A new subfamily of enlargements of a maximally monotone operator

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Fitzpatrick Workshop
SPCOM 2015, 10 February

Outline

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Monotone Inclusion Problem

Let $T : X \rightrightarrows X^*$ be maximal monotone. Many nonlinear problems are stated as:

$$\text{Given } z \in X^*, \text{ find } x \in X \; : \boxed{z \in T(x)} \quad (P_0)$$

Equivalently:

$$\text{Given } z \in X^*, \text{ find } x \in X \; : \boxed{(x, z) \in G(T)}$$

solving (P_0) $\quad\leftrightsquigarrow\quad$ requires to know $G(T)$

Main Ingredients I: multivalued mappings

For $T : X \rightrightarrows X^*$ we define

- its graph as $G(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$,
- its domain as $D(T) := \{x \in X : T(x) \neq \emptyset\}$,
- its range as $R(T) := \bigcup\{T(x) : x \in D(T)\}$,

We say that T is

- *monotone* if

$$\langle y - x, y^* - x^* \rangle \geq 0 \qquad \forall (x, x^*), (y, y^*) \in G(T).$$

- *maximally monotone* if T has no monotone extension in the sense of graph inclusion.

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Main Ingredients II: subdifferentials

For $\varphi : X \to \mathbb{R}_\infty$ convex and lsc, we define

- $\mathrm{Dom}\varphi := \{x \ : \ \varphi(x) < \infty\}$, and
- we say that φ is proper when $\mathrm{Dom}\varphi \neq \emptyset$.
- the subdifferential of φ is the multivalued mapping $\partial\varphi : X \rightrightarrows X^*$ defined by

$$\partial\varphi(x) := \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x\rangle, \forall\, y \in X\},$$

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Fenchel Young inequality

Let $\varphi : X \to \mathbb{R}_\infty$ be convex and lsc, $\varphi^* : X^* \to \mathbb{R}_\infty$

$$\varphi^*(v) := \sup_{x \in X}\{\langle x, v\rangle - \varphi(x)\}$$

is the *conjugate of* φ. The *Fenchel Young inequality* states

$$\begin{aligned} \varphi(x) + \varphi^*(v) &\geq \langle x, v\rangle, \ \forall\ x \in X, v \in X^* \\ \varphi(x) + \varphi^*(v) &= \langle x, v\rangle, \iff v \in \partial\varphi(x). \end{aligned}$$

Notation: $\varphi^{FY}(x, v) := \varphi(x) + \varphi^*(v)$

Fitzpatrick Theory: the family $\mathcal{H}(T)$

In 1988 Fitzpatrick defined the family $\mathcal{H}(T)$ consisting of all $h : X \times X^* \to \mathbb{R}_\infty$ convex and lsc such that:

$$\begin{aligned} h(x,v) &\geq \langle x,v\rangle, \ \forall\, x \in X, v \in X^* \\ h(x,v) &= \langle x,v\rangle, \iff v \in T(x). \end{aligned}$$

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A key member of $\mathcal{H}(T)$

Fitzpatrick defined $\mathcal{F}_T : X \times X^* \to \mathbb{R}_\infty$ as

$$\mathcal{F}_T(x, x^*) := \sup_{(y,y) \in G(T)} \langle y, x^* \rangle + \langle x - y, y^* \rangle$$

which verifies

- $\mathcal{F}_T \in \mathcal{H}(T)$
- $\mathcal{F}_T \leq h \leq (\mathcal{F}_T)^* =: \sigma_T$ for all $h \in \mathcal{H}(T)$

Historical note: N.V.Krylov defined in 1980 $\mathcal{F}_T$ for T point-to-point monotone in finite dimensions.

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  if $x \in \mathrm{Dom}\varphi$. Otherwise, $\partial_\varepsilon\varphi(x) = \emptyset$.

$\breve{\partial}\varphi(\varepsilon, x) := \partial_\varepsilon\varphi(x)$ Brøndsted-Rockafellar enlargement (1965)

$\breve{\partial}\varphi$ characterized by *Fenchel Young ineq.*:

$$\langle x, v\rangle \leq \varphi^{FY}(x, v) = \varphi(x) + \varphi^*(v) \quad \leq \quad \langle x, v\rangle + \varepsilon \iff v \in \breve{\partial}\varphi(\varepsilon, x).$$

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The family $\mathbb{E}(T)$ of enlargements of T

$E : \mathbb{R}_+ \times X \rightrightarrows X^*$ is in $\mathbb{E}(T)$ when

(E_1) $T(x) \subset E(\varepsilon, x)$ for all $\varepsilon \geq 0, x \in X$;

(E_2) If $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $E(\varepsilon_1, x) \subset E(\varepsilon_2, x)$ for all $x \in X$;

(E_3) The transportation formula holds: Whenever $v^1 \in E(\varepsilon_1, x^1)$, $v^2 \in E(\varepsilon_2, x^2)$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, $\bar{x} := \alpha_1 x^1 + \alpha_2 x^2$, $\bar{v} := \alpha_1 v^1 + \alpha_2 v^2$ and $\bar{\varepsilon} := \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2\langle v^1 - v^2, x^1 - x^2\rangle$, then

$\bar{\varepsilon} \geq 0$ and $\bar{v} \in E(\bar{\varepsilon}, \bar{x})$.

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From enlargements to convex functions:

$$(E_3) \Longleftrightarrow \widetilde{G}(E) \text{ convex},$$

where

$$\begin{aligned} G(E) &:= \{(x, v, \varepsilon) \ : \ v \in E(\varepsilon, x)\} \\ & \qquad\qquad \searrow \\ \widetilde{G}(E) &:= \{(x, v, \varepsilon + \langle x, v \rangle) \ : \ v \in E(\varepsilon, x)\} \end{aligned}$$

From $\mathbb{E}(T)$ to $\mathcal{H}(T)$

$E \in Enl(T) \iff \widetilde{G}(E)$ is the $\left\{ \begin{array}{l} \text{epigraph of a lsc.} \\ \text{convex function} \\ \text{on } X \times X^*. \end{array} \right.$

This convex function is given by

$$h_E(x, v) := \inf\{ t \,:\, (x, v, t) \in \widetilde{G}(E)\}$$

Moreover, $h_E \in \mathcal{H}(T)$ for all $E \in \mathbb{E}(T)$!

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Given $h \in \mathcal{H}(T)$ define $L^h : \mathbb{R}_+ \times X \rightrightarrows X^*$ as

$$L^h(\varepsilon, x) := \{v \in X^* \;:\; h(x, v) \leq \langle x, v\rangle + \varepsilon\}$$

Then $L^h \in \mathbb{E}(T)$ for all $h \in \mathcal{H}(T)$!

$$\mathcal{H}(T) \underset{\textit{bijection}}{\longleftrightarrow} \mathbb{E}(T)$$

B.-Svaiter, 2002.

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Case $T = \partial\varphi$

Recall $\varphi^{FY}(x, v) = \varphi(x) + \varphi^*(v)$, then $\varphi^{FY} \in \mathcal{H}(\partial\varphi)$

Extreme members in the families

$\mathcal{H}(T)$ has a smallest and a largest element

$\mathcal{F}_T \leq h \leq \sigma_T = (\mathcal{F}_T)^*$, $\mathbb{E}(T)$ has largest element:

$$T^{BE}(\varepsilon, x) := \{v \in X^* \ : \ \langle x - y, v - u\rangle \geq -\varepsilon, \forall (y, u) \in G(T)\},$$

and smallest $T^{SE}(\varepsilon, x) = \cap_{E \in \mathbb{E}(T)} E(\varepsilon, x)$,

Related through $L^{\mathcal{F}_T} = T^{BE}$, and $L^{\sigma_T} = T^{SE}$

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Question: Can we identify a property that singles out "nice" enlargements?

Extreme members in the families

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Set $\mathbb{E}_a(T) := \{E \in \mathbb{E}(T)\ :\ E \text{ additive}\}$

$\breve{\partial}\varphi$ is additive, i.e., $\breve{\partial}\varphi \in \mathbb{E}_a(\partial\varphi)$

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Example of Max-Additivity

If $T = \partial\varphi$ then $\breve{\partial}\varphi$ is max-add (Svaiter, 2000)

If T arbitrary, then T^{SE} is always additive, but not necesarily max-add!

Max-additivity detects those elements in $\mathbb{E}_a(T)$ which have even more in common with $\breve{\partial}\varphi$!

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Example of mutual additivity

If T arbitrary, then T^{SE} and T^{BE} are always mutually additive (Svaiter, 2000)

Questions: How to identify additive elements $\mathbb{E}(T)$? How to identify max-add elements whithin $\mathbb{E}_a(T)$? How to characterize mutual additivity?

We will address these using convex functions!

From convex functions to T and viceversa

Let $f : X \times X^* \to \mathbb{R}_\infty$ be convex, Fitzpatrick (1988) defined $T_f : X \rightrightarrows X^*$ as

$$T_f(x) := \{v \in X^* \ : \ (v, x) \in \partial f(x, v)\} \quad \star$$

Fitzpatrick proved that T_f mon, and for T monotone and $f := \mathcal{F}_T$:

- $\forall x \in X, \ T(x) \subseteq T_{\mathcal{F}_T}(x)$.
- T maximal $\Longrightarrow \ T = T_{\mathcal{F}_T}$

Can recover T as a diagonal slice of the $\partial\mathcal{F}_T$!

Question: What happens if we use $\breve{\partial} f$ in $\star$?
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Let $h \in \mathcal{H}(T)$, define $\mathcal{J} : \mathcal{H}(T) \to \mathcal{H}(T)$ as

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I.e., $\mathcal{J}h$ swaps the variables of h^* Define $\mathcal{A} : \mathcal{H}(T) \to \mathcal{H}(T)$ as

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Fact: $\mathcal{A}h \in \mathcal{H}(T)$ if $h \in \mathcal{H}(T)$.

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An induced subfamily of enlargements

Let T be max-mon and fix $h \in \mathcal{H}(T)$. We define $\breve{T}_h : \mathbb{R}_+ \times X \rightrightarrows X^*$ as

$$\breve{T}_h(\varepsilon, x) := \{v \in X^* \ : \ (v, x) \in \breve{\partial} h(2\varepsilon, x, v)\} \quad \star$$

- $T_h(x) = \breve{T}_h(0, x) = T$
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Characterizing Mutual and Maximal Additivity

Let $E, E' \in \mathbb{E}(T)$, consider $h_E, h_{E'} \in \mathcal{H}(T)$ the corresponding functions (i.e., $E = L^{h_E}$ and $E' = L^{h_{E'}}$)

- $E \sim_a E'$ iff $\mathcal{J} h_E \leq h_{E'}$. Hence, $E \in \mathbb{E}_a(T)$ iff $\mathcal{J} h_E \leq h_E$.
- $h_E = \mathcal{J} h_E$ iff E is max-add
- In particular, $E \sim_a L^{\mathcal{J} h_E}$
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Conmutative diagram

Taking conjugates in $\mathcal{H}(T)$ is order reversing, and its effect in $\mathbb{E}(T)$ is to map E into its additive complement.

$$\begin{array}{ccc}
 & \mathcal{J} & \\
\mathcal{H}(T) & \longrightarrow & \mathcal{H}(T) \\
\Big\downarrow \begin{array}{c} h \\ \downarrow \\ L^h \end{array} & & \Big\downarrow \begin{array}{c} \mathcal{J}h \\ \downarrow \\ L^{\mathcal{J}h} \end{array} \\
 & E \ \rightarrow \ L^{\mathcal{J}h_E} & \\
\mathbb{E}(T) & \longrightarrow & \mathbb{E}(T)
\end{array}$$

Fixed points of $\mathcal{J}$ correspond to max-add elements!

Relation w/previous facts

Recall $\varphi^{FY}(x, v) = \varphi(x) + \varphi^*(v)$, since $\mathcal{J}\varphi^{FY} = \varphi^{FY}$ we confirm the fact that

$$\breve{\partial}\varphi \text{ is max-add}$$

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Fix $h \in \mathcal{H}(\partial\varphi)$ and $h \leq \varphi + \varphi^* = \varphi^{FY}$

- $\forall \varepsilon > 0$, $x \in \mathrm{Dom}\varphi$ we have

$$\breve{T}_h(\varepsilon/2, x) \subseteq \breve{\partial\varphi}(\varepsilon, x)$$

- If $h = \varphi^{FY}$ we must have $\breve{\partial\varphi} = \breve{T}_{\varphi^{FY}}$.
- If $h = \mathcal{F}_{\partial\varphi}$

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Open problems

- Does the subfamily $\mathbb{E}_{\mathcal{H}}(T)$ contain all additive enlargements of T?
- We have also seen that elements of $\mathbb{E}_{\mathcal{H}}(T)$ are max-add when $h = \mathcal{J}h$. Are these all the max-add enlargements of T?
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