Spaces of Convex Sets

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Abstract

Throughout this presentation, $(X, \|\cdot\|)$ will be a real normed linear space, and $\mathcal{C}(X)$ will be the set of closed, bounded, non-empty, convex subsets of X. We examine a method for embedding $\mathcal{C}(X)$ into a normed linear space $\mathcal{R}(X)$.

We also examine properties of $\mathcal{R}(X)$. In particular, we examine dimension, completeness, reflexivity, and separability of $\mathcal{R}(X)$, as well as some of its notable subspaces.

Finally, we examine the dual $\mathcal{R}(X)^*$, with particular reference to the Krein-Kakutani Theorem.

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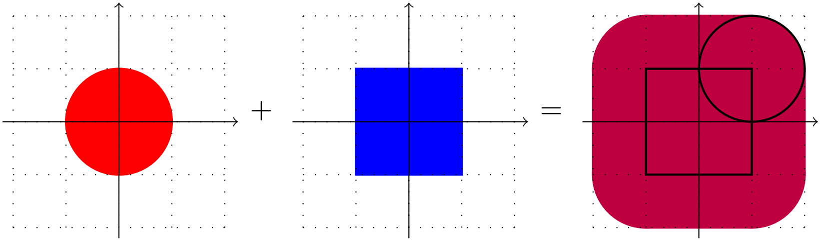

Closed Minkowski Sum

If $A, B \in \mathcal{C}(X)$, then $A + B$ is non-empty, convex, and bounded. However, if X is not reflexive, then $A + B$ need not be closed. For this reason, we define the **Closed Minkowski Sum**

$$\oplus : \mathcal{C}(X) \times \mathcal{C}(X) : (A, B) \mapsto \overline{A + B}$$

$\mathcal{C}(X)$ is closed under $\oplus$. It is also associative, commutative, and has the identity $\{0\}$.

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Order Cancellation Law

The Closed Minkowski Sum satisfies the following order cancellation law:

Proposition

For any $A, B, C \in \mathcal{C}(X)$,

$$A \oplus C \subseteq B \oplus C \implies A \subseteq B.$$

Hans Rådström observed that this cancellation law allowed us to embed $\mathcal{C}(X)$ into a normed linear space. The additive group of this space is the Grothendieck group of the monoid $\mathcal{C}(X)$.

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Using the cancellation law, we may define an equivalence relation on $\mathcal{C}(X) \times \mathcal{C}(X)$

$$\sim = \{((A, B), (C, D)) : A \oplus D = B \oplus C\}.$$

We define $\mathcal{R}(X) = (\mathcal{C}(X) \times \mathcal{C}(X))/\sim$, and denote by $A \ominus B$ the equivalence class of (A, B) in $\mathcal{R}(X)$.

We can embed $\mathcal{C}(X)$ into $\mathcal{R}(X)$ by identifying $C \mapsto C \ominus \{0\}$. We can then extend $\oplus$ to $\mathcal{R}(X)$ in the expected way, and $\mathcal{R}(X)$ forms an abelian group under $\oplus$, with $\ominus$ its inverse operation.

We also define a scalar multiplication operation:

$$\lambda(C \ominus D) = \begin{cases} |\lambda|C \ominus |\lambda|D & : \quad \lambda \geq 0 \\ |\lambda|D \ominus |\lambda|C & : \quad \lambda < 0 \end{cases}$$

This makes $\mathcal{R}(X)$ a real linear space.

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Further structure of $\mathcal{R}(X)$

We also can extend the subset partial order on $\mathcal{C}(X)$ to $\mathcal{R}(X)$ by

$$A \ominus B \leq C \ominus D \iff A \oplus D \subseteq B \oplus C.$$

This partial order makes $\mathcal{R}(X)$ a linear lattice. $\mathcal{R}(X)$ is also a Kakutani space:

Definition

A linear lattice V is a Kakutani space if

- V is *Archimedean*: $\forall v, w \in V$, if $nv \leq w \ \forall n \in \mathbb{N}$, then $v \leq 0$.
- $\exists e \in V, \forall v \in V, \exists n \in \mathbb{N}, -ne \leq v \leq ne$

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Kakutani spaces induce a canonical norm:

$$\|v\| = \inf\{\lambda \geq 0 : -\lambda e \leq v \leq \lambda e\}.$$

The canonical norm on a Kakutani space is a lattice norm, and furthermore, the associated operator norm on the dual is also a lattice norm.

In the case of $\mathcal{R}(X)$, this norm is derived from the Hausdorff distance:

$$\|A \ominus B\| = \mathcal{H}(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

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An example: $\mathcal{R}(\mathbb{R})$

We have

- $\mathcal{C}(\mathbb{R}) = \{[a,b] : a \leq b\}$
- $[a,b] \ominus [c,d] = \begin{cases} [a-c, b-d] & : & a-c \leq b-d \\ \ominus[c-a, d-b] & : & a-c > b-d \end{cases}$
- $\mathcal{R}(\mathbb{R}) = \mathcal{C}(\mathbb{R}) \cup \ominus\mathcal{C}(\mathbb{R})$
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Theorem

$\mathcal{R}(X)$ is separable if and only if X is finite-dimensional.
If $\dim(X) > 1$, then

- *$\mathcal{R}(X)$ is incomplete, and hence not reflexive or finite-dimensional.*
- *The completion $\overline{\mathcal{R}(X)}$, or equivalently the dual $\mathcal{R}(X)^*$, is not reflexive.*

Induced Transformations and Subspaces

For a bounded linear transformation $T : X \to Y$, we may induce a bounded linear transformation from $\mathcal{R}(X)$ to $\mathcal{R}(Y)$:

$$\rho_T : \mathcal{R}(X) \to \mathcal{R}(Y) : A \ominus B \mapsto \overline{T(A)} \ominus \overline{T(B)}.$$

If we consider the inclusion map T of a subspace Y into X, the image of the induced transformation ρ_T is an isometric embedding of $\mathcal{R}(Y)$ into $\mathcal{R}(X)$. This gives us a subspace structure of $\mathcal{R}(X)$ inherited from the subspace structure of X.

Proposition

Regardless of whether Y is closed in X, $\rho_T(\mathcal{R}(Y))$ is closed in $\mathcal{R}(X)$. Moreover, if Y is dense in X, then ρ_T is surjective, proving $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ are isometrically isomorphic.

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Induced Functionals

Equipped with our knowledge of $\mathcal{R}(\mathbb{R})$, we can induce functionals in $\mathcal{R}(X)^*$ from functionals in X^*. We may consider ρ_f for $f \in X^*$ as a bounded linear transformation of $\mathcal{R}(X)$ into $\mathbb{R}^2$. We can then compose ρ_f with functionals on $\mathbb{R}^2$ to obtain a two dimensional subspace of $\mathcal{R}(X)^*$ spanned by the following functionals:

$$\sigma_f(A \ominus B) = \sup f(A) - \sup f(B)$$
$$\iota_f(A \ominus B) = \inf f(A) - \inf f(B).$$

We say that a functional in $\operatorname{span}\{\sigma_f, \iota_f\}$ is **induced** by f. Note in particular that $-\iota_f = \sigma_{-f}$.

Structure of $\mathcal{R}(X)^*$

Theorem

Suppose $\phi \in \mathcal{R}(X)^*$*. Then there exist unique scalars* $(c_f)_{f \in S_{X^*}} \in \mathbb{R}$*, all but countably many of which are* 0*, and* $\psi \in \mathcal{R}(X)^*$ *that is lattice-orthogonal to* σ_f *for all* $f \in X^*$ *such that*

$$\phi = \psi + \sum_{f \in S_{X^*}} c_f \sigma_f.$$

Moreover, ϕ *is monotone if and only if* $c_f \geq 0$ *for all* $f \in S_{X^*}$ *and* ψ *is monotone.*

The Krein-Kakutani Theorem

Theorem (Krein, Kakutani)

If V is a Kakutani space, then for some compact Hausdorff topological space K, V embeds densely, isometrically, and monotonically into $C(K)$, in such a way that e maps to the constant function 1.

[Coppel, pp. 196–209] provides a self-contained proof of this form of the theorem.

In particular, the space $K = \mathrm{ext}(V_+^* \cap S_{V^*})$, under the weak* topology. That is, K is the extreme points of the set of norm 1, monotone functionals in V^*. This is one good reason to study $\mathcal{R}(X)^*$.

Applying The Krein-Kakutani Theorem

Theorem

Suppose $K = \mathrm{ext}(\mathcal{R}(X)^*_+ \cap S_{\mathcal{R}(X)^*})$, as per the Krein-Kakutani Theorem. Then $\{\sigma_f : f \in S_{X^*}\} \subseteq K$, with equality holding if and only if X is finite-dimensional.

If X is finite-dimensional, there is therefore a natural bijection between K and S_{X^*}. This bijection turns out to be a homeomorphism, with S_{X^*} under the standard Euclidean topology. Therefore,

Corollary

If X is finite-dimensional, then $\mathcal{R}(X)$ embeds densely, isometrically, and monotonically into $C(S_{X^*})$.

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References

- Hans Rådström, 1952: An Embedding Theorem for Spaces of Convex Sets. *Proceedings of the American Mathematical Society*, **3**, 165–169.
- W. A. Coppel, 1998: *Foundations of Convex Geometry*. Cambridge University Press, 222 pp.
- Klaus D. Schmidt, 1986: Embedding Theorems for Classes of Convex Sets. *Acta Applicandae Mathematicae*, **5**, 209–237.
- J. Grzybowki, and H. Przybycień, 2013: Completeness in Minkowski-Rådstroöm-Hörmander spaces. *Optimization: A Journal of Mathematical Programming and Operations Research*, doi:10,1080/02331934.2013.793330
- Diethard Pallaschke, and Ryszard Urbaǹski, 2003: *Pairs of Compact Convex Sets*. Kluwer Academic Publishers, 282 + xii pp.