Vectorization in Set Optimization

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UNIVERSITÄT
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Contents

1 Introduction

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Set optimization problem:

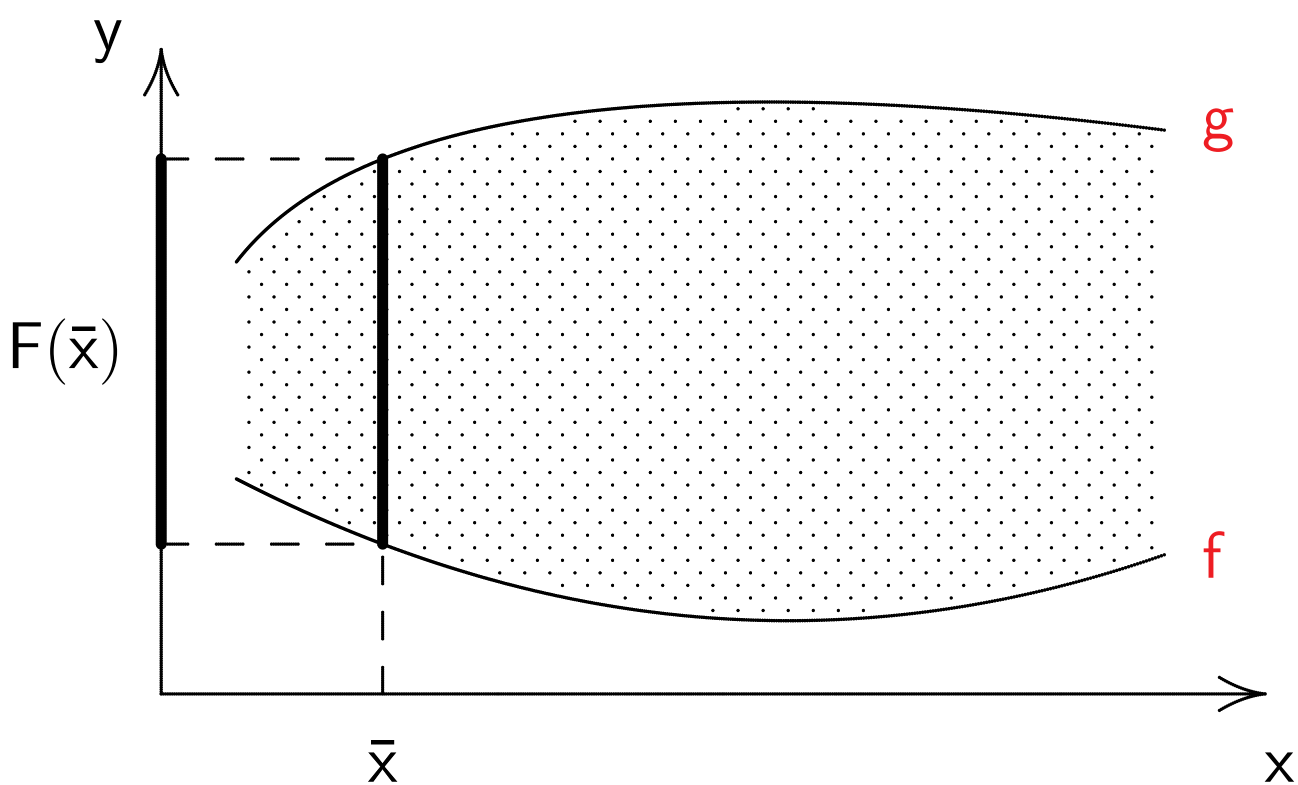
$$\boxed{\min_{x \in S} \; F(x).}$$

Early contributions:

J. Borwein (1977), W. Oettli (1980)

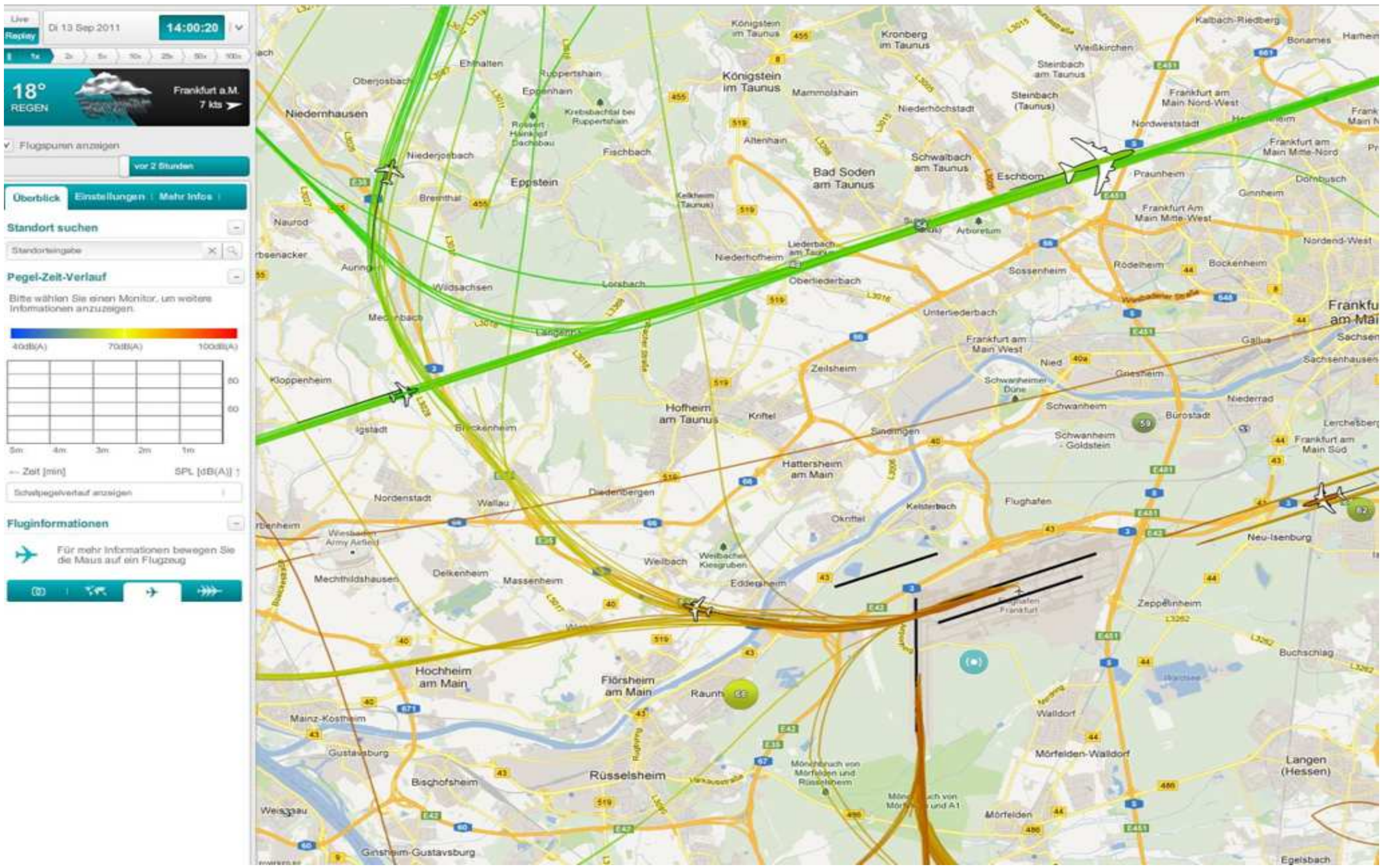
Example 1.2 (perturbed optimization problem):

$$\min_{x \in S} \varphi(x) \quad \curvearrowright \quad \min_{x \in S} F(x) \;\text{ with } F(x) := [f(x), g(x)] \;\; \forall\, x \in S$$



1 Introduction

Example 1.3 (socio-economic problem):



Aircraft noise at Frankfurt airport (by N. Neukel, 2011).

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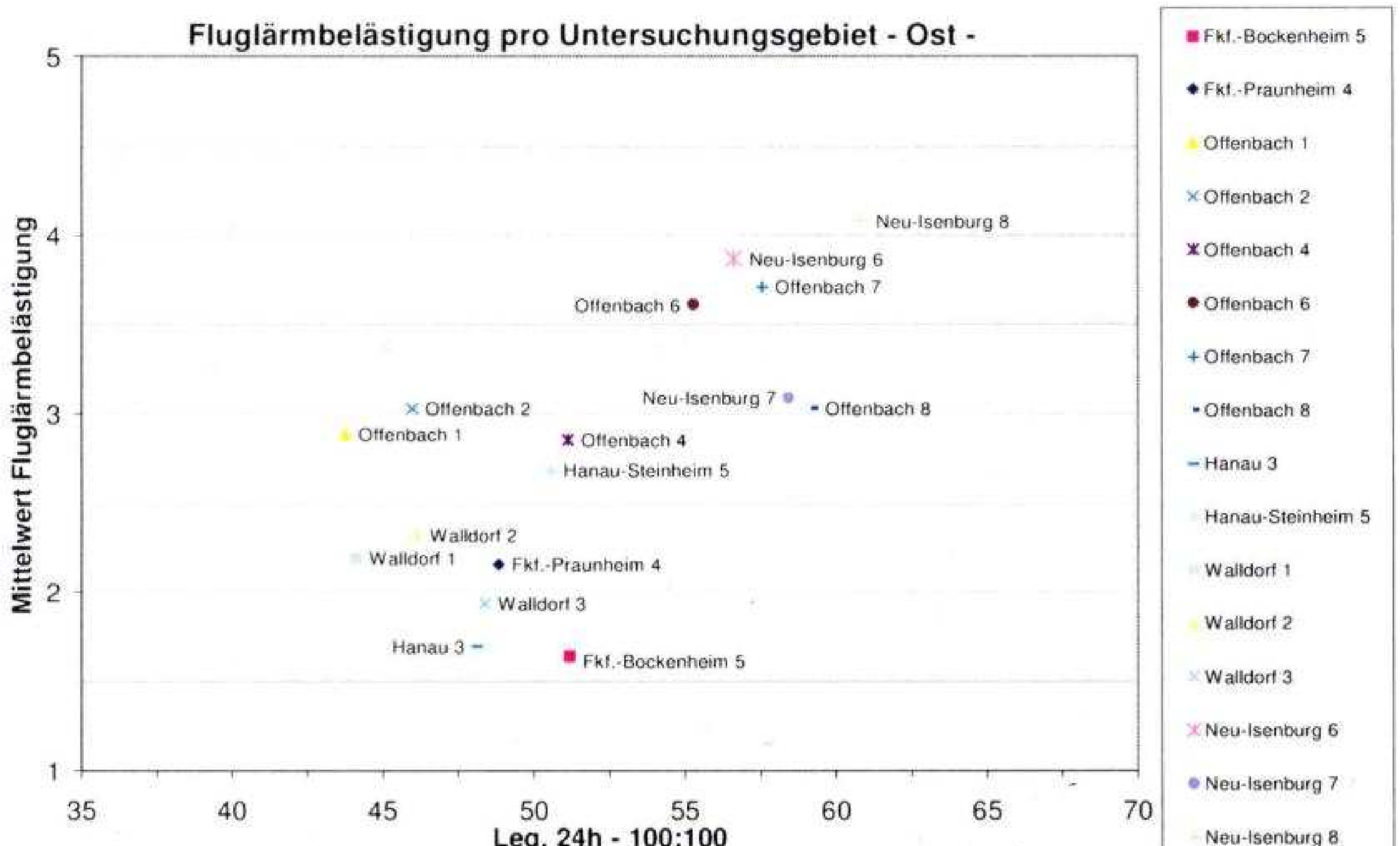

equivalent continuous sound level

Noise annoyance east of Frankfurt (by N. Neukel, 2011).

Definition 1.4:

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$(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is called a minimizer of

$\boxed{\min_{x \in S} F(x)}$ iff for $F(S) := \bigcup_{x \in S} F(x)$

$$(\{\bar{y}\} - C) \cap F(S) \subset \{\bar{y}\} + C,$$

i.e. $\bar{y} \in \min F(S)$.

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$$(\{\bar{y}\} - C) \cap F(S) \subset \{\bar{y}\} + C,$$

i.e. $\bar{y} \in \min F(S)$.

Serious disadvantage of this definition:
In general, only <u>one</u> element does not imply that the whole set $F(\bar{x})$ is in a certain sense minimal with respect to <u>all</u> sets $F(x)$ with $x \in S$.

1 Introduction

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But more than 90% of the papers on set optimization work with minimizers or variants of this notion.

Two pioneering approaches:

- by D. Kuroiwa (1998) in set optimization (see also Z.G. Nishnianidze in fixed point theory (1984) and R.C. Young in algebra (1931))
- by A. Chiriaev/G.W. Walster (1998) in interval arithmetic (implemented in the FORTRAN compiler f95 of SUN Microsystems)

Definition 2.1:

2.1 Known order relations of sets

Definition 2.1:
Let $A, B \subset Y$ be nonempty sets. Then the set less or KNY order relation $\preccurlyeq_s$ is defined by

$$\boxed{\begin{aligned} A \preccurlyeq_s B \quad :\Longleftrightarrow \quad & \big(\forall\, a \in A\ \exists\, b \in B : a \leq b\big) \text{ and} \\ & \big(\forall\, b \in B\ \exists\, a \in A : a \leq b\big). \end{aligned}}$$

2.1 Known order relations of sets

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Definition 2.2:
Let $A, B \subset Y$ be nonempty sets. Then the certainly less order relation $\preccurlyeq_c$ is defined by

$$\boxed{A \preccurlyeq_c B \;:\Longleftrightarrow\; (A = B) \text{ or } (A \neq B,\ \forall\, a \in A\ \forall\, b \in B : a \leq b).}$$

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Definition 2.3:
Let $A, B \subset Y$ be nonempty sets. Then the possibly less order relation $\preccurlyeq_p$ is defined by

$$\boxed{A \preccurlyeq_p B \quad :\Longleftrightarrow \quad \exists\, a \in A\ \exists\, b \in B : a \leq b.}$$

2.1 Known order relations of sets

Example 2.4:

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For order intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ we have

(a) $A \preccurlyeq_s B \iff a_1 \le b_1$ and $a_2 \le b_2$

(b) $A \preccurlyeq_c B \iff a_2 \le b_1$

(c) $A \preccurlyeq_p B \iff a_1 \le b_2$.

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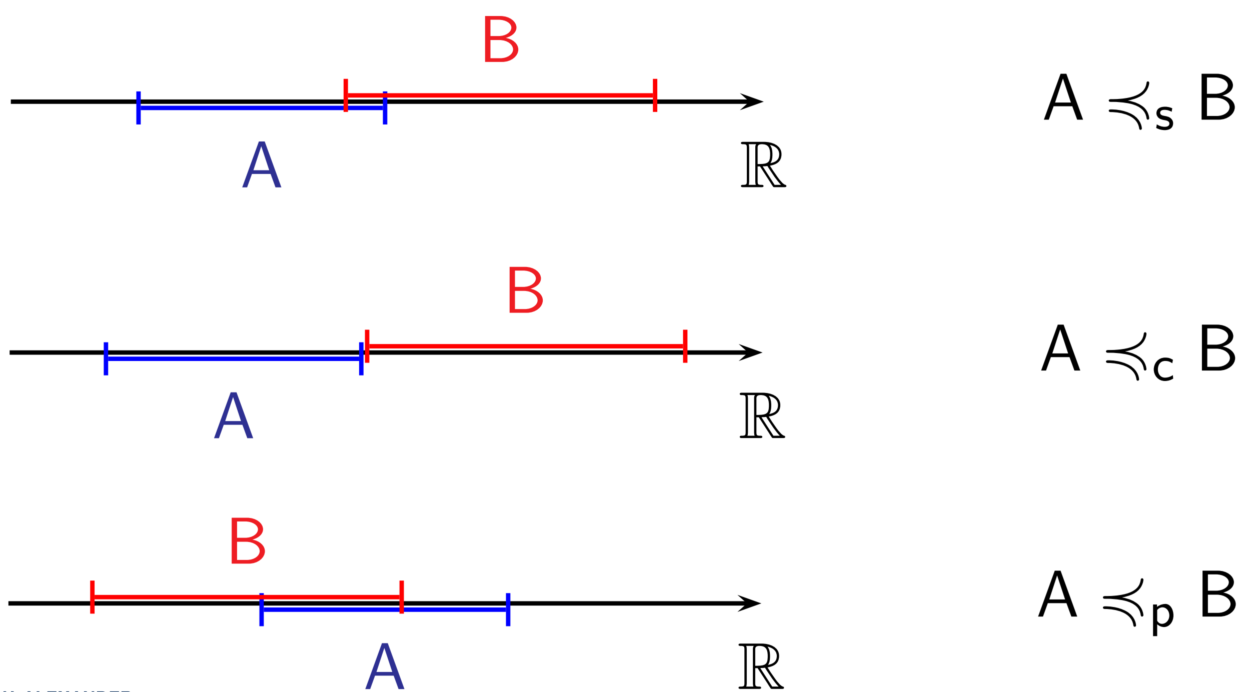

2.2 New order relations of sets

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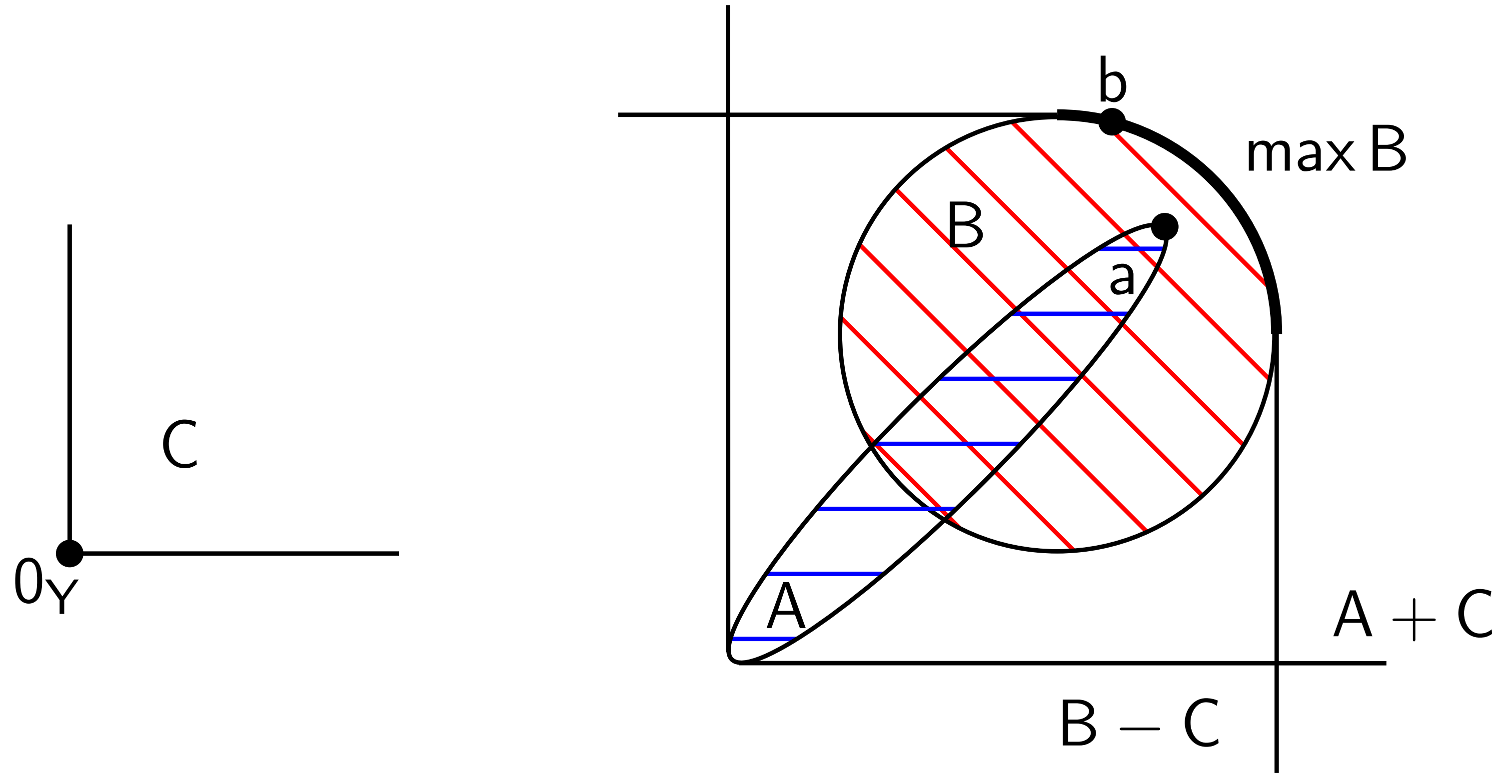

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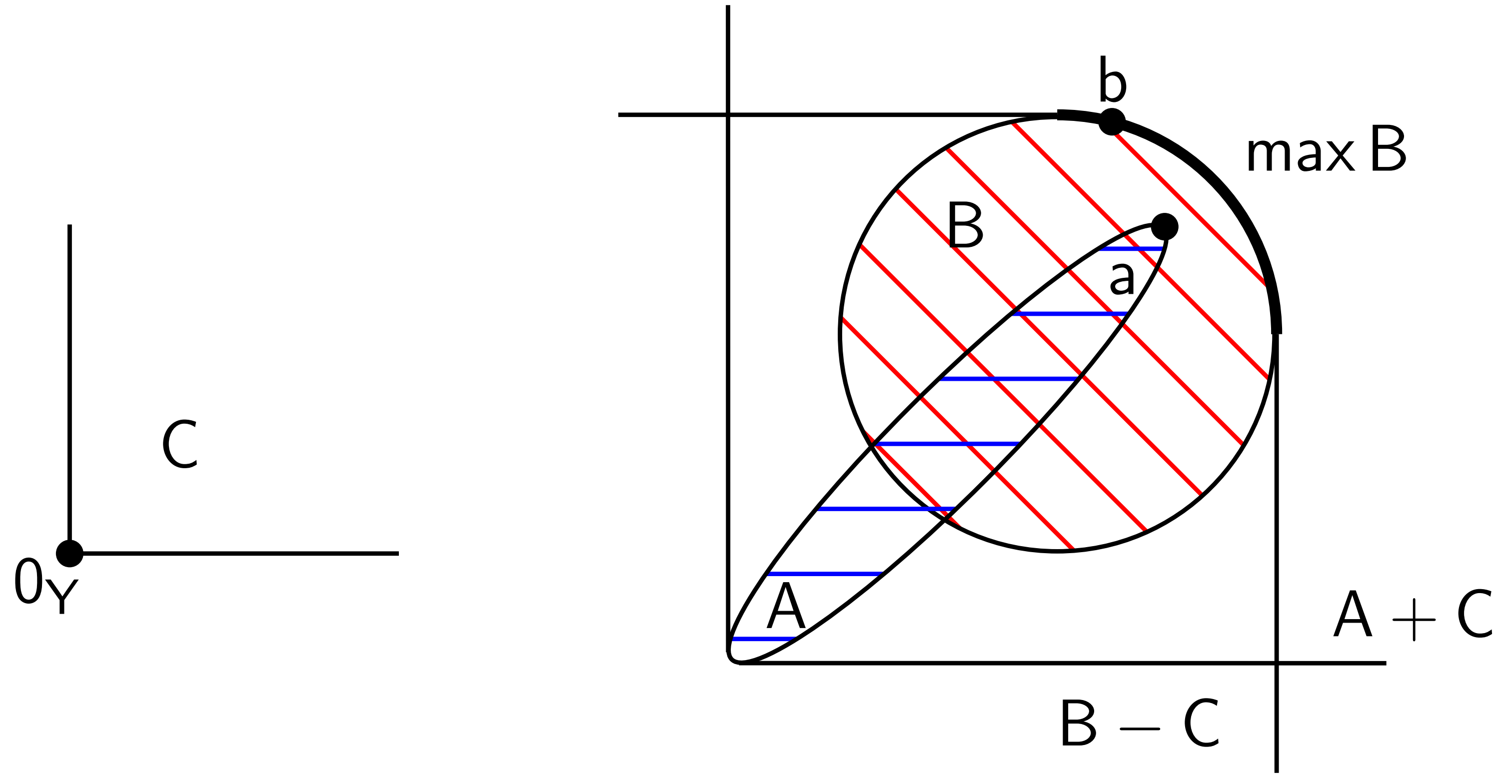

We have $A \preccurlyeq_s B$ but $a \not\leq b$ and $b \not\leq a$ for $a \in \max A$ and $b \in \max B$.

Definition 2.5:

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Let $A, B \subset Y$ be nonempty sets with at least one minimal and one maximal element. Then the minmax less order relation $\preccurlyeq_m$ is defined by

$$\boxed{A \preccurlyeq_m B \quad :\Longleftrightarrow \quad \min A \preccurlyeq_s \min B \text{ and } \max A \preccurlyeq_s \max B.}$$

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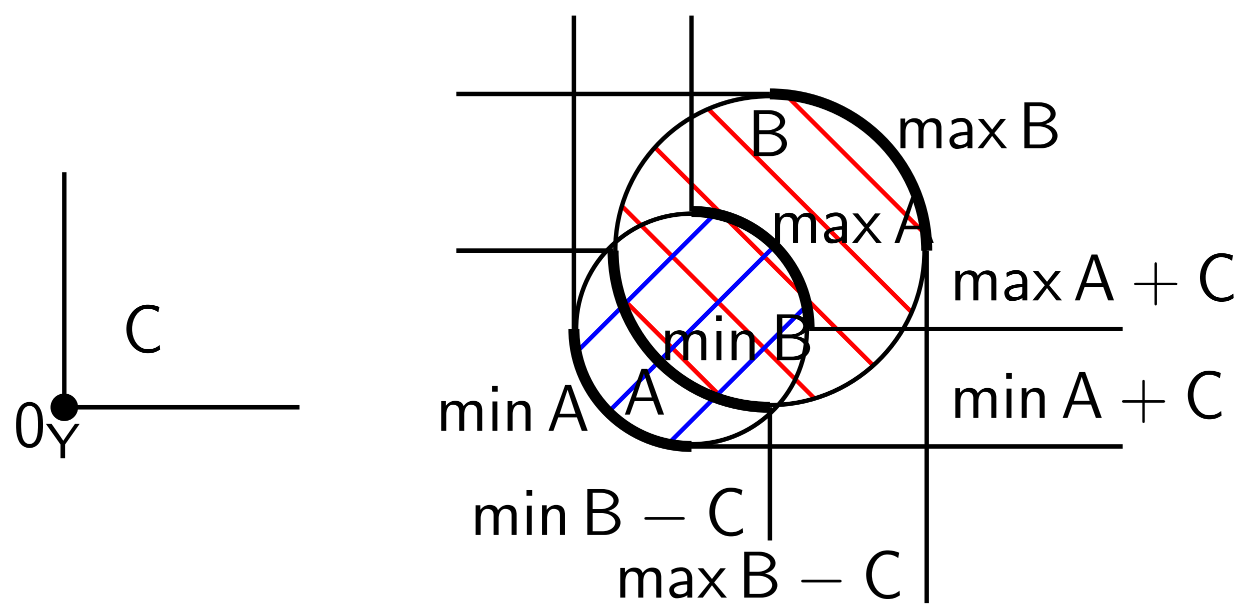

Definition 2.6:

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Let $A, B \subset Y$ be nonempty sets with at least one minimal and one maximal element. Then the minmax certainly less order relation $\preccurlyeq_{mc}$ is defined by

$$A \preccurlyeq_{mc} B \;:\Longleftrightarrow\; (A = B) \text{ or } \\ (A \neq B, \min A \preccurlyeq_c \min B \text{ and } \max A \preccurlyeq_c \max B).$$

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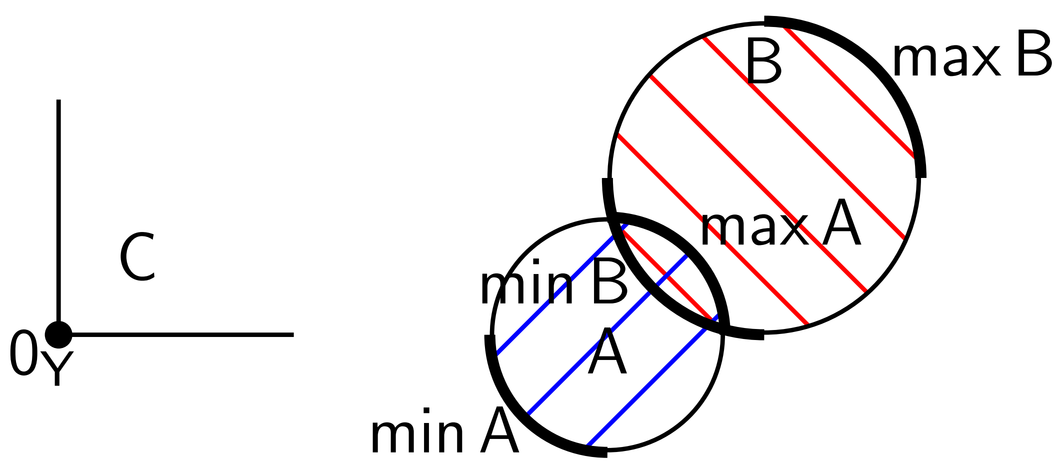

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2.2 New order relations of sets

Definition 2.7:
Let $A, B \subset Y$ be nonempty sets with at least one minimal and one maximal element. Then the minmax certainly nondominated order relation $\preccurlyeq_{mn}$ is defined by

$$\boxed{A \preccurlyeq_{mn} B \quad :\Longleftrightarrow \quad (A = B) \text{ or } (A \neq B, \max A \preccurlyeq_s \min B).}$$

Proposition 2.8 (Properties of $\preccurlyeq_{m}$ and $\preccurlyeq_{mc}$):

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(b) $(A \preccurlyeq_{m} B \text{ and } B \preccurlyeq_{m} A)$

$$\iff \quad \big(\min A + C = \min B + C,\ \min A - C = \min B - C,$$
$$\max A + C = \max B + C,\ \max A - C = \max B - C\big).$$

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$$\iff \quad (\min A + C = \min B + C,\ \min A - C = \min B - C,$$
$$\max A + C = \max B + C,\ \max A - C = \max B - C).$$

(c) If C is pointed, then:
$(A \preccurlyeq_{mc} B$ and $B \preccurlyeq_{mc} A)$
$\iff$ $(\min A = \min B$ and $\max A = \max B)$.

Proposition 2.9 (Properties of $\preccurlyeq_{mn}$):

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If C is pointed, then $\preccurlyeq_{mn}$ is reflexive, transitive, antisymmetric and compatible with the multiplication with nonnegative real numbers.

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A nonempty set $A \subset Y$ is said to have the quasi domination property iff $A \subset \min A + C$ and $A \subset \max A - C$.

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Proposition 2.11 (Comparing known and new order relations):
Let $A, B \subset Y$, $A \neq B$, be nonempty sets with at least one minimal and one maximal element. If A, B have the quasi domination property, then:

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$$
\begin{tikzcd}
 & A \preccurlyeq_{mc} B \arrow[dr, Rightarrow] & & \\
A \preccurlyeq_{c} B \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & A \preccurlyeq_{m} B \arrow[r, Rightarrow] & A \preccurlyeq_{s} B. \\
 & A \preccurlyeq_{mn} B \arrow[ur, Rightarrow] & &
\end{tikzcd}
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In the following let $\preccurlyeq$ be any pre-order (reflexive and transitive).

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Definition 3.1:
$\bar{x} \in S$ is called a minimal solution of the set optimization problem $\min\limits_{x \in S} F(x)$ w.r.t. the pre-order $\preccurlyeq$ iff

$$F(x) \preccurlyeq F(\bar{x}) \text{ for some } x \in S \implies F(\bar{x}) \preccurlyeq F(x).$$

3 Existence result

Definition 3.2:
Let S be a subset of a topological space X. F is said to be semicontinuous w.r.t. the pre-order $\preccurlyeq$ at $\bar{x} \in S$ iff $F(\bar{x}) \in \mathcal{V}$, where $\mathcal{V} = \{T \subset Y \mid T \not\preccurlyeq V\}$ for some $V \subset Y$, then there is a neighborhood U of $\bar{x}$ in X so that

$$F(x) \in \mathcal{V} \text{ for all } x \in U.$$

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$$F(x) \in \mathcal{V} \text{ for all } x \in U.$$

F is said to be semicontinuous w.r.t. the pre-order $\preccurlyeq$ on S, iff F is semicontinuous w.r.t. the pre-order $\preccurlyeq$ at every $\bar{x} \in S$.

Theorem 3.3 (existence result):

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Let S be a subset of a topological space X. If S is compact and F is semicontinuous w.r.t. the pre-order $\preccurlyeq$ on S, then the set optimization problem $\min\limits_{x \in S} F(x)$ has a minimal solution w.r.t. the pre-order $\preccurlyeq$.

4 Vectorization

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Vectorization means the replacement of a set optimization problem by a suitable vector optimization problem.

Lemma 4.1 (key lemma):
Y locally convex, $\emptyset \neq A, B \subset Y$, $C \subset Y$ convex cone, $A + C$ closed and convex. Then:

$$\boxed{B \subset A + C} \iff \forall\, \ell \in C^* \backslash \{0_{Y^*}\} : \inf_{a \in A} \ell(a) \leq \inf_{b \in B} \ell(b)$$

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$$\iff \underbrace{\sup_{\ell \in C^* \backslash \{0_{Y^*}\}} \left(\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b) \right)}_{=:\varphi(A,B)} \leq 0.$$

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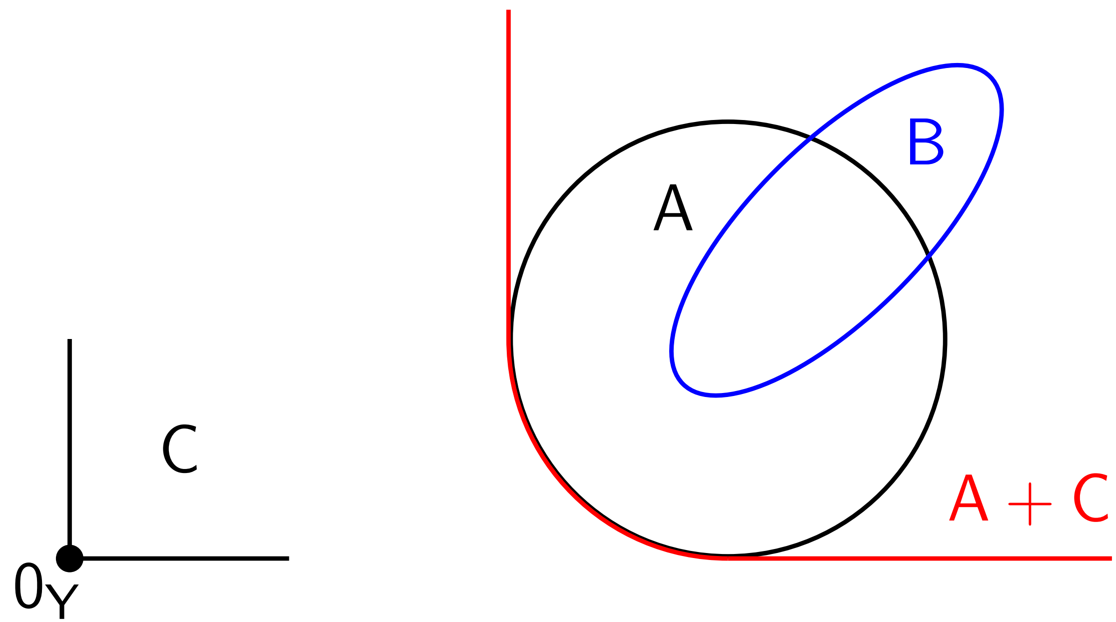

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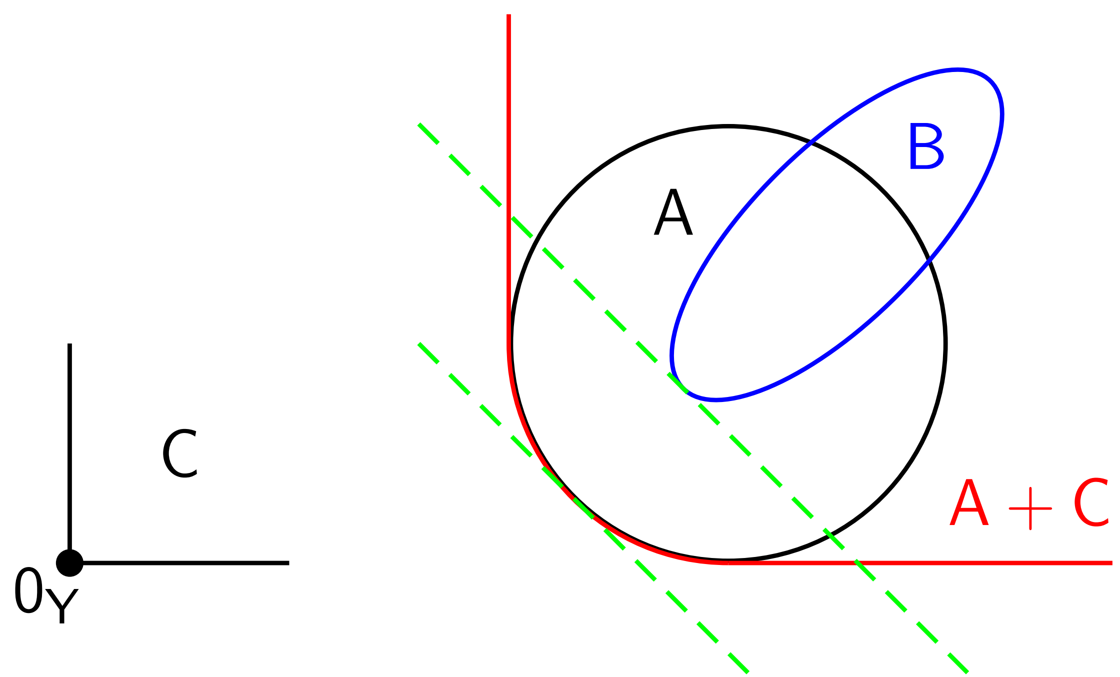

level sets of $\ell \in C^* \backslash \{0_{Y^*}\}$

Remark 4.2:
Under the assumptions of Lemma 4.1 we have

$$\boxed{B \subset A - C} \quad \Longleftrightarrow \quad \forall\, \ell \in C^{*}\backslash\{0_{Y^{*}}\} : \sup_{a \in A} \ell(a) \geq \sup_{b \in B} \ell(b).$$

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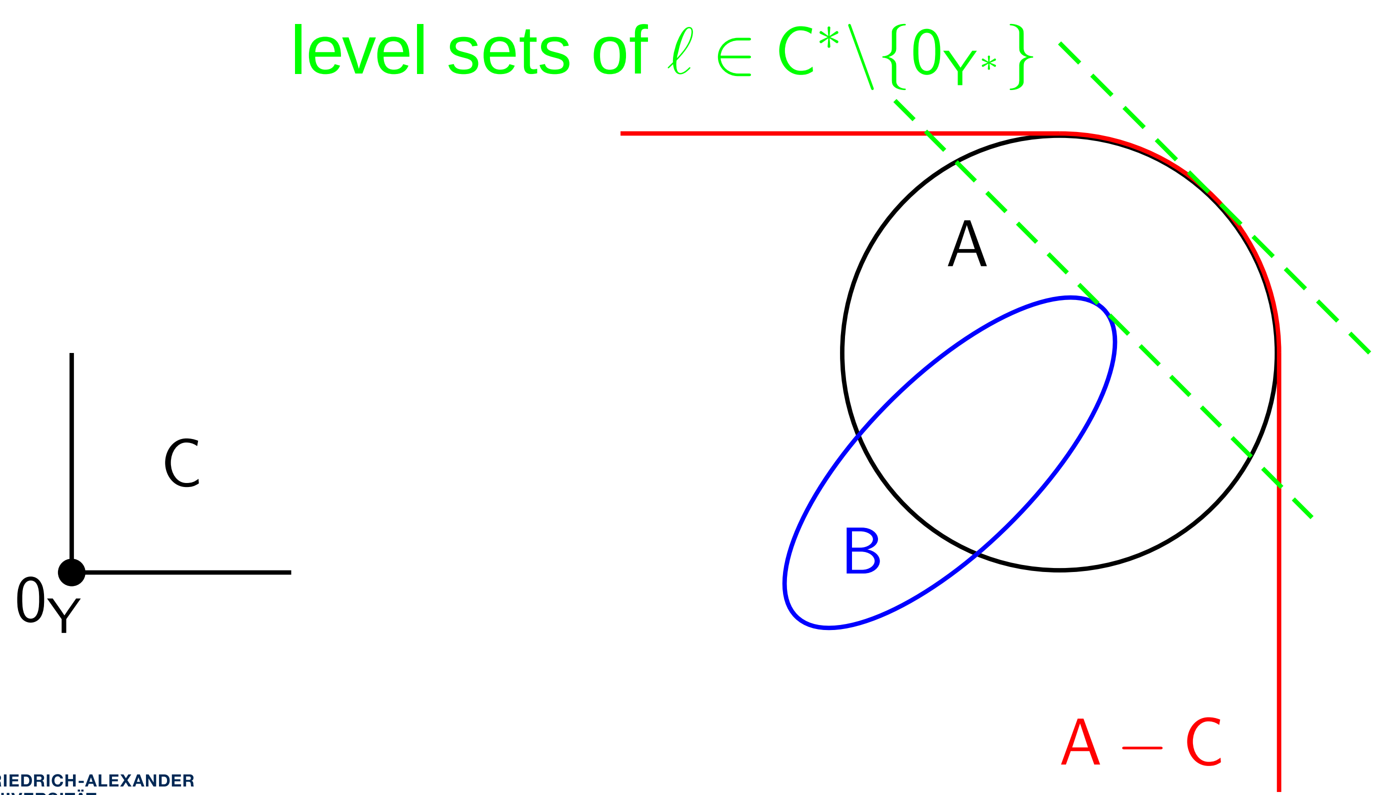

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Remark 4.3:

C^* in Lemma 4.1 can be reduced to its base (if it exists) or to other appropriate subsets.

Theorem 4.4:
Let the assumptions of Lemma 4.1 be satisfied and let $A + C$ and $B - C$ be closed and convex. Then

$$\boxed{A \preccurlyeq_s B} \iff \forall\, \ell \in C^* \backslash \{0_{Y^*}\} : \inf_{a \in A} \ell(a) \leq \inf_{b \in B} \ell(b)$$

$$\text{and } \sup_{a \in A} \ell(a) \leq \sup_{b \in B} \ell(b)$$

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$$\iff \max\left(\sup_{\ell \in C^* \backslash \{0_{Y^*}\}} \left(\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b) \right), \sup_{\ell \in C^* \backslash \{0_{Y^*}\}} \left(\sup_{a \in A} \ell(a) - \sup_{b \in B} \ell(b) \right) \right) \leq 0.$$

Corollary 4.5:
Let the assumptions of Lemma 4.1 be satisfied, let the sets $\min A$, $\min B$, $\max A$ and $\max B$ be nonempty and let the sets $A + C$, $A - C$, $B + C$ and $B - C$ are closed and convex. If $C + \min A = A + C$, $-C + \max A = A - C$, $C + \min B = B + C$ and $-C + \max B = B - C$, then

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$$\boxed{A \preccurlyeq_m B} \iff \forall\, \ell \in C^* \backslash \{0_{Y^*}\} : \inf_{a \in A} \ell(a) \le \inf_{b \in B} \ell(b),$$

$$\sup_{a \in \min A} \ell(a) \le \sup_{b \in \min B} \ell(b),$$

$$\inf_{a \in \max A} \ell(a) \le \inf_{b \in \max B} \ell(b)$$

$$\text{and } \sup_{a \in A} \ell(a) \le \sup_{b \in B} \ell(b).$$

Definition 4.6

$\mathcal{A}$ system of nonempty subsets of Y,
$\mathcal{R}^n(C^*\backslash\{0_{Y^*}\})$ space of functions on $C^*\backslash\{0_{Y^*}\}$ with values in $\overline{\mathbb{R}}^n$.
The map $v : \mathcal{A} \to \mathcal{R}^2(C^*\backslash\{0_{Y^*}\})$ is pointwise defined by

$$v(A)(\ell) := \begin{pmatrix} \inf\limits_{a\in A} \ell(a) \\ \\ \sup\limits_{a\in A} \ell(a) \end{pmatrix} \text{ for all } A \in \mathcal{A} \text{ and all } \ell \in C^*\backslash\{0_{Y^*}\}$$

4 Vectorization

and the map $\mathsf{w} : \mathcal{A} \to \mathcal{R}^4(C^* \backslash \{0_{Y^*}\})$ is pointwise defined by

$$w(A)(\ell) := \begin{pmatrix} \inf\limits_{a \in A} \ell(a) \\ \sup\limits_{a \in \min A} \ell(a) \\ \inf\limits_{a \in \max A} \ell(a) \\ \sup\limits_{a \in A} \ell(a) \end{pmatrix} \text{ for all } A \in \mathcal{A} \text{ and all } \ell \in C^* \backslash \{0_{Y^*}\}$$

($\min A$ and $\max A$ are assumed to be nonempty for all $A \in \mathcal{A}$).

Theorem 4.7 (first vectorization theorem):
Let the sets $F(x)+C$ and $F(x)-C$ be closed and convex for every $x \in S$. Then

$$\bar{x} \text{ minimal solution of } \boxed{\min_{x \in S} F(x)} \text{ with respect to } \boxed{\preccurlyeq_s}$$

$$\Longleftrightarrow \quad \bar{x} \text{ minimal solution of } \boxed{\min_{x \in S} \; v(F(x))} \text{ with respect to } \preccurlyeq$$

($\preccurlyeq$ denotes the componentwise and pointwise ordering of vector functions).

Theorem 4.8 (second vectorization theorem):
Let the sets $F(x)+C$ and $F(x)-C$ be closed and convex and $\min F(x)$ and $\max F(x)$ be nonempty for every $x \in S$. If $C+\min F(x) = F(x)+C$ and $-C+\max F(x) = F(x)-C$ for every $x \in S$, then

$$\bar{x} \text{ minimal solution of } \boxed{\min_{x\in S} F(x)} \text{ with respect to } \preccurlyeq_m$$

$$\Longleftrightarrow \quad \bar{x} \text{ minimal solution of } \boxed{\min_{x\in S} \; w(F(x))} \text{ with respect to } \preccurlyeq.$$

Example 4.9:

Consider $S := [-1, 1]$, $Y := \mathbb{R}^2$, $C := \mathbb{R}^2_+$ and $F : S \rightrightarrows Y$ with

$$F(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - 2x^2)^2 + (y_2 - 2x^2)^2 \leq (x^2 + 1)^2\}$$

$$\forall\, x \in S$$

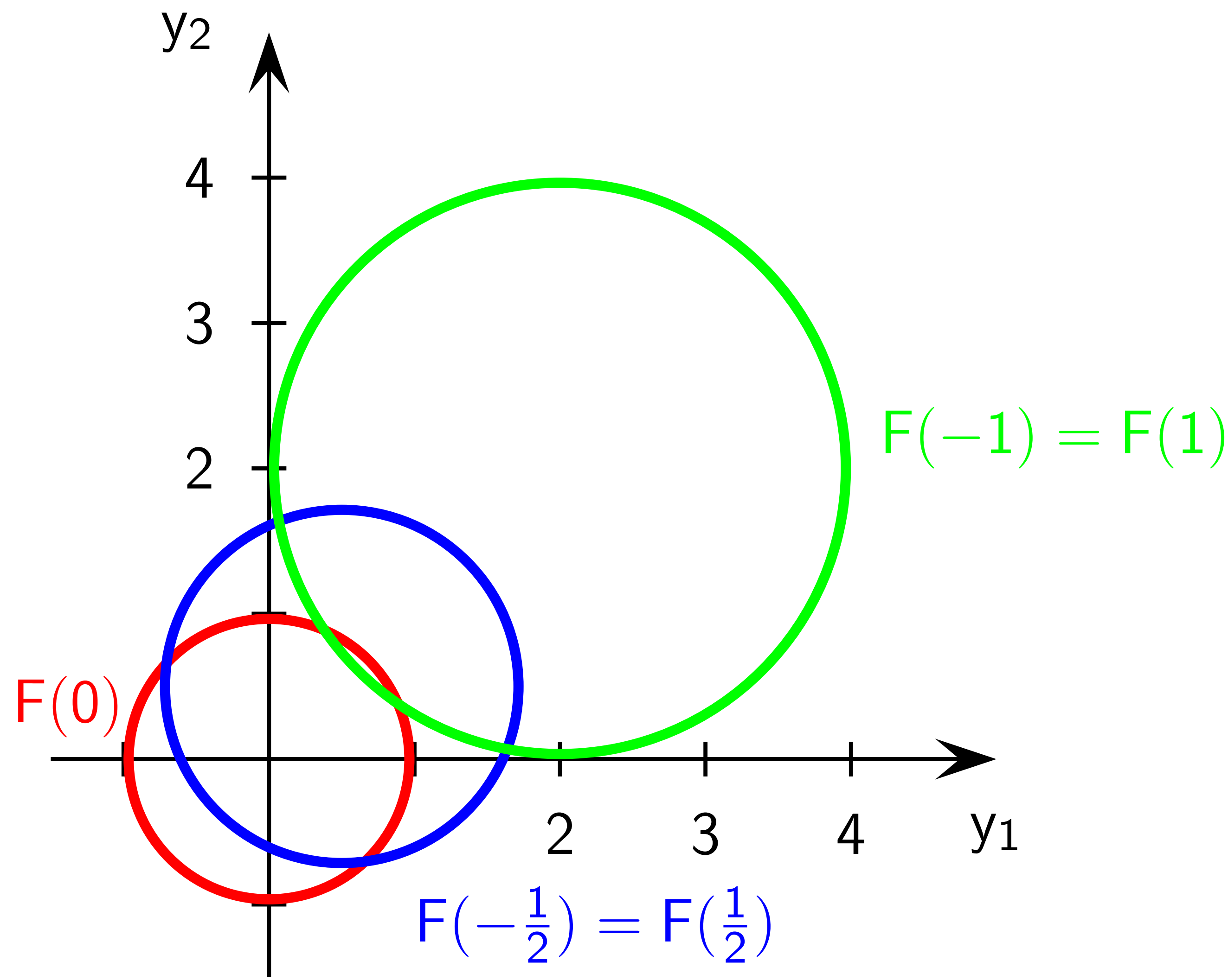

Illustration of the sets $F(x)$ for some $x \in S$.

4 Vectorization

The vector function v at $F(x)$ for some $x \in S$ is pointwise given by

$$v(F(x))(\ell) = \begin{pmatrix} x^2(2(\ell_1 + \ell_2) - 1) - 1 \\ x^2(2(\ell_1 + \ell_2) + 1) + 1 \end{pmatrix} \ \forall\, \ell \in \mathbb{R}^2_+ \text{ with } ||\ell|| = 1.$$

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The vector function v at $F(x)$ for some $x \in S$ is pointwise given by

$$v(F(x))(\ell) = \begin{pmatrix} x^2(2(\ell_1 + \ell_2) - 1) - 1 \\ x^2(2(\ell_1 + \ell_2) + 1) + 1 \end{pmatrix} \ \forall\, \ell \in \mathbb{R}^2_+ \text{ with } ||\ell|| = 1.$$

We conclude for all $x \in S$ and all $\ell \in \mathbb{R}^2_+$ with $||\ell|| = 1$

$$v(F(0))(\ell) \leq v(F(x))(\ell).$$

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$$\mathsf{v}(\mathsf{F}(0))(\ell) \leq \mathsf{v}(\mathsf{F}(\mathsf{x}))(\ell).$$

Hence, $\bar{\mathsf{x}} := 0$ is a strongly minimal solution of the vector optimization problem $\min\limits_{\mathsf{x}\in\mathsf{S}} \mathsf{v}(\mathsf{F}(\mathsf{x}))$.

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We conclude for all $x \in S$ and all $\ell \in \mathbb{R}^2_+$ with $||\ell|| = 1$

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Hence, $\bar{x} := 0$ is a strongly minimal solution of the vector optimization problem $\min\limits_{x \in S} v(F(x))$.

Then $\underline{\bar{x} := 0 \text{ is a minimal solution}}$ of the set optimization problem $\min\limits_{x \in S} F(x)$ with respect to $\preccurlyeq_s$.

Definition 4.10:
$\bar{x} \in S$ is called a strongly minimal solution of the set optimization problem $\min_{x \in S} F(x)$ with respect to the set less order relation $\preccurlyeq_s$ iff

$$F(\bar{x}) \preccurlyeq_s F(x) \text{ for all } x \in S.$$

Theorem 4.11:

Let the sets $F(x) + C$ and $F(x) - C$ be closed and convex for every $x \in S$, and let the scalar optimization problems $\min\limits_{\substack{x \in S \\ y \in F(x)}} \ell(y)$ and $\min\limits_{x \in S} \max\limits_{y \in F(x)} \ell(y)$ be solvable.

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Let the sets $F(x)+C$ and $F(x)-C$ be closed and convex for every $x \in S$, and let the scalar optimization problems $\min\limits_{\substack{x\in S\\ y\in F(x)}} \ell(y)$ and $\min\limits_{x\in S} \max\limits_{y\in F(x)} \ell(y)$ be solvable.
$\bar{x} \in S$ is a <u>strongly minimal solution</u> of the set optimization problem $\min\limits_{x\in S} F(x)$ with respect to the set less order relation $\preccurlyeq_s$ if and only if for every $\ell \in C^* \backslash \{0_{Y^*}\}$ the element $\bar{x} \in S$ is a minimal solution of the two scalar optimization problems

$$\boxed{\min\limits_{\substack{x\in S\\ y\in F(x)}} \ell(y) \quad \text{and} \quad \min\limits_{x\in S} \max\limits_{y\in F(x)} \ell(y).}$$

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