the following three properties:*

1. *Value definiteness* $(\mathrm{VD})$*: $v$ is total, i.e., $v(P)$ defined for all $P \in \mathcal{O}$.*
2. *Noncontextuality* $(\mathrm{NC})$*: $v$ is a function of $P$ only.*
3. *Quantum mechanics predictions* $(\mathrm{QM})$*: For every context $C \subset \mathcal{O}$: $\sum_{P \in C} v(P) = 1$.*

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In this case *some* observables are *value indefinite*, hence some quantum measurements are indeterminate.

## How much value indefiniteness is reasonable?

- Rather than assuming that value indefiniteness apply uniformly, can we prove it from "simpler" assumptions?
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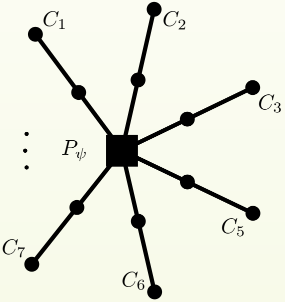

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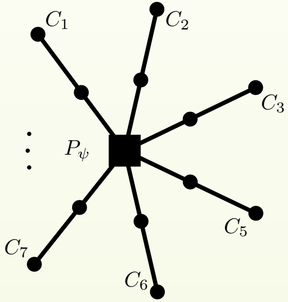
  - *One* direction of eigenvalue-eigenstate principle.
- Intuitively, expect everything outside this 'star' to be value indefinite.



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  - VD″: An observable $P$ is assigned 1, and a *non-compatible* observable $P'$ is value definite.

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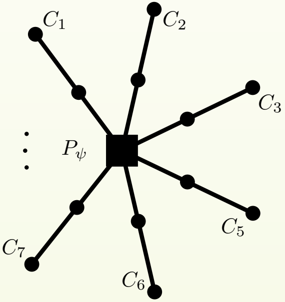
- We need explicit assumptions.



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### Admissibility of $v$

A value assignment function $v$ is admissible whenever for every context $C \subset \mathcal{O}$:

(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$.

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Classical Greechie orthogonality diagrams proving the Kochen-Specker theorem fail to prove this statement.

**Theorem 1.** *Let* $n \geq 3$ *and* $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ *be states such that* $0 < |\langle\psi|\phi\rangle| < 1$. *Then we effectively construct a finite set of observables* $\mathcal{O}$ *containing* $P_\psi$ *and* $P_\phi$ *for which there is no admissible value assignment function on* $\mathcal{O}$ *such that* $v(P_\psi) = 1$ *and* $P_\phi$ *is value definite.*

## Proof

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2. We prove a reduction for $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$ to the first case.
3. We prove a reduction for the last case of $\frac{1}{\sqrt{2}} < |\langle\psi|\phi\rangle| < 1$ case.

## Almost all observables are value indefinite

**Theorem 2.** The set of value indefinite observables has constructive measure 1.

These results are purely mathematical. How should we interpret them physically?

### Eigenstate value definiteness

If a system is in a state $|\psi\rangle$, then $v(P_\psi) = 1$ for any *admissible* value assignment function $v$.

### Interpretation

If a system is in a state $|\psi\rangle$, then the result of measuring an observable $A$ is indeterministic unless $|\psi\rangle$ is an eigenstate of $A$.

We assumed one direction of the eigenvalue-eigenstate principle, but derived the other direction.

The Kochen-Specker theorem shows (via the adopted interpretation) that quantum-mechanics is indeterministic.

**Theorem 1** shows the *extent* of this indeterminism and indicates precisely which observables are value indefinite.

Indeterminism does not imply randomness. However, unpredictability is a requirement of randomness. So,

*are quantum mechanical measurements unpredictable?*

Consider a physical experiment $E$ producing a single bit.

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With a particular trial (instantiation) of $E$ we associate the real parameter $\lambda$ which fully describes it. While $\lambda$ is not in its entirety an obtainable quantity, it contains any information that may be pertinent to prediction and we may have practical access to finite aspects of this information.

An extractor is a physical device selecting a finite amount of information included in $\lambda$ without altering the experiment $E$. Mathematically, an extractor is a (deterministic) function $\lambda \mapsto \xi(\lambda) \in \{0,1\}^*$ where $\xi(\lambda)$ is a finite string of bits.

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$P_E$ can utilise as input the information $\xi(\lambda)$, but, *as required by* EPR, must be passive, that is, it must not disturb or interact with $E$ in any way.

A predictor $P_E$ provides a correct prediction using the extractor $\xi$ for an instantiation of $E$ with parameter $\lambda$ if, when taking as input $\xi(\lambda)$, it outputs $\mathbf{0}$ or $\mathbf{1}$ (i.e. it does not refrain from making a prediction) and this output is equal to $x$, the result of the experiment.

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The predictor $P_E$ is $k$-correct for $\xi$ if there exists an $n \geq k$ such that when $E$ is repeated $n$ times with associated parameters $\lambda_1, \ldots, \lambda_n$ producing the outputs $x_1, x_2, \ldots, x_n$, $P_E$ outputs the sequence

$$P_E(\xi(\lambda_1)), P_E(\xi(\lambda_2)), \ldots, P_E(\xi(\lambda_n))$$

with the following two properties:

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1. no prediction in the sequence is incorrect, and
2. in the sequence there are $k$ correct predictions.

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The outcome $x$ of a single trial of the experiment $E$ performed with parameter $\lambda$ is predictable (with certainty) if there exist an extractor $\xi$ and a predictor $P_E$ which is correct for $\xi$, and $P_E(\xi(\lambda)) = x$.

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Accordingly, $P_E$ correctly predicts the outcome $x$, never makes an incorrect prediction, and can produce arbitrarily many correct predictions.

**Theorem 3.** *If $E$ is an experiment measuring a quantum value indefinite observable, then for every predictor $P_E$ using any extractor $\xi$, $P_E$ is not correct for $\xi$.*

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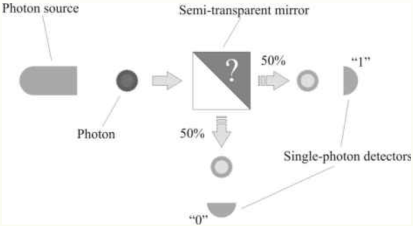
**Theorem 4.** *In an infinite repetition of the experiment $E$ measuring a quantum value indefinite observable which generates the infinite sequence $x_1x_2\ldots$, no single bit $x_i$ can be predicted with certainty.*

## An open problem

- Assume noncontextuality.
- Theorem 1 doesn't hold in two-dimensional Hilbert space.



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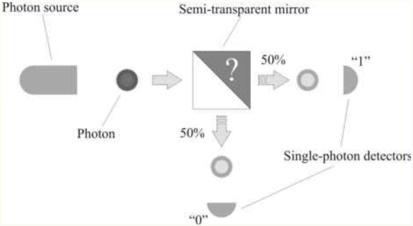
- Assume noncontextuality.
- Theorem 1 doesn't hold in two-dimensional Hilbert space.
- Does Theorem 4 hold in two-dimensional Hilbert space?



References


- A. Albert, B. Podolsky and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? 47,10: 777–780, 1935
- S. Kochen and E. Specker. The problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics* 17:59–87, 1967 .
- A. A. Abbott, C. S. Calude, J. Conder and K. Svozil. Strong Kochen-Specker theorem and incomputability of quantum randomness, *Physical Review A* 86, 6 (2012), PhysRevA.86.062109.
- A. A. Abbott, C. S. Calude and K. Svozil. Value-indefinite observables are almost everywhere, *Physical Review A*, 89, 3 (2014), 032109–032116, PhysRevA.89.032109.
- A. A. Abbott, C. S. Calude and K. Svozil. On the unpredictability of individual quantum measurement outcomes, arXiv:1403.2738, 2014.
- A. A. Abbott, C. S. Calude and K. Svozil. A variant of the Kochen-Specker theorem localising value indefiniteness, arXiv:1503.01985, 2015.