C^*-algebras from self-similar actions, and their states

24 and 25 July 2017

Presented by
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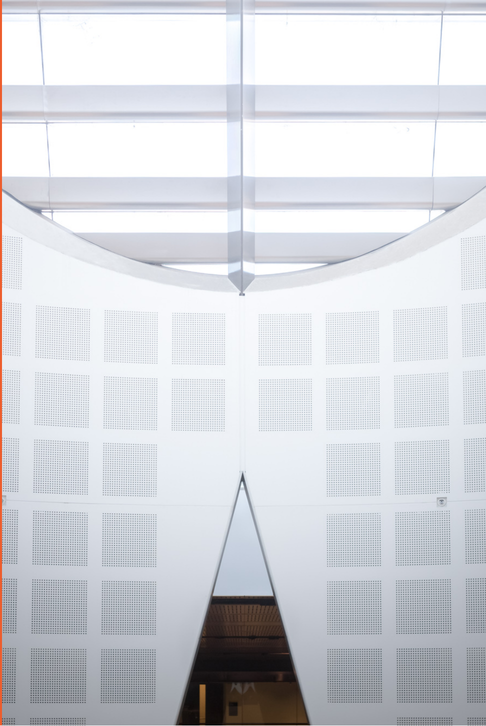
THE UNIVERSITY OF SYDNEY



Overview

Background

Self-similar actions of groups

The actions

The algebras

The states

The bigger picture

A path space perspective

The obstruction to generalisation

Self-similar actions of groupoids

Transformation groupoids

Constructing self-similar groupoid actions

The algebras

The states

The significance of examples

Because of topological closure, the study of C^*-algebras does not reduce to the study of simple components.

Simple algebras are still important, but the classification of C^*-algebras is, by necessity, an examples-driven endeavour.

Techniques used in the study of C^*-algebras include

- the construction of large classes of examples which can be studied using common tools.
- the development of invariants that allow you to readily(?!) identify two examples as being isomorphic.

One way automorphisms of graphs can help

Suppose we have a graph with a countable number of vertices.

Each vertex can represent a basis vector.

A graph automorphism represents a linear map sending each basis vector to the image of the corresponding vertex.

The edges between the vertices restrict the operators that can be represented by automorphisms.

Basis of these talks

Volodymyr Nekrashevych, **Self-similar groups**,
AMS Mathematical Surveys and Monographs 117 (2005).

Volodymyr Nekrashevych, C^***-algebras and self-similar groups**,
J. reine andgew. Math. 630 (2009) 59–123.

M. Laca, I. Raeburn, J. Ramagge, and M.F. Whittaker,
Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions, *J. Funct. Anal.* 266 (2014) 6619–6661.

R. Exel, E. Pardo, **Self-similar graphs, a unified treatment of Katsura and Nekrashevych** C^***-algebras**, *Adv. Math.* 306 (2017) 1046–1129.

M. Laca, I. Raeburn, J. Ramagge, and M.F. Whittaker,
Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs, arXiv:1610.00343.

2 Self-similar actions of groups

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The alphabet X and the tree T_X

Suppose X is a finite set, X^k is the set of k-tuples in$X$, with $X^0 = \{*\}$, and define $X^* := \bigsqcup_{k \geq 0} X^k = \{\text{finite words in } X\}$.

$T = T_X$ is an infinite homogeneous rooted tree with

- vertex set $T_X^0 = X^* = \{\mu \in X^*\}$
- edge set $T_X^1 = \{\{\mu, \mu x\} \colon \mu \in X^* \text{ and } x \in X\}$
- root the empty word, $*$

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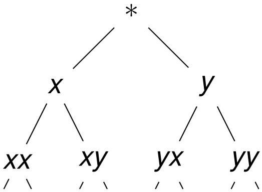
We label

- edges in T_X with elements of X
- paths in T_X with elements of X^*.

$T_{\{x,y\}}$



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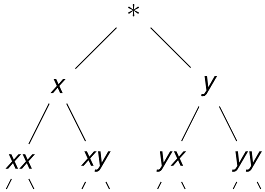

The boundary X^ω of T_X can be identified with semi-infinite words in X starting at $*$, so $\quad X^\omega = \{x_1 x_2 \ldots \ : x_i \in X\}$.

Automorphisms of $T = T_X$

From a traditional graph-theoretic perspective, an automorphism α of T consists of a family of bijections $\alpha_k \colon X^k \to X^k$ for $k \geq 0$ such that for all $\mu, \nu \in X^*$

$$\{\alpha_k(\mu), \alpha_{k+1}(\nu)\} \in T^1 \Leftrightarrow \{\mu, \nu\} \in T^1.$$

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If $\beta = \{\beta_k\}$ is an automorphism, then each $\{\beta_k(\mu), \beta_{k+1}(\mu x)\}$ is an edge in T, and hence $\beta_{k+1}(\mu x) \in \beta_k(\mu) X$.

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So an automorphism satisfies the, ostensibly weaker, property (**??**)

Lemma
Suppose $\alpha \colon T^0 \to T^0$ is a bijection satisfying

$$\alpha(X^k) = X^k \textit{ for all } k, \textit{ and } \alpha(\mu x) \in \alpha(\mu)X \textit{ for all } \mu \in X^k \textit{ and } x \in X.$$

*Define $\alpha_k := \alpha|_{X^k}$. Then $\{\alpha_k\}$ is an automorphism α of T. The inverse is also an automorphism of T, and also satisfies (**??**).*

Action of a group on T_X

A group G acts (by automorphisms) on T_X if it preserves adjacency (and hence depth).

Consider actions on X^* consistent with an action on T_X.

In particular, the action of $g \in G$ can not split a path apart, but its action on an edge labelled $x \in X$ may differ depending on the level.

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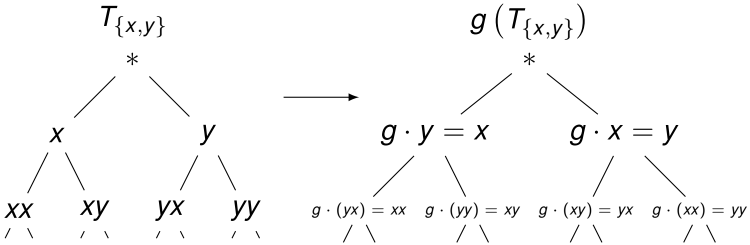
In particular, the action of $g \in G$ can not split a path apart, but its action on an edge labelled $x \in X$ may differ depending on the level.

So, in general, $\quad g \cdot (vw) \neq (g \cdot v)(g \cdot w) \quad$ for $g \in G$, $v, w \in X^*$.



Definition of a self-similar action

A self-similar action is a pair (G, X) consisting of a group G and a finite alphabet X with a faithful action of G on X^* satisfying $g \cdot \varnothing = \varnothing$ and

$$\text{for all } (g, x) \in G \times X, \text{ there exist } (h, y) \in G \times X \text{ such that}$$

$$g \cdot (xw) = y(h \cdot w) \quad \text{for all } w \in X^*$$

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$$\text{for all } g \in G, v \in X^*, \text{ there exists a unique } h \in G \text{ such that}$$

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Call this $h \in G$ the restriction of g at v and write $h = g|_v$.

An example - the odometer

Let $G = \mathbb{Z} = \langle a \rangle$ and $X = \{0, 1\}$.

Define an action of $\mathbb{Z}$ on X^* recursively by

$$\begin{aligned} a \cdot (0w) &= 1w \\ a \cdot (1w) &= 0(a \cdot w) \end{aligned}$$

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This corresponds to the diadic adding machine;
it coincides with the rule of adding one to a diadic integer
(with place value increasing towards the right).

Another example - the Basilica group

Let $X = \{0, 1\}$ and

$$G = \langle a, b\colon \sigma^n([a, a^b]) \text{ for all } n \in \mathbb{N}\rangle$$

where σ is the substitution $\sigma(b) = a$ and $\sigma(a) = b^2$.

Define an action of G on X^* recursively by

$$\begin{aligned} a \cdot (0w) &= 1(b \cdot w) \qquad & b \cdot (0w) &= 0(a \cdot w) \\ a \cdot (1w) &= 0w \qquad & b \cdot (1w) &= 1w \end{aligned}$$

The Basilica group is an iterated monodromy group with many interesting properties, including being amenable.

Other interesting examples

- Iterated monodromy groups
- The Grigorchuk group
- Branch groups

The nucleus

A nucleus of a self-similar action (G, X) is a minimal set $\mathcal{N} \subseteq G$ satisfying the property

for each $g \in G$, there exists $N \in \mathbb{N}$ such that

$$g|_v \in \mathcal{N} \text{ for all words } v \in X^n \text{ with } n \geq N.$$

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A self-similar action is contracting if it has a finite nucleus.

For a contracting self-similar action (G, X), the nucleus is unique and equal to

$$\mathcal{N} = \bigcup_{g \in G} \bigcap_{n \geq 0} \{g|_v \colon v \in X^*, |v| \geq n\}$$

The bimodule

Given a self-similar action (G, X), let $C^*(G)$ be the full group C^*-algebra of G and define

$$M = M_{(G,X)} = \bigoplus_{x \in X} C^*(G).$$

So

$$M = \{(m_x)_{x \in X} \mid m_x \in C^*(G)\}.$$

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For each $y \in X$ we write e_y for the element of M satisfying

$$(e_y)_x = \begin{cases} 1_{C^*(G)} & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

M is a free right Hilbert $C^*(G)$-module with

- right $C^*(G)$-action

$$(m_x)_{x \in X} \cdot a = (m_x a)_{x \in X}$$

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- basis $\{e_x \mid x \in X\}$ is a $C^*(G)$-basis for M, so

$$M = \operatorname{span}\{e_x \cdot a \mid x \in X, a \in C^*(G)\}.$$

The representation of G

Let (G, X) be a self-similar action and $M = \bigoplus_{x \in X} C^*(G)$.

Denote by $\delta_g \in C^*(G)$ the point mass at $g \in G$.

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For each $g \in G$, define a linear operator U_g on M via

$$U_g(e_x \cdot a) = e_{g \cdot x} \cdot (\delta_{g|_x} a)$$

for $x \in X$ and $a \in C^*(G)$.

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Lemma

The map $U \colon G \to \mathcal{UL}(M)$ given by $g \mapsto U_g$ is a faithful nondegenerate unitary representation of G with $(U_g)^* = U_{g^{-1}}$.

The Cuntz-Pimsner algebra of (G, X)

Theorem (Nekrashevych)

Let (G, X) be a self-similar action and $M = \bigoplus_{x \in X} C^*(G)$.

The Cuntz-Pimsner algebra $\mathcal{O}(G, X) := \mathcal{O}(M)$ is the universal C^*-algebra generated by a unitary representation $u \colon G \to \mathcal{UL}(M)$ and a family of isometries $\{s_x \colon x \in X\}$ satisfying

$$\textbf{(CR)} \quad \sum_{x \in X} s_x s_x^* = 1$$

$$\textbf{(SSR1)} \quad s_x^* s_x = 1 \quad \text{and} \quad s_x^* s_y = 0 \text{ if } x \neq y.$$

$$\textbf{(SSR2)} \quad u_g s_x = s_{g \cdot x} u_{g|_x}$$

for all $g \in G$ and $x \in X$.

If (G, X) is contracting with nucleus $\mathcal{N}$ then $\mathcal{O}(G, X)$ is generated by $\{u_g, s_x \colon g \in \mathcal{N}, x \in X\}$.
If (G, X) is contracting with nucleus $\{e\}$ then $\mathcal{O}(G, X) = \mathcal{O}_{|X|}$.

The Toeplitz algebra of (G, X)

Theorem (Laca, R., Raeburn, Whittaker)
Let (G, X) be a self-similar action and $M = \bigoplus_{x \in X} C^*(G)$.

The Toeplitz algebra $\mathcal{T}(G, X) := \mathcal{T}(M)$ is the universal C^* algebra generated by a unitary representation $u\colon G \to \mathcal{UL}(M)$ and a family of isometries $\{s_x\colon x \in X\}$ satisfying

(SSR1) $s_x^* s_x = 1$ and $s_x^* s_y = 0$ if $x \neq y$.

(SSR2) $u_g s_x = s_{g \cdot x} u_{g|_x}$

for all $g \in G$ and $x \in X$.

Moreover, $\mathcal{T}(G, X) = \overline{\operatorname{span}}\{s_\mu u_g s_\nu^* : \mu, \nu \in X^*, \ g \in G\}$ where $s_\mu := s_{\mu_1} \dots s_{\mu_n}$ for $\mu = \mu_1 \cdots \mu_n \in X^n$.

The states

States are linear functionals on algebras that satisfy properties of significance in statistical mechanics. A state ϕ of a system $(B, \mathbb{R}, \alpha)$ is a KMS$_\beta$ state if $\phi(ab) = \phi(b\alpha_{i\beta}(a))$ for all a, b in a family of analytic elements spanning a dense subspace of B.

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Given $\mu = \mu_1 \dots \mu_n \in X^*$, define $s_\mu := s_{\mu_1} \cdots s_{\mu_n} \in M$.

There is an action $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(G, X)$ given by

$$\sigma_t(s_\mu) = e^{it|\mu|} s_\mu \qquad\qquad \sigma_t(u_g) = u_g$$

for $\mu \in X^*$ and $g \in G$.

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for $\mu \in X^*$ and $g \in G$. A state $\phi : \mathcal{T}(G, X) \to \mathbb{C}$ is a KMS$_\beta$ state iff

$$\begin{aligned}\phi((s_v u_g s_w^*)(s_y u_h s_z^*)) &= \phi((s_y u_h s_z^*)\sigma_{i\beta}(s_v u_g s_w^*)) \\ &= e^{-\beta(|v|-|w|)}\phi((s_y u_h s_z^*)(s_v u_g s_w^*)).\end{aligned}$$

States on the Cuntz-Pimsner algebra

Lemma

Let (G, X) be a self-similar action.
If ϕ is a KMS$_\beta$ state on $\mathcal{O}(G, X)$, then $\beta = \ln |X|$.

States on the Cuntz-Pimsner algebra

Lemma

Let (G, X) be a self-similar action.
If ϕ is a KMS_β state on $\mathcal{O}(G, X)$, then $\beta = \ln|X|$.

Lemma

Let (G, X) be a contracting self-similar action with nucleus $\mathcal{N}$. For each $g \in \mathcal{N} \setminus \{e\}$, let

$$F_g^n = \{\mu \in X^n \colon g \cdot \mu = \mu \text{ and } g|_\mu = e\}.$$

The sequence $\{|X|^{-n}|F_g^n|\}$ is increasing and converges to a limit c_g satisfying $0 \leq c_g < 1$ and there is a unique $KMS_{\ln|X|}$ state ϕ for $\mathcal{O}(G, X)$ satisfying

$$\phi(u_g) = c_g.$$

States on the Toeplitz algebra

Theorem

Let (G, X) be a self-similar action, $M = \bigoplus_{x \in X} C^*(G)$ and $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(G, X)$ satisfy $\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)} s_v u_g s_w^*$ for $v, w \in X^*$ and $g \in G$.

1. *For $\beta < \ln |X|$, there are no KMS_β states.*

States on the Toeplitz algebra

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1. *For $\beta < \ln |X|$, there are no KMS_β states.*
2. *For $\beta = \ln |X|$, every $KMS_{\ln |X|}$ state satisfies $\phi_{\ln |X|}(u_g u_h) = \phi_{\ln |X|}(u_h u_g)$ for all $g, h \in G$,*

$$\phi_{\ln |X|}(s_v u_g s_w^*) = \begin{cases} e^{-(\ln |X|)|v|} \phi_{\ln |X|}(u_g) & \textit{if } v = w \\ 0 & \textit{otherwise,} \end{cases}$$

and factors through $\mathcal{O}(G, X)$.

States on the Toeplitz algebra

Theorem

Let (G, X) be a self-similar action, $M = \bigoplus_{x \in X} C^*(G)$ and $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(G, X)$ satisfy $\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)} s_v u_g s_w^*$ for $v, w \in X^*$ and $g \in G$.

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   and factors through $\mathcal{O}(G, X)$.
3. *For $\beta > \ln |X|$, the simplex of KMS_β-states of $\mathcal{T}(M)$ is homeomorphic to the simplex of normalized traces on $C^*(G)$ via an explicit construction $\tau \mapsto \psi_{\beta,\tau}$.*

States on the Toeplitz algebra: ψ_{β,τ_e}

Suppose that (G, X) is a self-similar action and $\beta > \ln|X|$.
Suppose τ_e is the trace on $C*(G)$ satisfying

$$\tau_e(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in G$ and $k \geq 0$, we set

$$F_g^k := \{\mu \in X^k : g \cdot \mu = \mu \text{ and } g|_v = e\}.$$

Then there is a KMS$_\beta$ state ψ_{β,τ_e} on $(\mathbb{T}(G, X), \sigma)$ such that

$$\psi_{\beta,\tau_e}(s_v u_g s_w^*) = \begin{cases} e^{-\beta|v|}(1 - |X|e^{-\beta})\sum_{k=0}^{\infty} e^{-\beta k}|F_g^k| & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

States on the Toeplitz algebra: ψ_{β,τ_1}

Suppose that (G, X) is a self-similar action and $\beta > \ln|X|$.
Suppose $\tau_1 : C^*(G) \to \mathbb{C}$ is the integrated form of the trivial representation sending $g \mapsto 1$ for all $g \in G$.
For $g \in G$ and $k \geq 0$, we set

$$G_g^k := \{\mu \in X^k : g \cdot \mu = \mu\}.$$

Then there is a KMS$_\beta$ state ψ_{β,τ_1} on $(\mathbb{T}(G, X), \sigma)$ such that

$$\psi_{\beta,\tau_1}(s_v u_g s_w^*) = \begin{cases} e^{-\beta|v|}(1 - |X|e^{-\beta})\sum_{k=0}^{\infty} e^{-\beta k}|G_g^k| & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

Computing F_g^k and G_g^k: the Moore diagram

Suppose (G, X) is a self-similar action.

A Moore diagram is a directed graph whose vertices are elements of G and edges are labelled by pairs of elements of X.

In a Moore diagram the arrow

$$g \xrightarrow{(x,y)} h$$

means that $g \cdot x = y$ and $g|_x = h$.

We can draw a Moore diagram for any subset S of G that is closed under restriction.

The Moore diagram of the nucleus helps us calculate F_g^k and G_g^k; we look for labels of the form (x, x), called stationary paths.

Computing the nucleus

Proposition

Suppose (G, X) is a self-similar action and S is a subset of G that is closed under restriction. Every vertex in the Moore diagram of S that can be reached from a cycle belongs to the nucleus.

Proof.

Suppose $g \in G$ is a vertex in the Moore diagram of S, and there is a cycle of length $n \geq 1$ consisting of edges labelled $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ with $s(x_1, y_1) = g$, $r(x_i, y_i) = s(x_{i+1}, y_{i+1})$, and $r(x_n, y_n) = g$. By definition of the Moore diagram we have $g \cdot (x_1 \cdots x_n) = y_1 \cdots y_n$ and $g|_{x_1 \cdots x_n} = g$. Thus $g = g|_{(x_1 \cdots x_n)^m}$ for all $m \in \mathbb{N}$ and

$$g \in \bigcap_{n \geq 0} \{g|_v : v \in X^*, |v| \geq n\} \implies g \in \mathcal{N}.$$

A similar argument shows that if g can be reached from a cycle, then there are arbitrarily long paths ending at g. $\square$

Example: basilica group

Recall the basilica group

$$G = \langle a, b \colon \sigma^n([a, a^b]) \text{ for all } n \in \mathbb{N} \rangle,$$

where σ is the substitution $\sigma(b) = a$ and $\sigma(a) = b^2$, with a self-similar action (G, X) where $X = \{0, 1\}$ satisfying

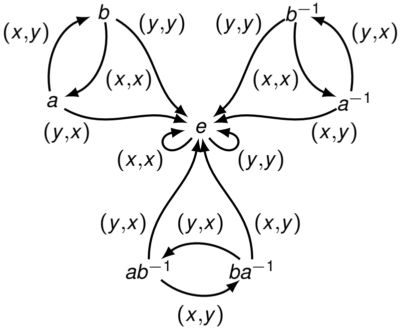
$$\begin{aligned} a \cdot (0w) &= 1(b \cdot w) & \qquad b \cdot (0w) &= 0(a \cdot w) \\ a \cdot (1w) &= 0w & \qquad b \cdot (1w) &= 1w \end{aligned}$$

Proposition

The basilica group action (G, X) is contracting, with nucleus

$$\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\};$$

the Moore diagram of $\mathcal{N}$ is



Example: basilica group

The critical value for KMS$_\beta$ states is $\beta_c = \ln|X| = \ln 2$.

Proposition

The system $(\mathcal{O}(G,X),\sigma)$ *has a unique* KMS$_{\ln 2}$ *state, which is given on the nucleus* $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ *by*

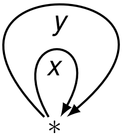
$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

The proof relies on the fact that there are no stationary paths from $g \in \{a, a^{-1}, ab^{-1}, ba^{-1}\}$ to e, so for such g we have $F_g^k = \varnothing$ for all k and $\phi(u_g) = c_g = 0$.
For $g \in \{b, b^{-1}\}$, the only stationary paths go straight from g to e, and there are 2^{k-1} of them; thus $|X|^{-k}|F_g^k| = 2^{-k}2^{k-1} = \frac{1}{2}$, and $\phi(u_g) = c_g = \frac{1}{2}$.

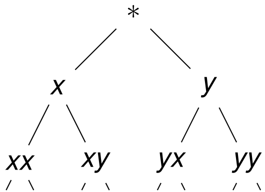
3 The bigger picture

Path space interpretation

The tree $T_{\{x,y\}}$ alongside represents the path space of the graph



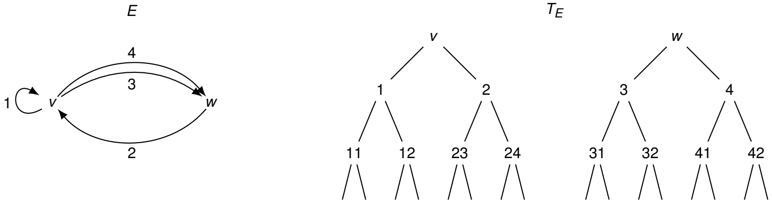
$T_{\{x,y\}}$



T_X represents the path space of a bouquet of $|X|$ loops.

More general path spaces: from trees to forests

The path space of a finite directed graph E is a forest T_E of rooted trees.

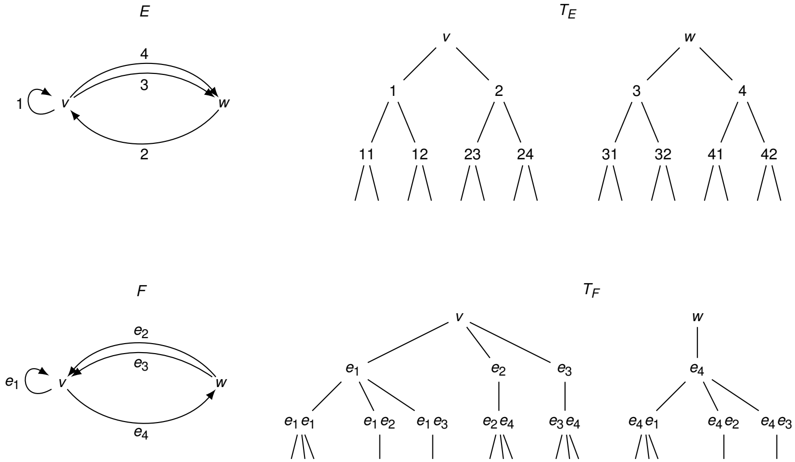


Problems arise:

- the trees in the forest are not necessarily homogeneous;
- restrictions need not be uniquely determined;
- automorphisms of T_E need not be graph automorphisms of E.

In particular, the source map may not be equivariant: $s(g \cdot e) \neq g \cdot s(e)$ in general. This distinguishes our work.

Small changes make big differences



4 Self-similar actions of groupoids

Path spaces of finite directed graphs, E

Generalise: replace X by edges E^1 in a finite directed graph E.

Suppose $E = (E^0, E^1, r, s)$ is a directed graph with vertex set E^0, edge set E^1, and range and source maps $r, s \colon E^1 \to E^0$. Write

$$E^k = \{\mu = \mu_1 \cdots \mu_k \colon \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$$

for the set of paths of length k in E, E^0 for the set of vertices, and define $E^* := \bigsqcup_{k \geq 0} E^k$.

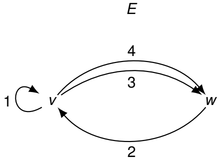
We recover the previous work by taking E to be the graph $(\{*\}, X, r, s)$ in which $r(x) = r(y) = s(x) = s(y) = *$ for all $x, y \in X = E^1$ and $E^* = X^*$.

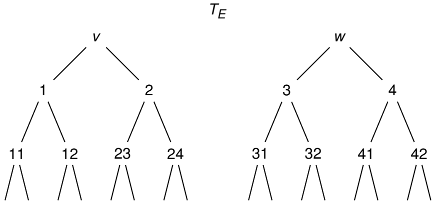
Path space T_E of finite directed graph E

The analogue of the tree T_X is the (undirected) graph T_E with vertex set $T^0 = E^*$ and edge set

$$T^1 = \big\{\{\mu, \mu e\} \colon \mu \in E^*, e \in E^1,\ \text{and } s(\mu) = r(e)\big\}.$$

The subgraph $vE^* = \{\mu \in E^* \colon r(\mu) = v\}$ is a rooted tree with root $v \in E^0$, and $T_E = \bigsqcup_{v \in E^0} vE^*$ is a disjoint union of trees, or *forest*.





Partial isomorphisms

Restrictions become problematic in this context; knowing an action on one tree in the forest doesn't constrain the action on other trees.

Suppose $E = (E^0, E^1, r, s)$ is a directed graph.

A partial isomorphism of T_E consists of two vertices $v, w \in E^0$ and a bijection $g : vE^* \to wE^*$ such that

$$g|_{vE^k} \text{ is a bijection onto } wE^k \text{ for } k \in \mathbb{N}, \text{ and}$$
$$g(\mu e) \in g(\mu)E^1 \text{ for all } \mu e \in vE^*.$$

For $v \in E^0$, we write $\mathrm{id}_v : vE^* \to vE^*$ for the partial isomorphism given by $\mathrm{id}_v(\mu) = \mu$ for all $\mu \in vE^*$.

Denote the set of all partial isomorphisms of T_E by $\mathrm{PIso}(E^*)$.

Define domain and codomain maps $d, c\colon\ \mathrm{PIso}(E^*) \to E^0$ so that $g : d(g)E^* \to c(g)E^*$.

Groupoids

Working with partial isomorphisms means working with groupoids.

A groupoid differs from a group in two main ways:

- the product in a groupoid is only partially defined, and
- a groupoid typically has more than one unit.

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A groupoid differs from a group in two main ways:

- the product in a groupoid is only partially defined, and
- a groupoid typically has more than one unit.

A groupoid is a small category with inverses.

Groupoids

So a groupoid consists of

- a set G^0 of objects (the unit space of the groupoid),
- a set G of morphisms,
- two functions $c, d : G \to G^0$, and
- a partially defined product $(g, h) \mapsto gh$ from

$$G^2 := \{(g, h) : d(g) = c(h)\} \text{ to } G$$

such that (G, G^0, c, d) is a category and such that each $g \in G$ has an inverse g^{-1}.

We write G to denote the groupoid. If $|G^0| = 1$, then G is a group.

$(\mathrm{PIso}(E^*), E^0, c, d)$ is a groupoid

Proposition

Suppose that $E = (E^0, E^1, r, s)$ *is a directed graph with associated forest* T_E.

Then $(\mathrm{PIso}(E^*), E^0, c, d)$ *is a groupoid in which:*

- *the product is given by composition of functions,*
- *the identity isomorphism at* $v \in E^0$ *is* $\mathrm{id}_v : vE^* \to vE^*$*, and*
- *the inverse of* $g \in \mathrm{PIso}(E^*)$ *is the inverse of the function* $g : d(g)E^* \to c(g)E^*$.

Groupoid action

Suppose that E is a directed graph and G is a groupoid with unit space E^0.

An action of G on the path space E^* is a (unit-preserving) groupoid homomorphism $\phi : G \to \mathrm{PIso}(E^*)$.

The action is faithful if ϕ is one-to-one.

If the homomorphism is fixed, we usually write $g \cdot \mu$ for $\phi_g(\mu)$.

This applies in particular when G arises as a subgroupoid of $\mathrm{PIso}(E^*)$.

Self-similar groupoid action (G, E)

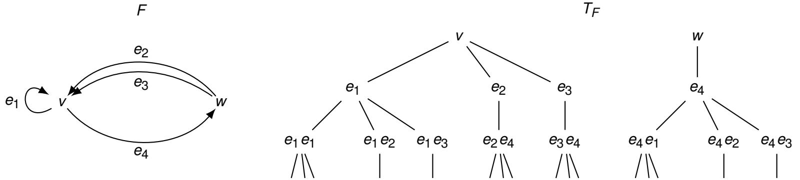
Definition

Suppose $E = (E^0, E^1, r, s)$ is a directed graph and G is a groupoid with unit space E^0 which acts faithfully on T_E.

The action is self-similar if for every $g \in G$ and $e \in d(g)E^1$, there exists $h \in G$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \text{ for all } \mu \in s(e)E^*. \tag{1}$$

Since the action is faithful, there is then exactly one such $h \in G$, and we write $g|_e := h$. Say (G, E) is a self-similar groupoid action.



Consequences of self-similar groupoid definition

Lemma

Suppose $E = (E^0, E^1, r, s)$ is a directed graph and G is a groupoid with unit space E^0 acting self-similarly on T_E.

Then for $g, h \in G$ with $d(h) = c(g)$ and $e \in d(g)E^1$, we have

- $d(g|_e) = s(e)$ *and* $c(g|_e) = s(g \cdot e)$,
- $r(g \cdot e) = g \cdot r(e) \quad$ *and* $\quad s(g \cdot e) = g|_e \cdot s(e)$,
- *if* $g = \mathrm{id}_{r(e)}$, *then* $g|_e = \mathrm{id}_{s(e)}$, *and*
- $(hg)|_e = (h|_{g \cdot e})(g|_e)$.

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- *if* $g = \mathrm{id}_{r(e)}$, *then* $g|_e = \mathrm{id}_{s(e)}$, *and*
- $(hg)|_e = (h|_{g \cdot e})(g|_e)$.

Note that in general $s(g \cdot e) \neq g \cdot s(e)$, ie the source map is not G-equivariant.
Indeed, $g \cdot s(e)$ will often not make sense: g maps $d(g)E^*$ onto $c(g)E^*$, and $s(e)$ is not in $d(g)E^*$ unless $s(e) = d(g)$.

Action on paths

We want to be able to deal with paths rather than just edges.

Proposition

Suppose $E = (E^0, E^1, r, s)$ is a directed graph and G is a groupoid with unit space E^0 acting self-similarly on T_E.

Then for all $g, h \in G$, $\mu \in d(g)E^*$, and $\nu \in s(\mu)E^*$ we have:

1. $g|_{\mu\nu} = (g|_\mu)|_\nu$,
2. $\mathrm{id}_{r(\mu)}|_\mu = \mathrm{id}_{s(\mu)}$,
3. $(hg)|_\mu = h|_{g\cdot\mu}\, g|_\mu$, *and*
4. $g^{-1}|_\mu = (g|_{g^{-1}\cdot\mu})^{-1}$.

Constructing self-similar groupoid actions

We use automata to construct self-similar groupoid actions.

An automaton over $E = (E^0, E^1, r_E, s_E)$ is

- a finite set A containing E^0, with
- functions $r_A, s_A \colon A \to E^0$ such that $r_A(v) = v = s_A(v)$ if $v \in E^0 \subset A$, and
- a function

$$\begin{array}{ccc} A \;{}_{s_A}\!\times_{r_E}\; E^1 & \to & E^1 \;{}_{s_E}\!\times_{r_A}\; A \\ (a, e) & \mapsto & (a \cdot e, a|_e) \end{array}$$

  such that:

  (A1) for every $a \in A$, $e \mapsto a \cdot e$ is a bijection $s_A(a)E^1 \to r_A(a)E^1$;
  (A2) $s_A(a|_e) = s_E(e)$ for all $(a, e) \in A \;{}_{s_A}\!\times_{r_E}\; E^1$;
  (A3) $r_E(e) \cdot e = e$ and $r_E(e)|_e = s_E(e)$ for all $e \in E^1$.

Constructing self-similar groupoid actions

Since $s_E(v) = r_E(v) = s_A(v) = r_A(v) = v$ for all $v \in E^0$, the range and source maps are consistent whenever they both make sense.

We can extend restriction to paths by defining

$$a|_\mu = (\cdots((a|_{\mu_1})|_{\mu_2})|_{\mu_3}\cdots)|_{\mu_k}.$$

The point is that $s_A(a|_{\mu_1}) = s_E(\mu_1) = r_E(\mu_2)$, for example, and hence $(a|_{\mu_1})|_{\mu_2}$ makes sense.

We use automata over E to construct subgroupoids of $\mathrm{PIso}(E^*)$.

Constructing self-similar groupoid actions

Proposition

Suppose that E is a directed graph and A is an automaton over E. We recursively define maps

$$f_{a,k} : s(a)E^k \to r(a)E^k$$

for $a \in A$ and $k \in \mathbb{N}$ by $f_{a,1}(e) = a \cdot e$ and

$$f_{a,k+1}(e\mu) = (a \cdot e) f_{a|_e,k}(\mu) \quad \text{for } e\mu \in s_A(a)E^{k+1}.$$

Then for every $a \in A$, $f_a = \{f_{a,k}\}$ is a partial isomorphism of $s(a)E^*$ onto $r(a)E^*$ so that $d(f_a) = s(a)$ and $c(f_a) = r(a)$.

For $a = v \in A \cap E^0$, we have $f_a = \mathrm{id}_v \colon vE^* \to vE^*$.

Constructing self-similar groupoid actions

Theorem

Suppose that E is a directed graph and A is an automaton over E.

For $a \in A$, let f_a be the partial isomorphism of T_E just described.

Let G_A be the subgroupoid of $\mathrm{PIso}(E^*)$ generated by $\{f_a : a \in A\}$. By convention this includes the identity morphisms $\{\mathrm{id}_v : v \in E^0\}$.

Then G_A acts faithfully on the path space E^*, and this action is self-similar.

The action of G_A is faithful because G_A is constructed as a subgroupoid of $\mathrm{PIso}(E^*)$.

It should be possible to construct unfaithful actions from some automata.

Toeplitz algebra of a self-similar groupoid action

Suppose that G is a (discrete) groupoid.

The groupoid elements $g \in G$ give point masses i_g in $C_c(G)$, and $C_c(G) = \operatorname{span}\{i_g : g \in G\}$.

For $g, h \in G$, the involution and product are determined by

$$i_g^* = i_{g^{-1}} \quad \text{and} \quad i_g * i_h = \begin{cases} i_{gh} & \text{if } d(g) = c(h) \\ 0 & \text{otherwise.} \end{cases}$$

Toeplitz algebra of a self-similar groupoid action

A function $U : G \to B(H)$ is a unitary representation of G if

- for $v \in G^0$, U_v is the orthogonal projection on a closed subspace of H,
- for each $g \in G$, U_g is a partial isometry with initial projection $U_{d(g)}$ and final projection $U_{c(g)}$, and
- for $g, h \in G$, we have

$$U_g U_h = \begin{cases} U_{gh} & \text{if } d(g) = c(h) \\ 0 & \text{otherwise.} \end{cases}$$

Note that each U_g is a unitary isomorphism $U_{d(g)}H \to U_{c(g)}H$.

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Note that each U_g is a unitary isomorphism $U_{d(g)}H \to U_{c(g)}H$.

There's a similar notion of unitary representation with values in a C^*-algebra, and then the map $i \colon g \mapsto i_g$ is a unitary representation of G in $C_c(G) \subset C^*(G)$.

The pair $(C^*(G), i)$ is universal for unitary representations of G.

Toeplitz algebra of a self-similar groupoid action

We construct a Hilbert bimodule over $C^*(G)$, M, and define the Toeplitz algebra of (G, E) to be $\mathcal{T}(G, E) := \mathcal{T}(M)$.

If E is a finite graph without sources, we show that $\mathcal{T}(G, E)$ is generated by $\{u_g : g \in G\} \cup \{p_v : v \in E^0\} \cup \{s_e : e \in E^1\}$ where

- u is a unitary representation of G with $u_v = p_v$ for $v \in E^0$;
- (p, s) is a Toeplitz-Cuntz-Krieger family in $\mathcal{T}(G, E)$, and $\sum_{v \in E^0} p_v$ is an identity for $\mathcal{T}(G, E)$;
- for $g \in G$ and $e \in E^1$,

$$u_g s_e = \begin{cases} s_{g \cdot e} u_{g|_e} & \text{if } d(g) = r(e) \\ 0 & \text{otherwise;} \end{cases}$$

- for $g \in G$ and $v \in E^0$,

$$u_g p_v = \begin{cases} p_{g \cdot v} u_g & \text{if } d(g) = v \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{T}(G, E) = \overline{\operatorname{span}}\{s_\mu u_g s_\nu^* : \mu, \nu \in E^*, g \in G \text{ and } s(\mu) = g \cdot s(\nu)\}.$$

Gauge action on $\mathcal{T}(G, E)$

Suppose that E is a finite graph with no sources and that (G, E) is a self-similar groupoid action over E.

There is a strongly continuous gauge action

$$\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{T}(G, E)$$

such that

$$\gamma_z(i_{C^*(G)}(a)) = i_{C^*(G)}(a)$$

and $\quad \gamma_z(i_M(m)) = z i_M(m)$ for $a \in C^*(G)$ and $m \in M$.

The gauge action gives rise to a periodic action σ of the real line (a dynamics) by the formula $\sigma_t = \gamma_{e^{it}}$. This dynamics satisfies

$$\sigma_t(s_\mu u_g s_\nu^*) = e^{it(|\mu|-|\nu|)} s_\mu u_g s_\nu^*. \tag{2}$$

KMS$_\beta$-states on $\mathcal{T}(G, E)$

Proposition

Let E be a finite graph with no sources and vertex matrix B, and let $\rho(B)$ be the spectral radius of B.

Suppose that (G, E) is a self-similar groupoid action.

Let $\sigma : \mathbb{R} \to \mathrm{Aut}\,\mathcal{T}(G, E), \quad \sigma_t(s_\mu u_g s_\nu^*) = e^{it(|\mu|-|\nu|)} s_\mu u_g s_\nu^*.$

- *For $\beta < \ln \rho(B)$, there are no KMS$_\beta$-states for σ.*
- *For $\beta \geq \ln \rho(B)$, a state ϕ is a KMS$_\beta$-state for σ if and only if $\phi \circ i_{C^*(G)}$ is a trace on $C^*(G)$ and*

$$\phi(s_\mu u_g s_\nu^*) = \delta_{\mu,\nu} \delta_{s(\mu),c(g)} \delta_{s(\nu),d(g)} e^{-\beta|\mu|} \phi(u_g)$$

for $g \in S$ and $\mu, \nu \in E^*$.

Questions?

Thank you for your attention.