SN spaces, r_L–density and maximal monotonicity

by

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Abstract

r_L–density is a concept that can be applied to subsets of $E \times E^*$, where E is a nonzero real Banach space. We start our discussion of it in the more general situation of subsets of SN spaces, where the notation is more concise. In the $E \times E^*$ case, every closed r_L–dense monotone set is maximally monotone, but there exist maximally monotone sets that are not r_L–dense. The graph of the subdifferential of a proper, convex lower semicontinuous function on E is r_L–dense. The graphs of certain subdifferentials of certain nonconvex functions are also r_L–dense. (This follows from joint work with Xianfu Wang.) The closed monotone and r_L–dense sets have a number of very desirable properties, including a sum theorem under both natural and unnatural constraint conditions, so r_L–density satisfies the ideal calculus rules. We also give a generalization of the Brezis–Browder theorem on linear relations.

Downloads

You can download files containing complete proofs and many references from the web. I will give you the link at the end of the talk.

SN spaces

Symmetric linear maps, the associated quadratic form q_L, and SN spaces.

L–positive sets.

The function r_L, r_L–density and maximality.

The function s_L, a criterion for the r_L–density of certain sets.

Polar subspaces.

The function Φ_A and the Fitzpatrick extension.

The $E \times E^*$ case

The tail.

Subdifferentials of convex and non convex functions.

A negative alignment criterion for r_L–density.

The convexity of $\overline{D(S)}$ and $\overline{R(S)}$.

Type (ANA).

Sum theorems and subdifferential perturbation theorems.

Strong maximality, type (FP) and type (NI).

Generalizations of the Brezis–Browder theorem.

Symmetric linear maps

Let B be a nonzero real Banach space. A linear map $L\colon B \to B^*$ is *symmetric* if, $\forall\ b, c \in B$, $\langle b, Lc\rangle = \big\langle c, Lb\big\rangle$. The quadratic form q_L on B is defined by

$$q_L(b) := \tfrac{1}{2}\big\langle b, Lb\big\rangle.$$

- We have the parallelogram law:

$$b, c \in B \quad \Longrightarrow \quad \tfrac{1}{2}q_L(b-c) + \tfrac{1}{2}q_L(b+c) = q_L(b) + q_L(c).$$

Definition of SN space

B $\big($more precisely, $(B, L)\big)$ is a *symmetric nonexpansion space (SN space)* if B is a nonzero real Banach space and $L\colon B \to B^*$ is a symmetric nonexpansive linear map from B into B^*.

Examples of SN spaces

(a) If B is a Hilbert space then B is an SN space with $Lc := c$. Then $q_L(b) = \frac{1}{2}\|b\|^2$.

(b) If B is a Hilbert space then B is an SN space with $Lc := -c$. Then $q_L(b) = -\frac{1}{2}\|b\|^2$.

(c) $\mathbb{R}^3$ is an SN space with $L(c_1, c_2, c_3) := (c_2, c_1, c_3)$. Then $q_L(b_1, b_2, b_3) = b_1 b_2 + \frac{1}{2}b_3^2$.

(d) $\mathbb{R}^3$ is **not** an SN space with $L(c_1, c_2, c_3) := (c_2, c_3, c_1)$ since

$$\big\langle(0,1,0), L(1,0,0)\big\rangle = 0 \qquad \text{but} \qquad \big\langle(1,0,0), L(0,1,0)\big\rangle = 1.$$

Definition of SN space

B (more precisely, (B, L)) is a *symmetric nonexpansion space (SN space)* if B is a nonzero real Banach space and $L\colon B \to B^*$ is a symmetric nonexpansive linear map from B into B^*.

An example of an SN space motivated by monotonicity

(e) Let E be a nonzero Banach space and $B := E \times E^*$ under the norm

$$\big\|(x, x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}.$$

Let $\big(E \times E^*, \|\cdot\|\big)^* = (E^* \times E^{**}, \|\cdot\|)$, with $\big\|(y^*, y^{**})\big\| := \sqrt{\|y^*\|^2 + \|y^{**}\|^2}$ and $\big\langle(x, x^*), (y^*, y^{**})\big\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle$. $\forall\ (y, y^*) \in B$, let

$$L(y, y^*) := (y^*, \widehat{y}),$$

where $\widehat{y}$ is the canonical image of y in E^{**}. Since

$$\big\langle(x, x^*), L(y, y^*)\big\rangle = \langle x, y^*\rangle + \langle x^*, \widehat{y}\rangle = \langle y, x^*\rangle + \langle y^*, \widehat{x}\rangle = \big\langle(y, y^*), L(x, x^*)\big\rangle,$$

B is an SN space, and

$$q_L(x, x^*) = \langle x, x^*\rangle.$$

Any finite dimensional SN space of this form must have even dimension. Thus odd dimensional cases of the examples considered on the previous slide cannot be of this form. In contrast to the three examples on the previous slide, if E is not reflexive then L is not surjective.

- E will always be a nonzero Banach space and so, with L as defined above, $(E \times E^*, L)$ is an SN space. B will be an SN space.

Definition of L–positive set

Let $A \subset B$. We say that A is *L–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q_L(b - c) \geq 0.$$

Examples of L–positive sets

(a) B is a Hilbert space with $Lc := c$: every nonempty subset of B is L–positive.

(b) B is a Hilbert space with $Lc := -c$: the L–positive subsets of B are the singletons.

(e) E is a nonzero Banach space, $B := E \times E^*$ and, $L(x, x^*) := (x^*, \widehat{x})$. Let $\emptyset \neq A \subset B$. Then A is L–positive when

$$(x, x^*), (y, y^*) \in A \quad \Longrightarrow \quad \langle x - y, x^* - y^* \rangle \geq 0.$$

That is to say,

$$A \text{ is } L\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

General notation

- Let X be a vector space and $f\colon X \mapsto]-\infty, \infty]$. Then $\operatorname{dom} f := \{x \in X\colon f(x) \in \mathbb{R}\}$.
- f is *proper* if $\operatorname{dom} f \neq \emptyset$.
- $\mathcal{PC}(X)$ is the set of all proper convex functions $f\colon X \mapsto]-\infty, \infty]$.
- If X is a Banach space, $\mathcal{PCLSC}(X) := \{f \in \mathcal{PC}(X)\colon f \text{ is lower semicontinuous}\}$.
- If $f, g\colon X \to [-\infty, \infty]$, then $\{X|f = g\}$ is the "equality set" $\{x \in X|f(x) = g(x)\}$.

SN space notation

- If (B, L) is a Banach SN space, $\mathcal{PC}_q(B) := \{f \in \mathcal{PC}(B)\colon\ f \geq q_L \text{ on } B\}$.
- If (B, L) is a Banach SN space, $\mathcal{PCLSC}_q(B) := \{f \in \mathcal{PCLSC}(B)\colon\ f \geq q_L \text{ on } B\}$.

The L–positive set given by a convex function

If $f \in \mathcal{PC}_q(B)$ and $\{B|f = q_L\} \neq \emptyset$ then $\{B|f = q_L\}$ is an L–positive subset of B.

Proof. Let $b, c \in B$, $f(b) = q_L(b)$ and $f(c) = q_L(c)$. Then, from the parallelogram law, the quadraticity of q_L, and the convexity of f,

$$\begin{aligned}\tfrac{1}{2}q_L(b-c) = q_L(b) + q_L(c) - \tfrac{1}{2}q_L(b+c) &= q_L(b) + q_L(c) - 2q_L\big(\tfrac{1}{2}(b+c)\big)\\ &\geq f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \geq 0.\end{aligned}$$

□

Definition of the function r_L

Let $b \in B$. Then $r_L(b) := \frac{1}{2}\|b\|^2 + q_L(b)$.

- Since $\|L\| \le 1$, $\quad b \in B \Longrightarrow \frac{1}{2}\|b\|^2 \ge -q_L(b)$. Consequently

$$b \in B \quad \Longrightarrow \quad r_L(b) \ge 0.$$

Examples of the function r_L

(a) B is a Hilbert space with $Lb := b$. Then

$$r_L(b) = \|b\|^2.$$

(b) B is a Hilbert space with $Lb := -b$. Then

$$r_L(b) = 0.$$

(e) E is a nonzero Banach space, $\quad B := E \times E^*\quad$ and, $\quad L(x, x^*) := (x^*, \widehat{x})$. Then

$$r_L(x, x^*) = \tfrac{1}{2}\|x\|^2 + \langle x, x^*\rangle + \tfrac{1}{2}\|x^*\|^2.$$

Definition of r_L–density

Let $A \subset B$. Then A is r_L–dense if

$$\forall\ b \in B, \quad \inf r_L(A - b) = 0.$$

- This means: $\forall\ b \in B$ and $\varepsilon > 0\ \exists\ a \in A$ such that $r_L(a - b) < \varepsilon$.

Definition of r_L–density

Let $A \subset B$. Then A is r_L–dense if

$$\forall\ b \in B, \quad \inf r_L(A - b) = 0.$$

- This means: $\forall\ b \in B$ and $\varepsilon > 0\ \exists\ a \in A$ such that $r_L(a - b) < \varepsilon$.

Theorem: r_L–density implies maximality

Let A be a closed r_L–dense L–positive subset of B. Then A is maximally L–positive (in the obvious sense).

Proof. Suppose that $b \in B$ and $A \cup \{b\}$ is L–positive. Let $\varepsilon > 0$. By hypothesis,

$$\exists\ a \in A \text{ such that } r_L(a - b) < \varepsilon.$$

Thus

$$\tfrac{1}{2}\|a - b\|^2 + q_L(a - b) < \varepsilon.$$

Since $A \cup \{b\}$ is L–positive, $q_L(a - b) \geq 0$, and so

$$\tfrac{1}{2}\|a - b\|^2 \leq \varepsilon.$$

However, A is closed. Thus, letting $\varepsilon \to 0$, we have

$$b \in A. \qquad \square$$

- As we will see later, the converse of the above result is false. There are maximally L–positive subsets of $E \times E^*$ that are not r_L–dense.

Definition of the function s_L

Let $b^* \in B^*$. We define $s_L(b^*) \in [\,-\infty, \infty\,]$ by

$$s_L(b^*) := \sup_{b \in B} \left[\langle b, b^*\rangle - q_L(b) - \tfrac{1}{2}\|Lb - b^*\|^2\right].$$

The reason for this strange definition will appear on the next slide.

Examples of the function s_L

(a) B is a Hilbert space with $Lb := b$. Then

$$s_L(b^*) = \tfrac{1}{2}\|b^*\|^2.$$

(e) E is a nonzero Banach space, $B := E \times E^*$ and, $L(x, x^*) := (x^*, \widehat{x})$. Then

$$s_L(x^*, x^{**}) = \sup_{(y,y^*) \in E \times E^*} \left[\langle y, x^*\rangle + \langle y^*, x^{**}\rangle - \langle y, y^*\rangle - \tfrac{1}{2}\|y^* - x^*\|^2 - \tfrac{1}{2}\|\widehat{y} - x^{**}\|^2\right].$$

Mercifully, this simplifies to the formula

$$\forall\ (x^*, x^{**}) \in E^* \times E^{**}, \quad s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle. \qquad (\star)$$

Criterion for $\{B|f = q_L\}$ to be r_L–dense in B

Let $f \in \mathcal{PCLSC}_q(B)$. Then:

$$\{B|f = q_L\} \text{ is } r_L\text{–dense in } B \iff f^* \geq s_L \text{ on } B^*.$$

Proof. One can prove that both conditions above are equivalent to

$$\forall\ c \in B,\ \inf_{b\in B}\big[(f - q_L)(b) + r_L(b - c)\big] \leq 0.$$

The proof of the equivalence with the left hand condition uses a completeness argument. The proof of the equivalence with the right hand condition uses Rockafellar's version of the Fenchel duality theorem on the two functions f and g_c, where

$$g_c(b) := -q_L(b) + r_L(b - c) = -\langle b, Lc\rangle + q_L(c) + \tfrac{1}{2}\|b - c\|^2,$$

which is continuous and convex. The definition of s_L was obtained by working backwards from this proof. For more details, see the material on the web. □

Polar subspace

- If Y is a linear subspace of a Banach space X, Y^0 is the "polar subspace of Y", that is to say the linear subspace $\big\{x^* \in X^*\colon\ \langle Y, x^*\rangle = \{0\}\big\}$ of X^*.

Theorem on the r_L-density of linear subspaces

Let (B, L) be an SN space and A be a closed linear L–positive subspace of B. Then
$$A \text{ is } r_L\text{-dense in } B \iff \sup s_L(A^0) \le 0.$$

Comment. We will see later that the above result leads to a generalization of the Brezis–Browder theorem on the monotonicity of the adjoint relation.

• We have shown how $f \in \mathcal{PC}_q(B)$ leads to the L–positive set, $\{B|f = q_L\}$.
• We now consider the converse problem: given an L–positive set, A, we show how to obtain a convex function, Φ_A, on B.

A convex function given by an L–positive set

Let A be an L–positive subset of B. We define $\Phi_A\colon\ B \mapsto \,]-\infty, \infty]$ by

$$\Phi_A(b) := \sup_A \big[Lb - q_L\big] = \sup_{a \in A} \big[\langle a, bLb\rangle - q_L(a)\big].$$

Nice property of Φ_A

Let A be a maximally L–positive subset of B. Then

$$\Phi_A \in \mathcal{PCLSC}_q(B) \quad \text{and} \quad \big\{B|\Phi_A = q_L\big\} = A.$$

Criterion for $\big\{B|f = q_L\big\}$ to be r_L–dense in B

Let $f \in \mathcal{PCLSC}_q(B)$. Then:

$$\big\{B|f = q_L\big\} \text{ is } r_L\text{–dense in } B \iff f^* \geq s_L \text{ on } B^*.$$

Theorem on the r_L–density of an L–positive set

A be a closed L–positive subset of B. Then A is r_L–dense in B if, and only if, A is maximally L–positive and $\Phi_A{}^* \geq s_L$ *on* B^*.

Proof. Immediate from the two results above with $f := \Phi_A$. □

Theorem on the r_L–density of an L–positive set

A be a closed L–positive subset of B. Then A is r_L–dense in B if, and only if, A is maximally L–positive and $\Phi_A{}^* \ge s_L$ on B^*.

Definition of the Fitzpatrick extension

Let A be a closed, r_L–dense L–positive subset of B and $s_L \circ L = q_L$. The Fitzpatrick extension of A is the set $A^{\mathbb{F}} := \{B^* | \Phi_A{}^* = s_L\}$.

- The reason that we use the word "extension" is that frequently $L^{-1}A^{\mathbb{F}} = A$.

Remember (☆)

For all $(x^*, x^{**}) \in E^* \times E^{**}$, $\quad s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle.$ (☆)

Multifunction notation

- For the rest of this paper, we will suppose that $B = E \times E^*$, as in case (e).
- If $S\colon E \rightrightarrows E^*$ let $D(S) := \{x \in E\colon Sx \ne \emptyset\}$ and $R(S) := \bigcup_{x\in E} Sx$.
- If $S\colon E \rightrightarrows E^*$ let $\varphi_S := \Phi_{G(S)}$. φ_S is known as the "Fitzpatrick function" of S.
- If $S\colon E \rightrightarrows E^*$ we say that S is *closed* if $G(S)$ is a closed subset of $E \times E^*$.
- If $S\colon E \rightrightarrows E^*$ we say that S is *r_L–dense* if $G(S)$ is an r_L–dense subset of $E \times E^*$.
- If $S\colon E \rightrightarrows E^*$ is closed, monotone and r_L–dense, we define $S^{\mathbb{F}}\colon E^* \rightrightarrows E^{**}$, by $G(S^{\mathbb{F}}) := G(S)^{\mathbb{F}}$. Using (☆), $x^{**} \in S^{\mathbb{F}}(x^*) \iff \varphi_S{}^*(x^*, x^{**}) = s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle$. We will call $S^{\mathbb{F}}$ the Fitzpatrick extension of S.

The tail...

Let $E = \ell^1$, and define T: $\ell^1 \to \ell^\infty = E^*$ by

$$(Tx)_n = \sum_{k=n}^{\infty} x_k.$$

T is the "tail" operator. Then T is maximally monotone but not r_L–dense.

Proof. It is well known that T is maximally monotone. Let

$$e^* := (1, 1, \ldots) \in {\ell_1}^* = \ell_\infty.$$

Let $x \in \ell_1$, and write $\sigma = \langle x, e^* \rangle = \sum_{n\geq 1} x_n$. Clearly, $\|x\| \geq \sigma$. Since $Tx \in c_0$, we also have $\|Tx - e^*\| = \sup_n |(Tx)_n - 1| \geq \lim_n |(Tx)_n - 1| = 1$. Thus

$$\begin{aligned}\langle x, Tx\rangle &= \sum_{n\geq 1} x_n \sum_{k\geq n} x_k = \sum_{n\geq 1} x_n^2 + \sum_{n\geq 1}\sum_{k>n} x_n x_k \\ &\geq \tfrac{1}{2}\sum_{n\geq 1} x_n^2 + \sum_{n\geq 1}\sum_{k>n} x_n x_k = \tfrac{1}{2}\sigma^2.\end{aligned}$$

It follows that

$$\begin{aligned} r_L\big((x, Tx) - (0, e^*)\big) &= \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|Tx - e^*\|^2 + \langle x, Tx - e^*\rangle \\ &\geq \tfrac{1}{2}\sigma^2 + \tfrac{1}{2} + \langle x, Tx\rangle - \sigma \geq \tfrac{1}{2}\sigma^2 + \tfrac{1}{2} + \tfrac{1}{2}\sigma^2 - \sigma \\ &= \sigma^2 + \tfrac{1}{2} - \sigma \geq \tfrac{1}{4}.\end{aligned}$$

Consequently, T is not r_L–dense. $\square$

Criterion for $\{B|f = q_L\}$ to be r_L–dense in B

Let $f \in \mathcal{PCLSC}_q(B)$. Then:

$$\{B|f = q_L\} \text{ is } r_L\text{–dense in } B \iff f^* \geq s_L \text{ on } B^*.$$

Remember (☆)

$$\text{For all } (x^*, x^{**}) \in E^* \times E^{**}, \quad s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle. \tag{☆}$$

Theorem on subdifferentials

Let $k \in \mathcal{PCLSC}(E)$. Then ∂k is r_L–dense in $E \times E^*$.

Proof. Define $f \in \mathcal{PCLSC}(E \times E^*)$ by $f(x, x^*) := k(x) + k^*(x^*)$. From the Fenchel–Young inequality,

$$f(x, x^*) \geq \langle x, x^*\rangle = q_L(x, x^*),$$

so $f \in \mathcal{PCLSC}_q(E \times E^*)$. By direct computation, $\forall\ (x^*, x^{**}) \in E^* \times E^{**}$,

$$f^*(x^*, x^{**}) := k^*(x^*) + k^{**}(x^{**}).$$

From the Fenchel–Young inequality again and (☆),

$$f^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle = s_L(x^*, x^{**}).$$

From the criterion above, $\{E \times E^*|f = q_L\}$ is r_L–dense in $E \times E^*$.
But this set is exactly $G(\partial k)$. □

Comment. Since $G(\partial k)$ is closed, this result is a strict generalization of Rockafellar's theorem on the maximal monotonicity of subdifferentials.

A brief digression to non convex subdifferentials and non monotone sets (joint work with Xianfu Wang)

Weak subdifferentials

A weak subdifferential, ∂_w, is a rule that associates with each proper lower semicontinuous function f: $E \to]-\infty, \infty]$ a multifunction $\partial_w f : E \rightrightarrows E^*$ such that,

- $0 \in \partial_w f(x)$ if f attains a strict global minimum at x.
- $\partial_w(f + h)(x) \subseteq \partial_w f(x) + \partial h(x)$ whenever h is a continuous convex real function on E (here ∂h is the subdifferential of h of convex analysis).

Comment. The abstract subdifferential introduced by Thibault and Zagrodny gives a weak subdifferential. This implies that a number of other subdifferentials that have been introduced over the years also give weak subdifferentials. In particular, the Clarke-Rockafellar subdifferential is a weak subdifferential. Also, Mordukhovich's limiting subdifferential is a weak subdifferential if we confine our attention to Asplund spaces.

The r_L–density of weak subdifferentials

Let ∂_w be a weak subdiferential and k: $E \to \mathbb{R}$ be proper, lower semicontinuous and bounded below by a continuous affine functional. Then

$$\partial_w k \text{ is } r_L\text{–dense in } E \times E^*.$$

Comment. Of course, $\partial_w k$ is not necessarily monotone if k is not convex.

For the rest of this talk, we return to the monotone case.

Sufficient conditions for r_L–density

Let $S\colon E \rightrightarrows E^*$ be maximally monotone.

- If $R(S) = E^*$ then S is r_L–dense.
- If E is reflexive then S is r_L–dense

- If X and Y are nonempty sets, define $\pi_1\colon X \times Y \mapsto X$ and $\pi_2\colon X \times Y \mapsto Y$ by

$$\pi_1(x, y) := x \quad \text{and} \quad \pi_2(x, y) := y.$$

Theorem on domain and range

Let $S\colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then

$$\overline{D(S)} = \overline{\pi_1(\operatorname{dom}\varphi_S)} \quad \text{and} \quad \overline{R(S)} = \overline{\pi_2(\operatorname{dom}\varphi_S)}.$$

Consequently,

$$\overline{D(S)} \quad \text{and} \quad \overline{R(S)} \quad \text{are convex.}$$

Comments. Gossez gave an example of a maximally monotone multifunction for which $\overline{R(S)}$ is not convex.

An example of a maximally monotone multifunction for which $\overline{D(S)}$ is not convex would lead to a counterexample for the sum problem!

A negative alignment criterion for r_L–density

Let $S\colon E \rightrightarrows E^*$ be closed and monotone. Then

$$S \text{ is } r_L\text{–dense}$$

$$\Updownarrow$$

$$\forall\ (w, w^*) \in E \times E^*,\ \exists\ \tau \geq 0 \text{ and a sequence } \{(s_n, s_n^*)\}_{n\geq 1} \text{ in } G(S) \text{ such that}$$

$$\lim_{n\to\infty} \|s_n - w\| = \tau, \quad \lim_{n\to\infty} \|s_n^* - w^*\| = \tau \quad \text{and} \quad \lim_{n\to\infty} \langle s_n - w, s_n^* - w^*\rangle = -\tau^2.$$

Comments. (⇑) is obvious. (⇓) is more delicate because the boundedness of the sequence is not obvious. For more details, see the material on the web.

Definition of type (ANA)

Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then S is of type (ANA) if, whenever $(w, w^*) \in E \times E^* \setminus G(S)$, there exists $(s, s^*) \in G(S)$ such that $s \neq w$, $s^* \neq w^*$, and

$$\frac{\langle s - w, s^* - w^* \rangle}{\|s - w\| \|s^* - w^*\|} \text{ is as near as we please to } -1.$$

- This was a property originally proved for subifferentials.
- We do not have an example of a maximally monotone multifunction that is not of type (ANA).

Theorem on type (ANA)

Let $S\colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then S is maximally monotone of type (ANA).

Proof. Immediate from the properties of τ on the previous slide. □

Partial episums

• Let X and Y be nonzero Banach spaces and $f, g \in \mathcal{PCLSC}(X \times Y)$. Then we define the functions $f \oplus_2 g$ and $f \oplus_1 g$ by

$$(f \oplus_2 g)(x, y) := \inf \big\{ f(x, y - \eta) + g(x, \eta) \colon\ \eta \in Y \big\}$$

and
$$(f \oplus_1 g)(x, y) := \inf \big\{ f(x - \xi, y) + g(\xi, y) \colon\ \xi \in X \big\}.$$

• We substitute the symbol $\oplus_2^e$ for $\oplus_2$ and $\oplus_1^e$ for $\oplus_1$ if the infimum is *exact*, that is to say, can be replaced by a minimum.

A bivariate version of the Fenchel duality theorem

Let $f, g \in \mathcal{PCLSC}(X \times Y)$, $f \oplus_2 g \in \mathcal{PC}(X \times Y)$ *and*

$$\textstyle\bigcup_{\lambda > 0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] \text{ be a closed subspace of } X.$$

Then
$$(f \oplus_2 g)^* = f^* \oplus_1^e g^* \text{ on } X^* \times Y^*.$$

• If $S, T \colon E \rightrightarrows E^*$ then, $\forall x \in E$, $(S + T)x := \big\{ x^* + y^* \colon\ x^* \in Sx,\ y^* \in Tx \big\}$.

• $S + T$ is known as the “Minkowski sum” of S and T.

Composite sum theorem

Let $S, T \colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)*:*
(a) $D(S) \cap \operatorname{int} D(T) \neq \emptyset$.
(b) $\bigcup_{\lambda>0} \lambda\big[D(S) - D(T)\big] = E$.
(c) $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_T\big]$ *is a closed subspace of* E.
(d) $S + T$ *is closed, monotone and r_L–dense.*

Composite parallel sum theorem

Let $S, T \colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)*:*
(a) $R(S) \cap \operatorname{int} R(T) \neq \emptyset$.
(b) $\bigcup_{\lambda>0} \lambda\big[R(S) - R(T)\big] = E^*$.
(c) $\bigcup_{\lambda>0} \lambda\big[\pi_2 \operatorname{dom} \varphi_S - \pi_2 \operatorname{dom} \varphi_T\big]$ *is a closed subspace of* E^*.
(d) *The multifunction* $y \mapsto (S^{\mathbb{F}} + T^{\mathbb{F}})^{-1}(\widehat{y})$ *is closed, monotone and r_L–dense.*

Comments. The two results above follow from the bivariate version of Fenchel duality. They are not immediate. These results are in stark contrast to the situation for maximally monotone multifunctions. Is is apparently still unknown whether $S + T$ is maximally monotone when S and T are maximally monotone and $D(S) \cap \operatorname{int} D(T) \neq \emptyset$.

Composite sum theorem

Let $S, T\colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)*:*
(a) $D(S) \cap \operatorname{int} D(T) \neq \emptyset$.
(b) $\bigcup_{\lambda>0} \lambda\big[D(S) - D(T)\big] = E$.
(c) $\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_T\big]$ *is a closed subspace of* E.
(d) $S + T$ *is closed, monotone and* r_L–*dense*.

Composite parallel sum theorem

Let $S, T\colon E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)*:*
(a) $R(S) \cap \operatorname{int} R(T) \neq \emptyset$.
(b) $\bigcup_{\lambda>0} \lambda\big[R(S) - R(T)\big] = E^*$.
(c) $\bigcup_{\lambda>0} \lambda\big[\pi_2 \operatorname{dom} \varphi_S - \pi_2 \operatorname{dom} \varphi_T\big]$ *is a closed subspace of* E^*.
(d) *The multifunction* $y \mapsto (S^{\mathbb{F}} + T^{\mathbb{F}})^{-1}(\widehat{y})$ *is closed, monotone and* r_L–*dense*.

Another comment. If $S\colon E \rightrightarrows E^*$ is closed, monotone and r_L–dense then it can be proved that $S^{\mathbb{F}}\colon E^* \rightrightarrows E^{**}$ is maximally monotone. This does not seem to be very easy. Our proof depends on the following result of Simon Fitzpatrick and myself:

On the biconjugate of a maximum

Let X be a nonzero Banach space, $m \geq 1$, $g_0 \in \mathcal{PCLSC}(X)$ and $g_1, \ldots, g_m$ be convex and continuous on X. Then

$$(g_0 \vee \cdots \vee g_m)^{**} = {g_0}^{**} \vee \cdots \vee {g_m}^{**}.$$

First subdifferential perturbation theorem

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Let $k \in \mathcal{PCLSC}(E)$ and **either** $D(S) \cap \operatorname{int} \operatorname{dom} k \neq \emptyset$ **or** $\operatorname{int} D(S) \cap \operatorname{dom} k \neq \emptyset$*. Then the multifunction $S + \partial k$ is closed, monotone and r_L–dense.*

Second subdifferential perturbation theorem

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Let $k \in \mathcal{PCLSC}(E)$ and **either** $R(S) \cap \operatorname{int} \operatorname{dom} k^* \neq \emptyset$ **or** $\operatorname{int} R(S) \cap \operatorname{dom} k^* \neq \emptyset$*. Then the multifunction*

$$y \mapsto (S^{\mathbb{F}} + \partial k^*)^{-1}(\widehat{y})$$

is closed, monotone and r_L–dense.

Comments. The first subdifferential perturbation theorem follows easily from the composite sum theorem. The second subdifferential perturbation theorem follows with a little more difficulty from the composite parallel sum theorem.

Strong maximality

Let $S\colon\ E \rightrightarrows E^*$ be monotone. We say that S is *strongly maximally monotone* if:

(a) Whenever C is a nonempty $w(E^*, E)$–compact convex subset of E^*, $w \in E$ and,

$$\forall\ (s, s^*) \in G(S),\ \exists\ w^* \in C \text{ such that } \langle s - w, s^* - w^* \rangle \geq 0$$

then $S(w) \cap C \neq \emptyset$.

(b) Whenever C is a nonempty $w(E, E^*)$–compact convex subset of E, $w^* \in E^*$ and,

$$\forall\ (s, s^*) \in G(S),\ \exists\ w \in C \text{ such that } \langle s - w, s^* - w^* \rangle \geq 0$$

then $w^* \in S(C)$.

Strong maximality theorem

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and r_L–dense. Then S is strongly maximally monotone.

Comments. Of course, if C is a singleton, these statements become exactly the statement of maximal monotonicity.

(a) follows from the First subdifferential perturbation theorem with k a support functional, and (b) follows from the Second subdifferential perturbation theorem with k an indicator function.

This was another property originally proved for subifferentials.

We do not have an example of a maximally monotone multifunction that is not strongly maximal.

Type (FP)

Let $S\colon E \rightrightarrows E^*$ be monotone. We say that S is *of type (FP)* provided that the following holds: if U is an open convex subset of E^*, $U \cap R(S) \neq \emptyset$, $(w, w^*) \in E \times U$ and

$$\langle s - w, s^* - w^* \rangle \geq 0 \text{ whenever } (s, s^*) \in A \text{ and } s^* \in U$$

then $(w, w^*) \in G(S)$.

• If we take $U = E^*$, we can see that every monotone multifunction of type (FP) is maximally monotone.

• This concept was originally introduced by Fitzpatrick and Phelps. Their term for it was "locally maximal monotone".

Type (FP) criterion for r_L–density

Let $S\colon E \rightrightarrows E^*$ be closed and monotone. Then S is of type (FP) $\iff$ S is r_L–dense.

Comments. "$\Longleftarrow$" follows from the Second subdifferential perturbation theorem with k a support functional.

"$\Longrightarrow$" follows from an adaptation of a proof of Bauschke, Borwein, Wang and Yao.

Polar subspace

• If Y is a linear subspace of a Banach space X, Y^0 is the "polar subspace of Y", that is to say the linear subspace $\big\{x^* \in X^*\colon\ \langle Y, x^*\rangle = \{0\}\big\}$ of X^*.

Theorem on the r_L-density of linear subspaces

Let (B, L) be an SN space and A be a closed linear L–positive subspace of B. Then

$$A \text{ is } r_L\text{-dense in } B \iff \sup s_L(A^0) \le 0.$$

• Let A be a linear subspace of $E \times E^*$ $\big($that is to say a linear relation$\big)$. The *adjoint subspace (adjoint linear relation)*, A^{T}, of $E^{**} \times E^*$, is defined by:

$$(y^{**}, y^*) \in A^{\mathrm{T}} \iff \text{for all } (a, a^*) \in A,\ \langle a, y^*\rangle = \langle a^*, y^{**}\rangle.$$

(We use the notation "A^{T}" rather than the more usual "A^*" to avoid confusion with the dual space of A.) It is clear that

$$(y^{**}, y^*) \in A^{\mathrm{T}} \iff (y^*, -y^{**}) \in A^0.$$

Our next result extends the Brezis-Browder theorem to nonreflexive spaces.

Generalization of results of Bauschke, Borwein, Wang and Yao

Let A be a closed linear monotone subspace of $E \times E^*$. Then the three conditions below are equivalent:

A is r_L–dense.

A^{T} is a monotone subspace of $E^{**} \times E^*$.

A^{T} is a maximally monotone subspace of $E^{**} \times E^*$.

Type (NI)

Let S: $E \rightrightarrows E^*$ be monotone. We say that S is *of type (NI)* if

$$\forall\ (x^*, x^{**}) \in E^* \times E^{**}, \quad \inf_{(s,s^*)\in G(S)} \langle s^* - x^*, \widehat{s} - x^{**}\rangle \leq 0.$$

Type (NI) criterion for r_L–density

Let S: $E \rightrightarrows E^*$ be closed and monotone. Then S is r_L–dense $\iff$ S is maximally monotone of type (NI).

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