Janko's Sporadic Simple Groups: a bit of history

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Fifty years ago: the discovery

In January 1965, a surprising announcement was communicated to the international mathematical community.

Zvonimir Janko, working as a Research Fellow at the Institute of Advanced Study within the Australian National University had constructed a new sporadic simple group.

Before 1965 only five sporadic simple groups were known. They had been discovered almost exactly one hundred years prior (1861 and 1873) by Émile Mathieu but the proof of their simplicity was only obtained in 1900 by G. A. Miller.

Finite simple groups: earliest examples

- The cyclic groups $\mathbb{Z}_p$ of prime order and the alternating groups $\mathrm{Alt}(n)$ of even permutations of $n \geq 5$ items were the earliest simple groups to be studied (Gauss, Euler, Abel, etc.)
- *Evariste Galois* knew about $PSL(2, p)$ and wrote about them in his letter to Chevalier in 1832 on the night before the duel.
- *Camille Jordan* (*Traité des substitutions et des équations algébriques*,1870) wrote about linear groups defined over finite fields of prime order and determined their composition factors. The 'groupes abéliens' of Jordan are now called symplectic groups and his 'groupes hypoabéliens' are orthogonal groups in characteristic 2.
- *Émile Mathieu* introduced the five groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} in 1861 and 1873.

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The classical groups, G_2 and E_6

- In his PhD thesis *Leonard Eugene Dickson* extended Jordan's work to linear groups over all finite fields and included the unitary groups.

  He published the results in his 1901 book *Linear groups with an exposition of the Galois field theory*.

  This established the existence of all finite classical simple groups.
- Dickson constructed the finite analogues of the Lie groups of type G_2 and E_6 (1901, 1905 and 1908).

In Janko's announcement (*Proc. Nat. Acad. Sci. USA* 53 (1965)) he mentioned that *Andrew Coppel* had shown that his new group is a subgroup of $G_2(11)$.

The sporadic simple groups

William Burnside coined the phrase *sporadic simple group*.

In Note N of the second edition (1911) of his 1897 book '*Theory of groups of finite order*' Burnside states (in reference to the Mathieu groups):

- These apparently sporadic simple groups would probably repay a closer examination than they have yet received.

Chevalley groups

In 1955 *Claude Chevalley* showed that to every simple complex Lie algebra $X_n(\mathbb{C})$ of rank n and every field K there is a corresponding *Chevalley group* $X_n(K)$, which, except for a few special cases, is simple.

In particular, for each finite field of q elements there is finite Chevalley group $X_n(q)$.

Types A_n, B_n, C_n and D_n correspond to the classical linear, odd-dimensional orthogonal, symplectic, and even-dimensional orthogonal groups.

The groups $G_2(q)$ and $E_6(q)$ had been constructed by Dickson but now we also had $E_7(q)$, $E_8(q)$ and $F_4(q)$.

Steinberg variations

Chevalley's construction produced most, but not all, of the classical groups; the unitary groups and some orthogonal groups were not included.

In 1958 *Robert Steinberg* extended Chevalley's work to take into account automorphisms of the Lie algebras.

This brought the finite unitary groups and the remaining orthogonal groups into the fold as well as exhibiting new 'twisted' simple groups of types ${}^3D_4(q)$ and ${}^2E_6(q)$. (The latter were found independently by *D. Hertzig* and *Jacques Tits*.)

Suzuki groups

The constructions of Chevalley and Steinberg were not the last of the new families of finite simple groups.

In 1960 *Michio Suzuki* announced the existence of yet another family. Suzuki provided explicit matrix generators for his groups, which exhibited them as subgroups of the four-dimensional symplectic groups $Sp(4, 2^{2a+1})$.

Suzuki discovered his groups in the course of determining the doubly transitive groups in which only the identity fixes three points (*Zassenhaus groups*).

He also showed that they are the only finite simple groups in which the centraliser of every involution is a 2-group.

Ree groups

Almost immediately (in 1960) *Rimhak Ree* (and Steinberg) saw that the Suzuki groups were the fixed points of an automorphism of $Sp(4, 2^{2a+1})$ and furthermore they could be constructed by a variant of Steinberg's methods.

Furthermore the groups $G_2(3^{2a+1})$ and $F_4(2^{2a+1})$ have similar automorphisms and thereby Ree constructed two new infinite families of finite simple groups: ${}^2G_2(3^{2a+1})$ and ${}^2F_4(2^{2a+1})$, now known as *Ree groups*.

It was by studying the Ree groups of type ${}^2G_2(3^{2a+1})$ that Janko was led to the discovery of J_1, his first sporadic group.

The importance of involutions

Definition

An *involution* is an element of order 2. The subgroup of elements which commute with the involution is its *centraliser*.

Burnside had suspected at the beginning of the 20th century that all (non-abelian) finite simple groups should contain involutions.

In Note M of his 1917 book he expressed this as follows:

- The contrast that these results show between groups of odd and even order suggests inevitably that simple groups of odd order do not exist.

Characterisations by centralisers of involutions

It was *Richard Brauer* who thought that the non-abelian simple groups could be classified in terms of their centralisers of involutions, which necessarily existed if Burnside were right.

At the 1954 ICM Brauer reported that as a consequence of results of his with his student *K. A. Fowler* there are only a finite number of simple groups of even order with an involution centraliser isomorphic to a given group.

Brauer also reported his classification of $PSL(3, q)$ and M_{11} by the centraliser of an involution and a beautiful result of Brauer, Suzuki and *G. E. [Tim] Wall* which was subsequently published in 1958.

Walter Feit and John Thompson

In his 1959 PhD thesis *John Thompson* proved a long-standing conjecture of Frobenius and later that year he began a collaboration with *Walter Feit* which, in 1962, resulted in their famous proof of the Odd Order Theorem (255 pages in the Pacific Journal of Mathematics).

Thus the program outlined by Brauer had a realistic chance of success.

In 1962 it was widely believed that the list of finite simple groups was complete and, except for those five pesky groups of Mathieu, they lay in infinite families with the wonderful structure of Lie algebras supporting them. So Brauer's program had a well defined plan of attack: one should look at the centralisers of involutions.

The Australian National University, ca. 1962

In 1962, *Bernhard Neumann* came to the first Chair of Pure Mathematics in the Institute of Advanced Studies of the Australian National University, bringing with him a bevy of remarkable mathematicians — including his wife *Hanna Neumann*, his former students *Mike Newman* and *Laci Kovács* and his former colleague from Manchester, *Kurt Mahler*.

He also managed to entice a less well-known Yugoslavian: *Zvonimir Janko*.

Zvonimir Janko

Zvonimir Janko was born in 1932 in Bjelovar, Croatia (then Yugoslavia). He studied mathematics at the University of Zagreb.

His first teaching position was at the high school in Lišticia (now called Siroki Brijeg) in Bosnia and Herzegovina where he taught physics.

He gained his PhD from the University of Zagreb in 1960 under the supervision Professor Vladimir Devidé who later visited him at Monash in 1967.

He published his first paper in 1959 and by the time he arrived at the ANU he had 12 publications.

Janko in 1964

First steps

Janko was in correspondence with some of the American mathematicians who were working on the classification of all finite simple groups.

An obvious first step in the classification was to classify the finite simple groups with abelian Sylow 2-subgroups.

Indeed, *John Walter* in Illinois was well advanced on that very problem and was sure that the complete proof was not far off at all.

Janko, Thompson and Sah

Janko thought he could prove a small first step on the way towards that goal, namely to classify the finite simple groups with an abelian Sylow 2-subgroup *and* an involution whose centraliser is the direct product of the involution by $PSL(2,q)$.

He thought that the only possibilities would be Ree groups of type G_2.

John Thompson in Chicago and Janko at the ANU corresponded. Thompson agreed with Janko that they should produce a joint paper on the subject with Janko to write it.

Chih-Han Sah had already published a special case of this result where he quoted an unpublished result of Thompson claiming that if $q > 3$, then $q = 3^{2n+1}$ for some n.

Problems

Accordingly Janko set about writing the paper which he would send to Thompson for publication.

When he set about this task, he found that he could not eliminate the first and simplest situation which could arise, namely when the centraliser of the involution t was $\langle t \rangle \times PSL(2,5)$.

Janko did not inform Thompson of the problems he was having with the easiest case and the more he worked on it the more it seemed that no contradiction was possible and that there may have been another group out there whose existence had been kept from us.

Terry Gagen

Terry Gagen arrived in Canberra in January 1964 and had been assigned to Janko as his first PhD student.

The first problem which Janko suggested he consider was the problem in which John Walter was very advanced, namely the problem of determining the simple groups whose Sylow 2-subgroups were abelian.

That morphed into a more accessible problem as time went by.

Existence?

Eventually, Janko knew enough about the group–if it existed–that

- it would have $175\,560 = 2^3 \times 3^5 \times 5 \times 7 \times 11 \times 19$ elements;
- it would be generated by a collection of 7×7 matrices with coefficients in $\mathbb{Z}_{11}$.

Janko had produced a set of generators which had to work, *if the group existed*.

Indeed, if they did not work, then the group did not exist and John Thompson would have been correct.

Matrix multiplication

What was required was to multiply those matrices out until all possibilities were listed.

That would be at least 175 560 matrices $\cdots$ an impossibly large number at that time.

Fortunately there was a student at ANU, one year in advance of Terry, namely *Martin Ward*, who knew about programming computers, and who was able to get the ANU's computer at the time to multiply matrices modulo 11.

Martin and Terry set aside a night in which to do this.

The group exists!

Luckily, it was not necessary to produce all 175 560 matrices. Janko knew that the matrices would generate a group of order at most that number and so all they needed to do was to show that the order of the group generated by the matrices was at least that number.

They were able to recognise the order of the matrix from its trace, so they selected two matrices, multiplied them, checked its trace, ticked off its order, then moved on to another two matrices.

By the end of the night—Terry's memory is that dawn was just breaking—they had found elements of orders 3, 5, 7, 11, 19 and with the other information they had, they knew that the new simple group existed as Janko had suspected.

At the time, there were no mobile phones and one could not contact Janko at home asleep in bed. Terry recalls that they must have waited until Janko arrived at the department.

He remembers shaking his hand and saying that the group exists!

Of course, before this, it seemed very clear that the group really existed and Janko had written to Thompson to say that he was having problems with the '$q = 5$ case'.

Thompson replied that he had a proof eliminating that case in front of him as he wrote and that he would take care of it, if Janko couldn't!

By the time Thompson's letter arrived in Canberra, they had the group.

Janko asked whether it was possible to print out the entire 175 560 matrices and mail them to Thompson, but of course that idea came to nothing, though it was not impossible at all!

The Ree group problem

Fresh from his discovery and construction of the new sporadic simple group Janko continued working on the characterisation of Ree groups of characteristic 3.

In January 1965 Janko and Thompson had completed a joint paper going someway to a characterisation. The Ree groups satisfied their conditions but they were unable to prove that there were no others.

Ultimately it came down to proving that a certain field automorphism σ of the field of $q = 3^{2n+1}$ elements satisfies $\sigma^2 = 3$. This problem remained open until 1980 when *Enrico Bombieri* proved that $\sigma^2 = 3$ provided $n > 41$.

The remaining cases were dealt using computer programs written by *David Hunt* and *Andrew Odlyzko*. Janko spent a considerable amount of time on this problem and even set it as an exercise in one of his classes!

The Monash years

In 1961 Monash University saw its first students and *Kevin Westfold* took up his appointment as the inaugural professor of Applied Mathematics; in the following year he became Dean of Science.

Gordon Preston, who arrived in 1963, was the first professor of Pure Mathematics.

Preston, with the support of Westfold, immediately set about establishing a world class department of mathematics and statistics. By 1967 he had appointed five professors of mathematics and statistics. In those days this was unprecedented.

One of the appointments was Zvonimir Janko who arrived at Monash in August 1965, moving from a position as Research Fellow at the ANU to a full professor.

More involution centralisers

Janko looked at hundreds if not thousands of possible centralisers of involutions and was rewarded with two new examples of sporadic simple groups.

The smallest of these (J_2) was constructed by *Marshall Hall* and its uniqueness was proved by *David Wales*.

Janko discovered J_2 and J_3 in 1966. In September 1967 Marshall Hall presented his construction for J_2 at a group theory conference in Oxford.

Donald Higman and *Charles Sims* were in the audience and by next morning they had used the same technique to construct another new simple group.

The Held group

In 1967 *Dieter Held* had joined the department and joined the program, first characterising the alternating groups of degrees 8 and 9 be the centraliser of an involution and then characterising the alternating group of degree 10 and the Mathieu group M_{22}.

In 1968 he had returned to Germany but I recall Janko bringing one of his letters to an advanced group theory class and reading it to us.

Held was studying groups with a centraliser of an involution isomorphic to those in M_{24}. He had shown that there were three possibilities for the fusion of involutions.

One led to $PSL(5,2)$, another led to M_{24} but he could not eliminate the third. Janko said "Well, he's found a new simple group."

The last sporadic simple group

In 1968 Janko went to the IAS, Princeton, briefly back to Monash and then, in 1969, to Ohio State University in Columbus and finally, in 1972, to Heidelberg until his retirement in 2000.

Janko thought it would be worthwhile looking for simple groups with an involution $\langle t \rangle$ whose centraliser is a non-splitting extension of $\langle t \rangle$ by $\mathrm{Alt}(n)$ for some $n > 8$. He showed that no such group exists for $n = 9$ and John Thompson showed that there are no groups with $n \geq 12$.

But it was Thompson's student Richard Lyons who got the prize. He discovered that $n = 10$ is not possible but for $n = 11$ he provided compelling evidence for a new simple group. The existence and uniqueness of Lyons' group was proved by Charlie Sims, using a computer.

There was one more group discovered by Janko. That was in 1975 and it turned out to be the last of the 26 sporadic simple groups.

Janko in 2007