Binary and Ternary Curves of Fixed Genus and Gonality with Many Points

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Work

This is ongoing joint work with Xander Faber.

manypoints.org

- For given (small) finite field and (small) genus, what is the maximum number of points a smooth, projective curve can have?

manypoints.org

- For given (small) finite field and (small) genus, what is the maximum number of points a smooth, projective curve can have?
- There is a fairly extensive database at `manypoints.org`.
- For $\mathbb{F}_2$ and $\mathbb{F}_3$ it looks like:

g	$N_2(g)$	$N_3(g)$
0	3	4
1	5	7
2	6	8
3	7	10
4	8	12
5	9	13

Gonality

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- Gonality 1 curves are isomorphic to $\mathbb{P}^1$, so coincide with genus 0 curves.
- Gonality 2 curves are **hyperelliptic**, and include elliptic curves (genus 1 and up).
- Gonality 3 curves are known as **trigonal** curves.

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  or

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  or
- It's fun and interesting.

Let's start a table for binary curves

- I'll be concentrating on $\mathbb{F}_2$ until further notice.
- Proposition: If $g = 0$, then $\gamma = 1$. If $g = 1$, then $\gamma = 2$. If $g \geq 2$, then $\gamma \leq g + 1$.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	?	?	?
3				?	?	?
4				?	?	?
5					?	?
6						?

Hyperelliptic curves

- Proposition: The number of points on a binary curve of gonality γ is $\leq 3\gamma$.
- Proposition: For each genus $g \geq 2$, there exists a hyperelliptic curve over $\mathbb{F}_2$ with 6 rational points.
- Proof: Look at $y^2 + [1 + x^g(x+1)]\ y = [x(x+1)]^{g-\delta}$, where $\delta = g \pmod 2$.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				?	?	?
4				?	?	?
5					?	?
6						?

Genus 3, Gonality 3

- Proposition: For a curve with rational points, $\gamma \leq g$.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	?	?
4				?	?	?
5					?	?
6						?

Genus 3, Gonality 4

- A genus-3 curve with rational points must have gonality ≤ 3.
- $(x^2 + xz)^2 + (x^2 + xz)(y^2 + yz) + (y^2 + yz)^2 + z^4 = 0$ has gonality 4 and no points.
- (Can prove non-hyperelliptic pointless curves of genus 3 have gonality 4.)

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	?	?
4				0	?	?
5					?	?
6						?

Fun facts about genus 4 non-hyperelliptic curves

- Non-hyperelliptic curves of genus 4 can be embedded as the intersection of a quadric surface and a cubic surface.
- If the quadric surface is $xy + zw = 0$ or $xy + z^2 = 0$, the curve is trigonal.
- If the quadric surface is $xy + z^2 + wz + w^2 = 0$, the curve is not.

Genus 4, Gonality 3

- Consider the curve:

$$\begin{aligned} xy + zw &= 0 \\ xy^2 + y^3 + x^2z + y^2z + xz^2 + x^2w + y^2w + xw^2 &= 0. \end{aligned}$$

- It has 8 points and is trigonal.

Genus 4, Gonality 4

- The surface $xy + z^2 + wz + w^2 = 0$ has only 5 rational points.
- Consider the curve:

$$\begin{aligned} xy + z^2 + zw + w^2 &= 0 \\ xy^2 + x^2z + y^2z + yz^2 + x^2w + z^2w &= 0. \end{aligned}$$

- It has 5 points.

Genus 4, Pointless Curves

- If a genus 4 curve has gonality 5, it must be pointless.
- Consider the curve:

$$\begin{aligned} xy + z^2 + zw + w^2 &= 0 \\ x^3 + y^3 + z^3 + y^2 w + xzw &= 0. \end{aligned}$$

- Not gonality 4; look at $\mathbb{F}_4$ and $\mathbb{F}_{16}$ points.

Updated Table

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	8	?
4				0	5	?
5					0	?
6						?

Genus 5, Gonality 3 and 4

- Trigonal curves of genus 5 are birationally equivalent to plane quintics with a multiplicity-2 singularity.
- This gives an upper bound of $2^2 + 2 + 1 + 1$ (based on the size of $\mathbb{P}^2$).
- We achieve this bound with

$$xyz^3 + x^3z^2 + y^3z^2 + x^4z + xy^3z + y^4z + x^4y + x^2y^3 = 0.$$

- The genus-5 curves on `manypoints.org` with 9 points have degree-4 morphisms and thus gonality 4.

Genus 5, Gonality 5 and 6

- Non-hyperelliptic, non-trigonal genus-5 curves are intersections of three quadric surfaces.
- Can exhaust over pointless genus-5 curves to rule out gonality 6.
- Can exhaust over all genus-5 curves by looking at possible divisors to show that the maximum number of points on a gonality-5 curve is 3.

Binary Table

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	8	8
4				0	5	9
5					0	3
6						

Start of Ternary Table

Applying what we learned in constructing the binary table, we start with the following for $\mathbb{F}_3$:

γ \ g	0	1	2	3	4	5
1	3					
2		7	?	?	?	?
3				10	?	?
4				?	?	?
5					?	?
6						?

Things we learned about ternary curves

- It's easy to find hyperelliptic curves of genus $3, 4, 5$ with 8 points.
- An example from [Howe-Lauter-Top, 2003] gives $N_3(3, 4) = 0$.
- We exhibit a genus-4 curve with gonality 3 and 12 points, thus $N_3(4, 3) = 12$.
- The relevant surface for gonality-4 curves has 10 points; we exhibit such a curve.
- A curve showing $N_3(4, 5) = 0$ is in [Castryck-Tuitman, 2017].
- A search finds a curve showing that $N_3(5, 3) = 12$.
- The `manypoints.org` example with $N_3(5) = 13$ has gonality 4.
- An exhaustive search found that $N_3(5, 5) = 4$.
- An exhaustive search showed there is no genus-5, gonality-6 curve.

Ternary Table

γ \ g	0	1	2	3	4	5
1	3					
2		7	8	8	8	8
3				10	12	12
4				0	10	13
5					0	4
6						

Things to Think About

- Things we are thinking about:
  - GF(4), GF(5) and GF(7).
  - Does there exist a genus-5, gonality-6 curve over a finite field?
  - General construction to show $N_q(g, k) = (q + 1)k$ for fixed k and large enough q. (Vermeulen has a trigonal construction for GF(2)).
  - `gonality.org`
- Things we are not thinking about:
  - Maximum number of points with a given gonality sequence.
  - $q > 7$ or $g > 5$ (except sporadically).