Chief series of locally compact groups

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A **topological group** is a group that is also a topological space, such that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.
G is **locally compact** if there is a compact neighbourhood of 1.
G is **compactly generated** if there is a compact subset of G that generates G as a group.

Examples of compactly generated locally compact groups:

- Finitely generated groups (with the discrete topology)
- Compact groups
- Any connected locally compact group (e.g. connected subgroups of $\mathrm{GL}(\mathbb{R}^n)$)
- Many examples of totally disconnected locally compact groups, e.g. the automorphism group of any connected locally finite graph with finitely many orbits of vertices

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A **normal factor** of a topological group G is a quotient K/L, such that K and L are closed normal subgroups of G. We say it is a **chief factor** if $K > L$ there does not exist $K > M > L$ such that M is closed and normal in G.

A **(finite) chief series** for G is a series

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

of closed normal subgroups of G, such that each G_{i+1}/G_i is a chief factor.

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- Every finite group has a chief series. Given any group G, any finite chief factor of G is the product of finitely many copies of a simple group.
- Connected Lie groups have something like a chief series: there is a finite series in which every factor is chief or abelian. Every non-abelian chief factor is a product of finitely many copies of a simple connected Lie group.
- Compact groups have descending chief series, but these are usually infinite.
- Finitely generated discrete groups can have a very complicated normal subgroup structure (e.g. a finitely generated free group), and chief series fail to capture this structure.

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Theorem 1 (Caprace–Monod 2011)

Let G be a compactly generated locally compact group with no non-trivial compact or discrete normal subgroups. Then every non-trivial closed normal subgroup of G contains a minimal one.

Theorem 2 (R.–Wesolek)

Let G be a compactly generated locally compact group.

(i) Let $G_1 < G_2 < G_3 \dots$ be an ascending chain of closed normal subgroups of G and let $K = \overline{\bigcup_i G_i}$. Then there exists i such that K/G_i is compact-by-discrete.

(ii) Let $G_1 > G_2 > G_3 \dots$ be a descending chain of closed normal subgroups of G and let $K = \bigcap_i G_i$. Then there exists i such that G_i/K is compact-by-discrete.

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Theorem 3 (R.–Wesolek)

For every compactly generated locally compact group G, there is an **essentially chief series**, i.e. a finite series

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

of closed normal subgroups of G, such that each G_{i+1}/G_i is compact, discrete or a chief factor of G.

Let G be a compactly generated locally compact group. Write G° for the largest connected subgroup of G.

Fact

G has an action on a graph Γ, called a **Cayley-Abels graph** for G, such that:

- G acts transitively on vertices;
- The degree of Γ (= maximum number of neighbours of a vertex) is finite;
- If U is the stabiliser of a vertex, then U is open in G (so $G^\circ \leq U$) and U/G° is compact.

If N is a closed normal subgroup of G, then Γ/N is a Cayley-Abels graph for G/N.

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Outline of proof of Theorem 2 (e.g. Theorem 2(i)):

- Fix a Cayley-Abels graph Γ for G and consider $\deg(\Gamma/G_i)$. By dividing out by a large enough G_i, can assume $\deg(\Gamma) = \deg(\Gamma/K)$. Then all the vertex stabilisers in K acting on Γ are equal, so K/N is a discrete group, where N is the kernel of the action of K.
- By dividing out by a compact group, can assume K° is a Lie group (solution to Hilbert's 5th problem).
- Use the structure of Lie groups to deduce that there exists i such that $K^\circ/(G_i)^\circ$ is compact.
- N is connected-by-compact, so $N/(G_i)^\circ$ is compact-by-compact = compact, and K/N is discrete, so $K/(G_i)^\circ$ is compact-by-discrete, and hence K/G_i is compact-by-discrete.

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Say the normal factors K_1/L_1 and K_2/L_2 are **associated** if

$$\overline{K_1L_2} = \overline{K_2L_1};\ K_i \cap \overline{L_1L_2} = L_i \text{ for } i = 1, 2.$$

E.g. for any closed normal subgroups A and B of G, $A/(A \cap B)$ is associated to $\overline{AB}/B$.

Proposition (R.–Wesolek)

For non-abelian chief factors, association is an equivalence relation. For each equivalence class, there is a canonical uppermost representative M/C, such that any chief factor associated to M/C is of the form $A/(A \cap C)$ such that $M = \overline{AC}$. In particular, there is a continuous injective homomorphism from $A/(A \cap C)$ to M/C with dense image.

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Theorem 4 (R.–Wesolek)

Let G be a Polish group and let

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \dots \leq G_n = G$$

be a series of closed normal subgroups for G and let K/L be a non-abelian chief factor of G. Then there exists a unique i and $G_i \leq B < A \leq G_{i+1}$ such that A/B is a non-abelian chief factor associated to K/L.

Say a chief factor is **non-negligible** if it is non-abelian and not associated to any compact or discrete chief factor.

Corollary

Given an essentially chief series

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for the compactly generated locally compact group G, then each association class of non-negligible chief factor is represented exactly once as a factor G_{i+1}/G_i.
Consequently, G has only finitely many association classes of non-negligible chief factors.

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