Equivalences of Stability Properties for Discrete-Time Nonlinear Systems and extensions

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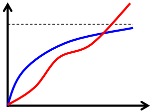
Overview

1. Comparison functions
2. Stability Analysis Approaches
3. Qualitative Equivalences
   - Systems without input
   - Systems with input
4. Future work

Comparison Functions

- Class-$\mathcal{K}$ functions: $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$:
  - continuous, zero at zero, and strictly increasing



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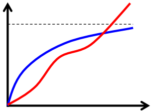
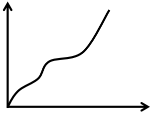


  - unbounded, it is of class-$\mathcal{K}_\infty$



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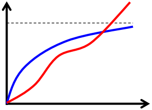

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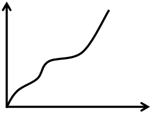

- Class-$\mathcal{L}$ functions: $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
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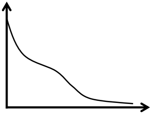

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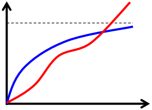

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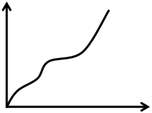

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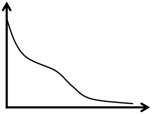


- Class-$\mathcal{KL}$ functions: $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
  - class-$\mathcal{K}$ in first argument, class-$\mathcal{L}$ in second.

$$\beta(s,t) = \tanh(s)e^{-t}$$

$$\beta(s,t) = se^{-t}$$

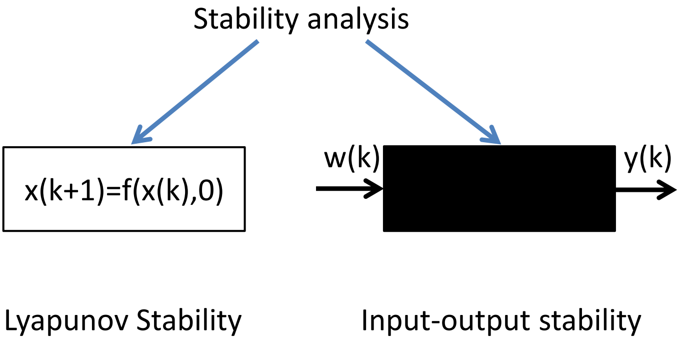
Dynamical systems

We consider discrete-time systems described by

$$x(k+1) = f(x(k), w(k)) \tag{1}$$

- System state $x(k) \in \mathbb{R}^n$, input $w(k) \in \mathbb{R}^m$.
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.
- equilibrium: $f(0, 0) = 0$.

Stability Analysis

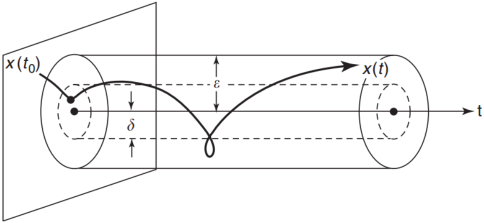


[1] A. M. Lyapunov." The general problem of the stability of motion", *Math. Soc. of Kharkov*, 1892.

[2] G. Zames. "On the input-output stability of time-varying nonlinear feedback systems part I: Conditions derived using concepts of loop gain, conicity, and passivity", *IEEE Transactions on Automatic Control*, 1966.

Lyapunov stability: Systems without input

- Stability in the sense of Lyapunov



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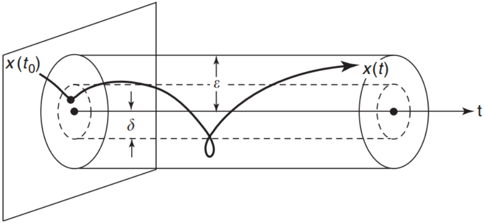
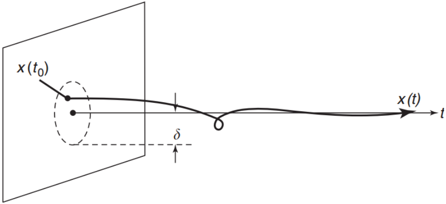
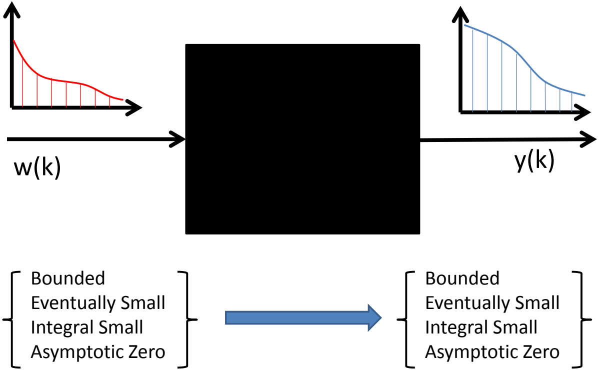


- Asymptotic Stability



Input-Output Stability and l_2-gain

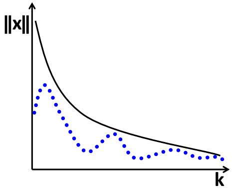


Lyapunov Stability and Extensions to Systems with Input

Definition System $x(k+1) = f(x(k), w(k))$

- has a **0-input globally asymptotically stable** equilibrium if for $w \equiv 0$ then there exists $\beta \in \mathcal{KL}$

$$\|x(k)\| \leq \beta(\|x_0\|, k)$$



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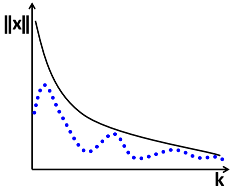
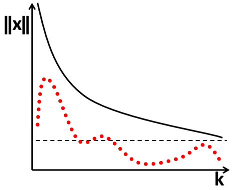


- is **input-to-state stable** (ISS) if, in addition, there exists $\sigma \in \mathcal{K}_\infty$

$$\|x(k)\| \leq \beta(\|x_0\|, k) + \sigma(\|w\|_\infty)$$



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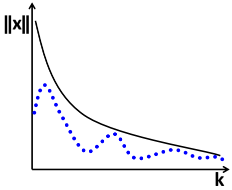

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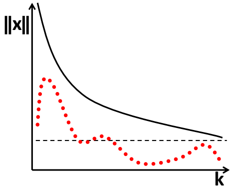

- is **integral ISS** if there exist $\alpha, \gamma, \sigma \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ such that

$$\alpha(\|x(k)\|) \leq \beta(\|x_0\|, k) + \sum_{\kappa=0}^{k-1} \sigma(\|w(\kappa)\|)$$

Lyapunov Theorem and Extensions

Theorem[1]*:* The equilibrium of $x(k+1) = f(x(k))$ is globally asymptotically stable if and only if there exists a smooth function function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2}$$

$$V(f(x,w)) - V(x) \leq -\alpha_3(\|x\|). \tag{3}$$

[1] Jiang, Z. P., Wang, Y., "A converse Lyapunov theorem for discrete time systems with disturbances", *Syst. Cont. Lett.*, 2001.

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Theorem[2]*:* System $x(k+1) = f(x(k), w(k))$ is Input-to-State stable if and only if there exists a smooth function function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

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l_2 Stability

Definition: System $x(k+1) = f(x(k), w(k))$

- is **0-input l_2-stable** if for $w \equiv 0$ then

$$\|x\|^2_{l_2[0,k]} \leq \gamma(\|x_0\|).$$

[1] l_2 norm: $\|z\|^2_{l_2[0,k]} := \sum_{\kappa=0}^{k} \|z(\kappa)\|^2$

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$$\|x\|^2_{l_2[0,k]} \leq \gamma(\|x_0\|) \;+\; \lambda \|w\|^2_{l_2[0,k-1]}.$$

- has **nonlinear l_2-gain** with transient and gain bound $\gamma, \sigma \in \mathcal{K}_\infty$ if

$$\|x\|^2_{l_2[0,k]} \leq \gamma(\|x_0\|) + \sigma\left(\|w\|^2_{l_2[0,k-1]}\right).$$

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0-input Systems: GAS and l_2-Stability Equivalence

Theorem

If system $x(k+1) = f(x(k))$ is l_2-stable then it is globally asymptotically stable (GAS). Conversely, if system $x(k+1) = f(x(k))$ is globally asymptotically stable then it is l_2-stable via a change of coordinates.

$$\text{GAS} \underset{\Longleftarrow}{\overset{\text{CoC}}{\Longrightarrow}} l_2\text{-stable}$$

GAS and l_2-Stability Equivalence: an Example

Consider the scalar system:

$$x^+ = \frac{x}{\sqrt{x^2+1}}$$

- Equilibrium **0** is GAS validated by Lyapunov function: $V(x) = x^2$:

$$V(x^+) - V(x) \leq -\frac{x^4}{x^2+1} =: -\alpha(\|x\|)$$

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- is not l_2-stable. Solution given by: $x_0 = \frac{x_0}{\sqrt{kx_0^2+1}}$, for $x_0 = 1$

$$\|x\|^2_{l_2[0,k]} = \sum_{\kappa=0}^{k} \frac{1}{\kappa+1} \;\not\leq\; \beta(1) \;\;\text{!!!}$$

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- change of coordinates $z := x^3$ leads to $\|z(k)\|^2 \leq \frac{1}{3k^2}\|z_0\|^{2/3}$, hence

$$\|z\|^2_{l_2[0,k]} \leq \|z_0\|^2 + \frac{\pi^2}{18}\|z_0\|^{2/3} =: \beta(\|z_0\|) \tag{6}$$

Systems with Input: ISS and Linear l_2-gain Equivalence

Theorem

If system $x(k+1) = f(x(k), w(k))$ has linear l_2-gain then it is ISS. Conversely, if system $x(k+1) = f(x(k), w(k))$ is ISS then it has linear l_2-gain via a change of coordinates.

$$\text{ISS} \underset{\Longleftarrow}{\overset{\text{CoC}}{\Longrightarrow}} \text{Linear } l_2\text{-gain}$$

ISS and Linear l_2-gain Equivalence: an Example

Consider the scalar system: $x^+ = \dfrac{x}{\sqrt{x^2+1}} + w$

- is ISS validated by ISS-Lyapunov function: $V(x) = x^2$:

$$V(x^+) - V(x) \leq -\frac{x^4}{x^2+1} + \left(2\|w\| + w^2\right)$$

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- has no linear l_2-gain by letting $w \equiv 0$ and $x_0 = 1$.

$$\|x\|^2_{l_2[0,k]} = \sum_{\kappa=0}^{k} \frac{1}{\kappa+1} \not\leq \beta(1) + \gamma(0) \quad !!!$$

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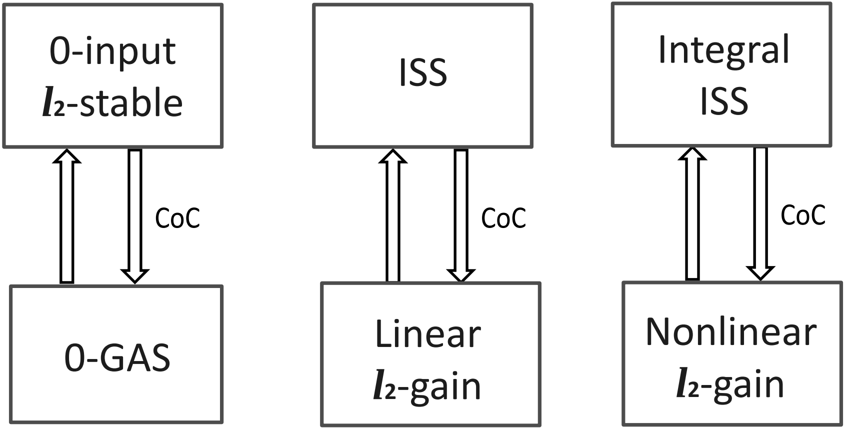
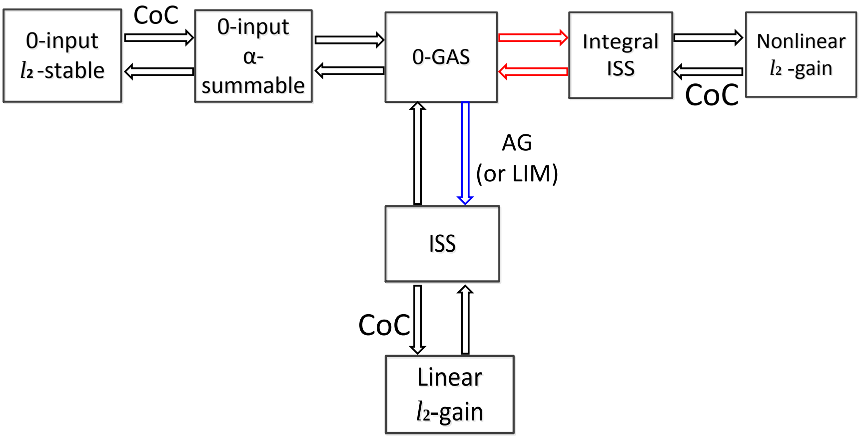
- changes of coordinates $z := \frac{x\|x\|}{\sqrt{x^2+1}}$, and $v := \mathrm{sign}(w)\sqrt{2\|w\| + w^2}$ do the job!

Qualitative Equivalences [3]



[3] D.N Tran, C.M Kellett, P.M Dower, "Equivalences of Stability Properties for Discrete-Time Nonlinear Systems", *IFAC MICNON Conf.*, Saint Petersburg, Russia, June 2015

Stability Relationships



[1] D.Angeli, "Intrinsic robustness of global asymptotic stability", *Syst. Cont. Lett.*, 1999

[2] K. Gao and Y. Lin, "On equivalent notions of input-to-state stability for nonlinear discrete time systems", *Proc. IASTED Intl. Conf. Cont. App.*, 2000.

Future Work

- Stability analysis approaches
  - Lyapunov approach
  - Operator approach
- Qualitative equivalences
- Future work: extensions to systems with general output

$$y(t) = h(x(t))$$

The End