Product system models for twisted C^*-algebras of topological higher-rank graphs

Becky Armstrong

The University of Sydney

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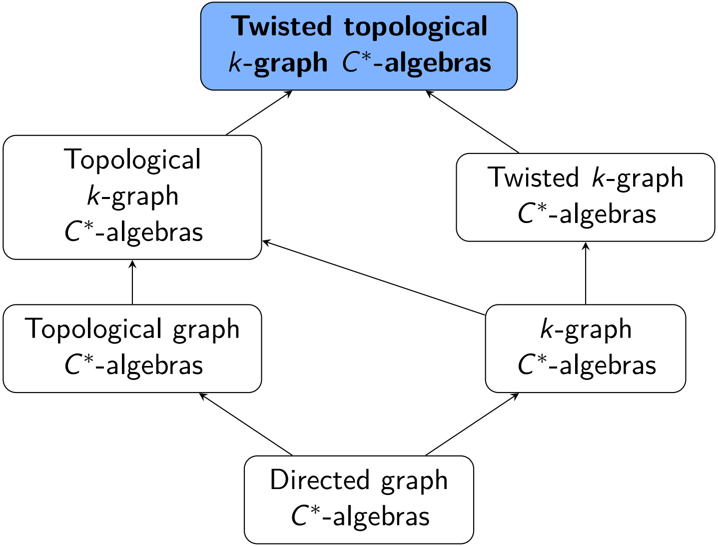
(Joint work with Nathan Brownlowe)

Directed graph C^*-algebras and their generalisations



Topological graph C^*-algebras

Definition (Katsura 2004)

A **topological graph** is a quadruple $E = (E^0, E^1, r, s)$, such that E^0 and E^1 are locally compact Hausdorff spaces, $r\colon E^1 \to E^0$ is a continuous map, and $s\colon E^1 \to E^0$ is a local homeomorphism.

The **topological graph correspondence of E** is a $C_0(E^0)$-correspondence $X(E) \subseteq C(E^1)$ with bimodule structure given by

$$(h \cdot f)(e) := h(r(e))\, f(e) \quad \text{and} \quad (f \cdot h)(e) := f(e)\, h(s(e)),$$

and

$$\langle f, g \rangle_{X(E)}(v) := \sum_{e \in s^{-1}(v)} \overline{f(e)} g(e).$$

Each topological graph E then has a Toeplitz algebra $\mathcal{T}(X(E))$, and a Cuntz–Pimsner algebra $\mathcal{O}(X(E))$.

Topological k-graphs

Definition (Yeend 2006)

Let $k \in \mathbb{N} \backslash \{0\}$. A **topological k-graph** is a pair (Λ, d) consisting of a small category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s, \circ)$ and a continuous functor $d \colon \Lambda \to \mathbb{N}^k$, called the **degree map**, which satisfy

(i) $\mathrm{Obj}(\Lambda)$ and $\mathrm{Mor}(\Lambda)$ are both second-countable, locally compact Hausdorff spaces;

(ii) $r, s \colon \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ are continuous, and s is a local homeomorphism;

(iii) the composition map

$$\circ \colon \Lambda \times_c \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda \mid s(\lambda) = r(\mu)\} \to \Lambda$$

is continuous and open, where $\Lambda \times_c \Lambda$ has the subspace topology inherited from the product topology on $\Lambda \times \Lambda$; and

(iv) the **unique factorisation property**: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exists a unique pair $(\mu, \nu) \in \Lambda \times_c \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$.

Topological k-graphs

We call the elements of $\mathrm{Obj}(\Lambda)$ **vertices**, and the elements of $\mathrm{Mor}(\Lambda)$ **paths**. We call r the **range** map and s the **source** map.

For each $n \in \mathbb{N}^k$, we define $\Lambda^n := d^{-1}(n)$. We have $\Lambda^0 = \mathrm{Obj}(\Lambda)$.

Given $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $m \leq n \leq d(\lambda)$, there is a unique path $\lambda(m, n) \in \Lambda^{n-m}$, such that $\lambda = \mu\, \lambda(m, n)\, \nu$, for some (unique) $\mu \in \Lambda^m$ and $\nu \in \Lambda^{d(\lambda)-n}$.

We say that Λ is **source-free** if, for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $r|_{\Lambda^n}^{-1}(v) \neq \emptyset$.

We say that Λ is **proper** if, for each $n \in \mathbb{N}^k$, $r|_{\Lambda^n}$ is a proper map, in the sense that for any compact subset V of Λ^0, $r|_{\Lambda^n}^{-1}(V)$ is a compact subset of Λ^n.

The k-graph Ω_k

Let Ω_k be the category with

- $\mathrm{Obj}(\Omega_k) := \mathbb{N}^k$;
- $\mathrm{Mor}(\Omega_k) := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$;
- $r(m, n) := m$;
- $s(m, n) := n$; and
- composition $(m, n)(n, p) := (m, p)$.

Define a functor $d\colon \Omega_k \to \mathbb{N}^k$ by $d(m, n) := n - m$.

Then (Ω_k, d) is a k-graph.

The infinite-path space

Definition

Let Λ be a proper, source-free topological k-graph. The **infinite-path space** of Λ is the set

$$\Lambda^\infty := \{x \colon \Omega_k \to \Lambda \mid x \text{ is a } k\text{-graph morphism}\}.$$

For any subset U of Λ, we define

$$Z(U) := \{x \in \Lambda^\infty \mid x(0, n) \in U \text{ for some } n \in \mathbb{N}^k\}.$$

Proposition (Yeend 2006)

Let Λ be a proper, source-free topological k-graph. The collection

$$\{Z(U) \mid U \textit{ is an open subset of } \Lambda^n \textit{ for some } n \in \mathbb{N}^k\}$$

is a basis for a locally compact Hausdorff topology on Λ^∞.

Building C^*-correspondences from topological k-graphs

For each $n \in \mathbb{N}^k$, there is a local homeomorphism $T^n \colon \Lambda^\infty \to \Lambda^\infty$ given by $T^n(x)(p,q) := x(p+n, q+n)$. We call each T^n a **shift map**.

Proposition

Let Λ be a proper, source-free topological k-graph. For each $n \in \mathbb{N}^k$, the quadruples $\Lambda_n := (\Lambda^0, \Lambda^n, r|_{\Lambda^n}, s|_{\Lambda^n})$ and $\Lambda_{\infty,n} := (\Lambda^\infty, \Lambda^\infty, T^0, T^n)$ are topological graphs.

For each $n \in \mathbb{N}^k$, let $X_n := X(\Lambda_n)$ and $Y_n := X(\Lambda_{\infty,n})$ be the topological graph correspondences associated to Λ_n and $\Lambda_{\infty,n}$, respectively. The homomorphisms implementing the left actions, $\phi_{X_n} \colon C_0(\Lambda^0) \to \mathcal{L}(X_n)$ and $\phi_{Y_n} \colon C_0(\Lambda^\infty) \to \mathcal{L}(Y_n)$, are both injective and have range in the compact operators.

Cocycles on topological k-graphs

Definition

A **continuous $\mathbb{T}$-valued 2-cocycle** on a topological k-graph Λ is a continuous map $c\colon \Lambda \times_c \Lambda \to \mathbb{T}$ satisfying

(C1) $c(\lambda, \mu)c(\lambda\mu, \nu) = c(\lambda, \mu\nu)c(\mu, \nu)$; and

(C2) $c(\lambda, s(\lambda)) = c(r(\lambda), \lambda) = 1$, for all $\lambda \in \Lambda$.

We define $\underline{Z}^2(\Lambda, \mathbb{T})$ to be the group of continuous $\mathbb{T}$-valued 2-cocycles on Λ.

Topological $(k+l)$-graphs

Example (A–Brownlowe 2017)

Let Λ be a topological k-graph, and β an action of $\mathbb{Z}^l$ by automorphisms of Λ.

We can form a topological $(k+l)$-graph $\Gamma := \Lambda \times_\beta \mathbb{Z}^l$ as follows:

- $\mathrm{Obj}(\Gamma) := \Lambda^0 \times \{0\}$, and $\mathrm{Mor}(\Gamma) := \Lambda \times \mathbb{N}^l$, both with the product topology;
- $r(\mu, m) := (r_\Lambda(\mu), 0)$, and $s(\mu, m) := (\beta_{-m}(s_\Lambda(\mu)), 0)$;
- composition is given by $(\mu, m)(\nu, n) := (\mu\beta_m(\nu), m+n)$, whenever $s_\Lambda(\mu) = r_\Lambda(\beta_m(\nu))$; and
- $d(\mu, m) := (d_\Lambda(\mu), m) \in \mathbb{N}^{k+l}$.

If Λ is proper and source-free, then so is Γ.

Cocycles on topological $(k + l)$-graphs

Example (A–Brownlowe 2017)

We can construct several continuous $\mathbb{T}$-valued 2-cocycles on Γ. For each $q \in \mathbb{N}\backslash\{0\}$ and $m \in \mathbb{N}^q$, we define $|m| := \sum_{i=1}^{q} m_i$.

- Let $f \colon \Lambda \to \mathbb{T}$ be a continuous functor such that $f \circ \beta_m = f$, for all $m \in \mathbb{N}^l$. For example, take $f(\mu) := e^{i|d(\mu)|}$. We define $c_f \in \underline{Z}^2(\Gamma, \mathbb{T})$ by
$$c_f((\mu, m), (\nu, n)) := f(\nu)^{|m|}.$$
- Let $\omega \colon \mathbb{N}^l \to \mathbb{T}$ be a continuous homomorphism. We define $c_\omega \in \underline{Z}^2(\Gamma, \mathbb{T})$ by
$$c_\omega((\mu, m), (\nu, n)) := \omega(m)^{|d(\nu)|}.$$

Product systems

Definition (Fowler 2002)

Let A be a C^*-algebra. A **product system** over $\mathbb{N}^k$ is a semigroup $X = \sqcup_{n \in \mathbb{N}^k} X_n$ such that

(i) each X_n is an A-correspondence, with the homomorphism implementing the left action denoted by $\phi_{X_n} \colon A \to \mathcal{L}(X_n)$;

(ii) the A-correspondence X_0 is a copy of ${}_A A_A$;

(iii) for each nonzero $m, n \in \mathbb{N}^k$, the map $X_m \times X_n \to X_{m+n}$ given by $x \otimes y \mapsto xy$ extends to an isomorphism of A-correspondences $X_m \otimes_A X_n \cong X_{m+n}$; and

(iv) $ax = a \cdot x$ and $xa = x \cdot a$, for each $x \in X$ and $a \in X_0$.

We say that X is **compactly aligned**, if, for all $S \in \mathcal{K}(X_m)$ and $T \in \mathcal{K}(X_n)$, we have

$$\big(S \otimes_A 1_{(m \vee n)-m}\big)\big(T \otimes_A 1_{(m \vee n)-n}\big) \in \mathcal{K}(X_{m \vee n}).$$

Representations of product systems

Definition (Fowler 2002)

A **representation** ψ of a product system X in a C^*-algebra B is a linear map $\psi\colon X \to B$ such that

(i) each (ψ_n, ψ_0) is a representation of X_n, where $\psi_n := \psi|_{X_n}$; and

(ii) $\psi_{m+n}(xy) = \psi_m(x)\psi_n(y)$, for all $x \in X_m$, $y \in X_n$.

For each $n \in \mathbb{N}^k$, there is a homomorphism $\psi^{(n)}\colon \mathcal{K}(X_n) \to B$ such that $\psi^{(n)}(\Theta_{x,y}) = \psi_n(x)\psi_n(y)^*$.

The Nica–Toeplitz algebra of a product system

Definition (Fowler 2002)

If X is a compactly aligned product system of A-correspondences, we say that a representation ψ of X is **Nica covariant** if, for each $S \in \mathcal{K}(X_m)$ and $T \in \mathcal{K}(X_n)$, we have

$$\psi^{(m)}(S)\psi^{(n)}(T) = \psi^{(m\vee n)}\Big(\big(S \otimes_A 1_{(m\vee n)-m}\big)\big(T \otimes_A 1_{(m\vee n)-n}\big)\Big).$$

Theorem (Fowler 2002)

*There is a universal C^*-algebra $\mathcal{NT}(X)$, called the **Nica–Toeplitz algebra of X**, which is generated by an isometric Nica-covariant representation $i_X \colon X \to \mathcal{NT}(X)$. That is, if ψ is a Nica-covariant representation of X, then there exists a homomorphism $\psi^{\mathcal{NT}}$ such that $\psi^{\mathcal{NT}} \circ i_X = \psi$.*

The Cuntz–Pimsner algebra of a product system

Definition (Fowler 2002)

Suppose that X is a product system of A-correspondences such that each left action ϕ_{X_n} is injective and has range in $\mathcal{K}(X_n)$. We say that a representation ζ of X is **Cuntz–Pimsner covariant** if

$$\zeta^{(n)}(\phi_{X_n}(a)) = \zeta_0(a),$$

for all $a \in A$ and $n \in \mathbb{N}^k$.

Theorem (Fowler 2002)

There is a universal C^*-algebra $\mathcal{O}(X)$, called the ***Cuntz–Pimsner algebra of X****, which is generated by an isometric Cuntz–Pimsner-covariant representation $j_X \colon X \to \mathcal{O}(X)$. That is, if ζ is a Cuntz–Pimsner-covariant representation of X, then there exists a homomorphism $\zeta^{\mathcal{O}}$ such that $\zeta^{\mathcal{O}} \circ j_X = \zeta$. There is a quotient map $q_X \colon \mathcal{NT}(X) \to \mathcal{O}(X)$ satisfying $j_X = q_X \circ i_X$.*

A product system built from finite paths

Let Λ be a proper, source-free topological k-graph, and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Recall that $\Lambda_n = (\Lambda^0, \Lambda^n, r|_{\Lambda^n}, s|_{\Lambda^n})$, and $X_n = X(\Lambda_n)$, for each $n \in \mathbb{N}^k$.

Proposition (A–Brownlowe 2017)

For $f \in X_m$ and $g \in X_n$, define $fg \colon \Lambda^{m+n} \to \mathbb{C}$ by

$$(fg)(\lambda) := c(\lambda(0,m), \lambda(m,m+n))\, f(\lambda(0,m))\, g(\lambda(m,m+n)).$$

Then $fg \in X_{m+n}$, and under this multiplication, the family

$$X := \bigsqcup_{n \in \mathbb{N}^k} X_n$$

of $C_0(\Lambda^0)$-correspondences is a compactly aligned product system over $\mathbb{N}^k$.

Definition (A–Brownlowe 2017)

We define the **twisted Toeplitz algebra** $\mathcal{T}C^*(\Lambda, c)$ to be the Nica–Toeplitz algebra $\mathcal{NT}(X)$.

We define the **twisted Cuntz–Krieger algebra** $C^*(\Lambda, c)$ to be the Cuntz–Pimsner algebra $\mathcal{O}(X)$.

A product system built from infinite paths

Let Λ be a proper, source-free topological k-graph, and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Recall that $\Lambda_{\infty,n} = (\Lambda^\infty, \Lambda^\infty, T^0, T^n)$, and $Y_n = X(\Lambda_{\infty,n})$, for each $n \in \mathbb{N}^k$.

Proposition (A–Brownlowe 2017)

For $f \in Y_m$ and $g \in Y_n$, define $fg \colon \Lambda^\infty \to \mathbb{C}$ by

$$(fg)(x) := c(x(0,m), x(m,m+n))\, f(x)\, g(T^m(x)).$$

Then $fg \in Y_{m+n}$, and under this multiplication, the family

$$Y := \bigsqcup_{n \in \mathbb{N}^k} Y_n$$

of $C_0(\Lambda^\infty)$-correspondences is a compactly aligned product system over $\mathbb{N}^k$.

A morphism of product systems

Proposition (A–Brownlowe 2017)

For each $n \in \mathbb{N}^k$, there is a map $\alpha_n \colon X_n \to Y_n$, given by $\alpha_n(f)(x) := f(x(0,n))$, for all $f \in X_n$ and $x \in \Lambda^\infty$. We have

(i) $\alpha_m(g \cdot f) = \alpha_0(g) \cdot \alpha_m(f)$, *for all* $f \in X_m$, $g \in C_0(\Lambda^0)$;

(ii) $\alpha_m(f \cdot g) = \alpha_m(f) \cdot \alpha_0(g)$, *for all* $f \in X_m$, $g \in C_0(\Lambda^0)$;

(iii) $\langle \alpha_m(f), \alpha_m(g) \rangle_{Y_m} = \alpha_0(\langle f, g \rangle_{X_m})$, *for all* $f, g \in X_m$;

(iv) $\alpha_{m+n}(fg) = \alpha_m(f)\alpha_n(g)$, *for all* $f \in X_m$, $g \in X_n$; *and*

(v) α_n *is injective, for each* $n \in \mathbb{N}^k$.

Main theorem

Theorem (A–Brownlowe 2017)

Let Λ be a proper, source-free topological k-graph, and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Then $\mathcal{T}C^*(\Lambda, c) = \mathcal{NT}(X)$ embeds into $\mathcal{NT}(Y)$, and $C^*(\Lambda, c) = \mathcal{O}(X)$ is isomorphic to $\mathcal{O}(Y)$, as illustrated by the following commuting diagram.

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[d, hook, "i_X"] \arrow[drr, "\psi"] \arrow[ddrr, "\zeta"'] \arrow[dd, bend right=60, hook, "j_X"'] & & Y \arrow[d, hook, "i_Y"'] \arrow[dd, bend left=60, "j_Y"] \\
\mathcal{T}C^*(\Lambda, c) \arrow[rr, hook, "\psi^{\mathcal{NT}}", "\ncong"'] \arrow[d, two heads, "q_X"] & & \mathcal{NT}(Y) \arrow[d, two heads, "q_Y"] \\
C^*(\Lambda, c) \arrow[rr, "\zeta^{\mathcal{O}}", "\cong"'] & & \mathcal{O}(Y).
\end{tikzcd}$$

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Thanks!