Totally disconnected locally compact groups and operator algebras

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CARMA

Totally disconnected locally compact groups

The locally compact group G is *totally disconnected* if the only connected components in G are singletons.

Theorem

Let G be a locally compact group. Then the connected component of the identity, N, is a closed normal subgroup of G and G/N is a t.d.l.c. group.

Theorem (van Dantzig, 1930's)

Suppose that G is a t.d.l.c. group and let $\mathcal{O} \ni 1$ be neighbourhood of the identity. Then there is a compact open subgroup $V \subseteq \mathcal{O}$.

Compact open subgroups are commensurated

Let U be a compact open subgroup of G. Then

$[U : U \cap xUx^{-1}] < \infty$ and $[xUx^{-1} : U \cap xUx^{-1}] < \infty$ for every $x \in G$,

i.e., U is a *commensurated* subgroup of G.

On the other hand, if G is any group and H is a commensurated subgroup of G, then there is a t.d.l.c. group $\widetilde{G}$ and a homomorphism $\varphi : G \to \widetilde{G}$ such that $\overline{\varphi(H)}$ is a compact open subgroup of $\widetilde{G}$.

$\widetilde{G}$ is the *relative profinite completion* of the pair (G, H),

see C. D. Reid and P. R. Wesolek, *Homomorphisms into totally disconnected, locally compact groups with dense image*, `arXiv:1509.00156v1` and references therein.

Weight functions and the scale function

For a fixed compact open $U \leq G$ define *weight* function

$$w_U(x) = [xUx^{-1} : U \cap xUx^{-1}], \quad (x \in G).$$

Then $w_U(xy) \leq w_U(x)w_U(y)$ for all $x, y \in G$, that is, w_U is *submultiplicative*.

The *scale* of $x \in G$ is the positive integer

$$s(x) = \min\left\{w_U(x) \mid U \leq G \text{ is compact and open}\right\}, \quad (x \in G).$$

Say that U is *minimising* if the minimum is attained at U.

It may be shown that, for every $U \leq G$ compact and open,

$$s(x) = \lim_{n\to\infty} w_U(x^n)^{\frac{1}{n}}.$$

A weighted convolution algebra

Given $U \leq G$ compact and open, let

$$L^1(G, w_U) = \left\{ f \in L(G) \mid \int_G f(x) w_U(x) \, \mathrm{d}x < \infty \right\}.$$

Then $L^1(G, w_U)$ is a Banach algebra under convolution.

- For $U, V \leq G$ compact and open, there is $B > 1$ such that

  $$B^{-1} w_V \leq w_U \leq B w_V.$$

  Hence $L^1(G, w_U)$ does not depend on U.
- w_U is bounded $\iff$ there is $V \triangleleft G$ compact and open.
- $s(x)$ is the spectral radius of the operator on $L^1(G, w_U)$ of translation by x.

A weighted convolution algebra 2

The convolution algebra $L^1(G, w_U)$ is just natural for the t.d.l.c. group G as is $L^1(G)$.

Problem

How do properties of the Banach algebra $L^1(G, w_U)$ reflect the structure of the totally disconnected, locally compact group G?

- Weighted convolution algebras $L^1(G, w)$ often have non-trivial cohomology (point derivations in the commutative case).
- Unlike $L^1(G)$ and C*-algebras, $L^1(G, w_U)$ does not have a unique natural norm.

A characterisation of minimising subgroups

Theorem

Let $x \in G$ and $U \leq G$ be compact and open. Put

$$U_+ = \bigcap_{k \geq 0} x^k U x^{-k} \text{ and } U_- = \bigcap_{k \leq 0} x^k U x^{-k}.$$

Then U is minimising for x if and only if

TA $U = U_+ U_-$, *and*

TB $U_{++} := \bigcup_{k \in \mathbb{Z}} x^k U_+ x^{-k}$ *is closed.*

In this case, $s(x) = [xU_+x^{-1} : U_+]$.

A compact open subgroup satisfying TA and TB is *tidy* for x.

The tree representation theorem

The group $V_{++} \ltimes \langle x \rangle$ is an HNN-extension and so Bass-Serre theory implies the following.

Theorem

Suppose that U is tidy for $x \in G$. Then $V_{++} \ltimes \langle x \rangle$ is a closed subgroup of G. There is a regular tree $\mathcal{T}_{q+1}$, where $q = s(x)$, and a homomorphism $\rho : V_{++} \ltimes \langle x \rangle \to Aut(\mathcal{T}_{q+1})$ such that:

- *$\rho(V_{++} \ltimes \langle x \rangle)$ is a closed subgroup of $Aut(\mathcal{T}_{q+1})$ fixing an end, ω, of the tree;*
- $\ker \rho$ *is the largest compact normal subgroup of $V_{++} \ltimes \langle x \rangle$; and*
- *$\rho(x)$ is a hyperbolic element of $Aut(\mathcal{T}_{q+1})$ which translates by distance 1 and has ω is its attracting end.*

Closed subgroups of $\mathrm{Aut}(\mathcal{T}_{q+1})$ fixing and end of the tree are key ingredients in the structure theory of t.d.l.c. groups corresponding to the $(ax + b)$-group and $\mathbb{R} \ltimes \mathbb{R}^+$ in Lie theory.

Representations and C*-algebras of $\rho(V_{++} \ltimes \langle x \rangle)$

Problem

What are the unitary representations of the closed subgroups of $\mathrm{Aut}(\mathcal{T}_{q+1})$ which fix an end of $\mathcal{T}_{q+1}$?

(There are uncountably many such groups.)

Problem

Induce representations of $\rho(V_{++} \ltimes \langle x \rangle)$ to representations of G (for certain groups G).

Problem

How much information about $\rho(V_{++} \ltimes \langle x \rangle)$ is retained by $C^*\rho(V_{++} \ltimes \langle x \rangle)$?

Contraction groups 1

Theorem (The Mautner phenomenon)

Let $\sigma : G \to U(\mathfrak{H})$ be a unitary representation of the locally compact group G and suppose that $\sigma(x)\xi = \xi$ for some $x \in G$ and non-zero $\xi \in V$. Then $\sigma(h)\xi = \xi$ for every $h \in G$ such that $x^n h x^{-n} \to 1$ as $n \to \infty$.

The set $\mathrm{con}(x) = \{h \in G \mid x^n h x^{-n} \to 1 \text{ as } n \to \infty\}$ is the *contraction subgroup* for x.
Thus the Mautner phenomenon says that, if ξ is fixed by x, then ξ is fixed by every $h \in \overline{\mathrm{con}(x)}$.

Contraction groups 2

Theorem (Baumgartner & W.)

Suppose that V is tidy for $x \in G$. Then $C := \overline{\mathrm{con}(x^{-1})}$ is a co-compact normal subgroup of V_{++} invariant under conjugation x and $s(x|_C) = s(x)$.

Theorem (Glöckner & W.)

Suppose that $H := \mathrm{con}(x)$ is closed. Then

- *$H = T \times D$, where T, D are closed x-invariant subgroups of H such that T is torsion and D is divisible;*
- *D is a direct sum of nilpotent p-adic Lie groups for a finite set of primes p; and*
- *T has a finite composition series of x-invariant subgroups where the composition factors are isomorphic to $(\prod_{n\geq 0} F_i) \times \sum_{n<0} F_i$, for some finite simple group F_i and the automorphism induced by x is the shift.*