

# Thinking critically about mathematics and thinking

Peter Ellerton, UQ Critical Thinking Project



THE UNIVERSITY  
OF QUEENSLAND  
AUSTRALIA

Create change





# Mathematics and Critical Thinking



an educational heritage



Modern educational systems, beginning in the early eighteen-hundreds, were subject to pressures of numbers and an increasing recognition that classical methods and topics of education were slow and unproductive. Jeremy Bentham's Chrestomathia school, developed for the growing middle classes and designed to address the need for faster and more immediately useful educational outcomes, is paradigmatic of the view that education is for utility, and a largely scientific utility at that (Bentham, 1816).

Both critics and supporters of Bentham acknowledged that schools were to be modelled on factory processes, and this was something of a selling point for many; but not for all. Elissa Itzkin (1978) notes correspondence from a school of the time expressing concerns that the roles of students and masters are too rigorously defined within this model.

“Indeed, the duties of each must be made perfectly mechanical. There must be no doubt or hesitation on the part of the master or pupil; for doubt would produce delay and dispute, and consequently throw the whole machine into disorder. Hence there can be no appeal to the reasoning powers; for reasoning, never can be reduced to mechanism . . . every boy must conform to the average motion of the School. “

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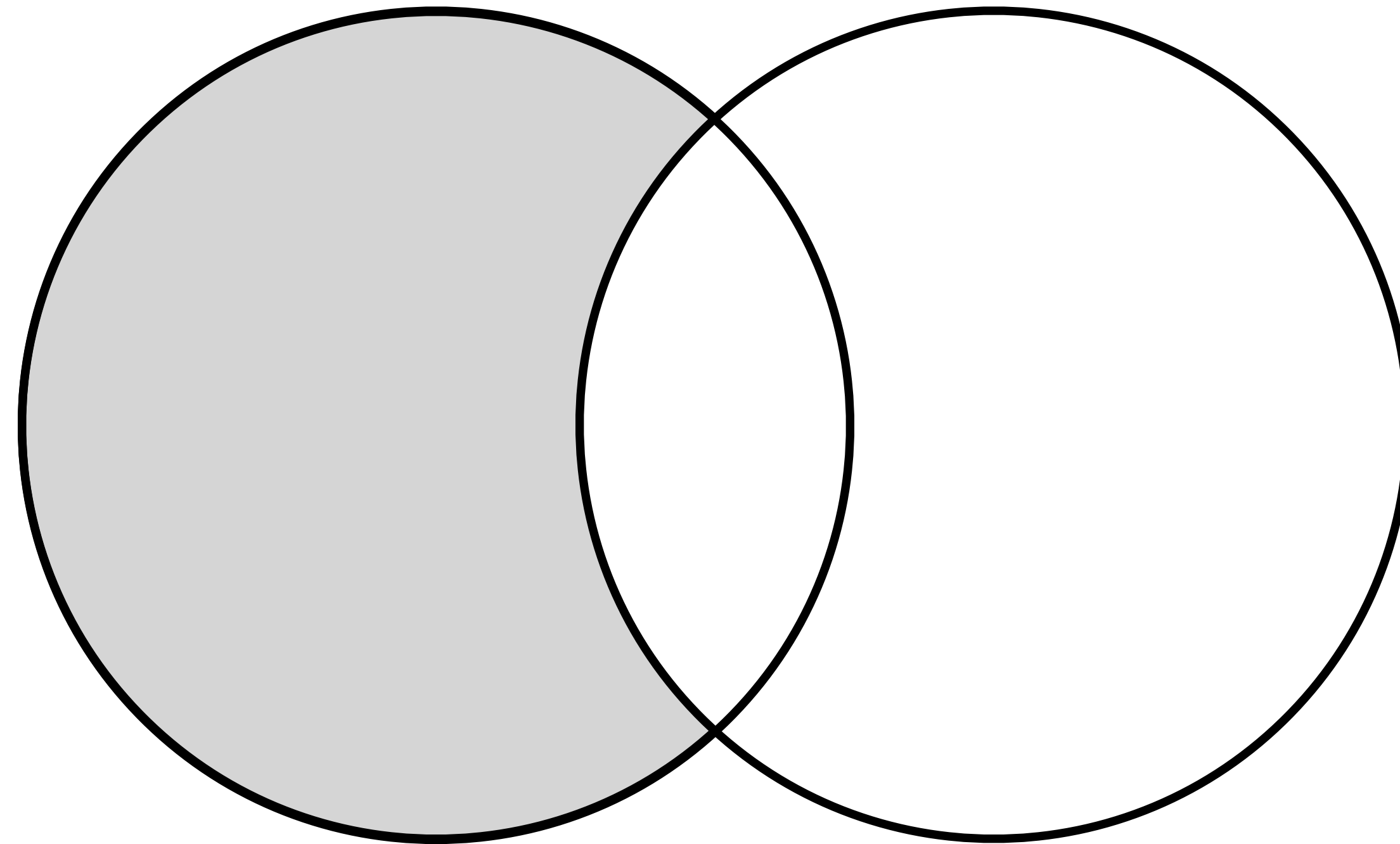


mathematics and critical thinking  
some possible models...

**Model One**

Mathematics

Critical  
Thinking

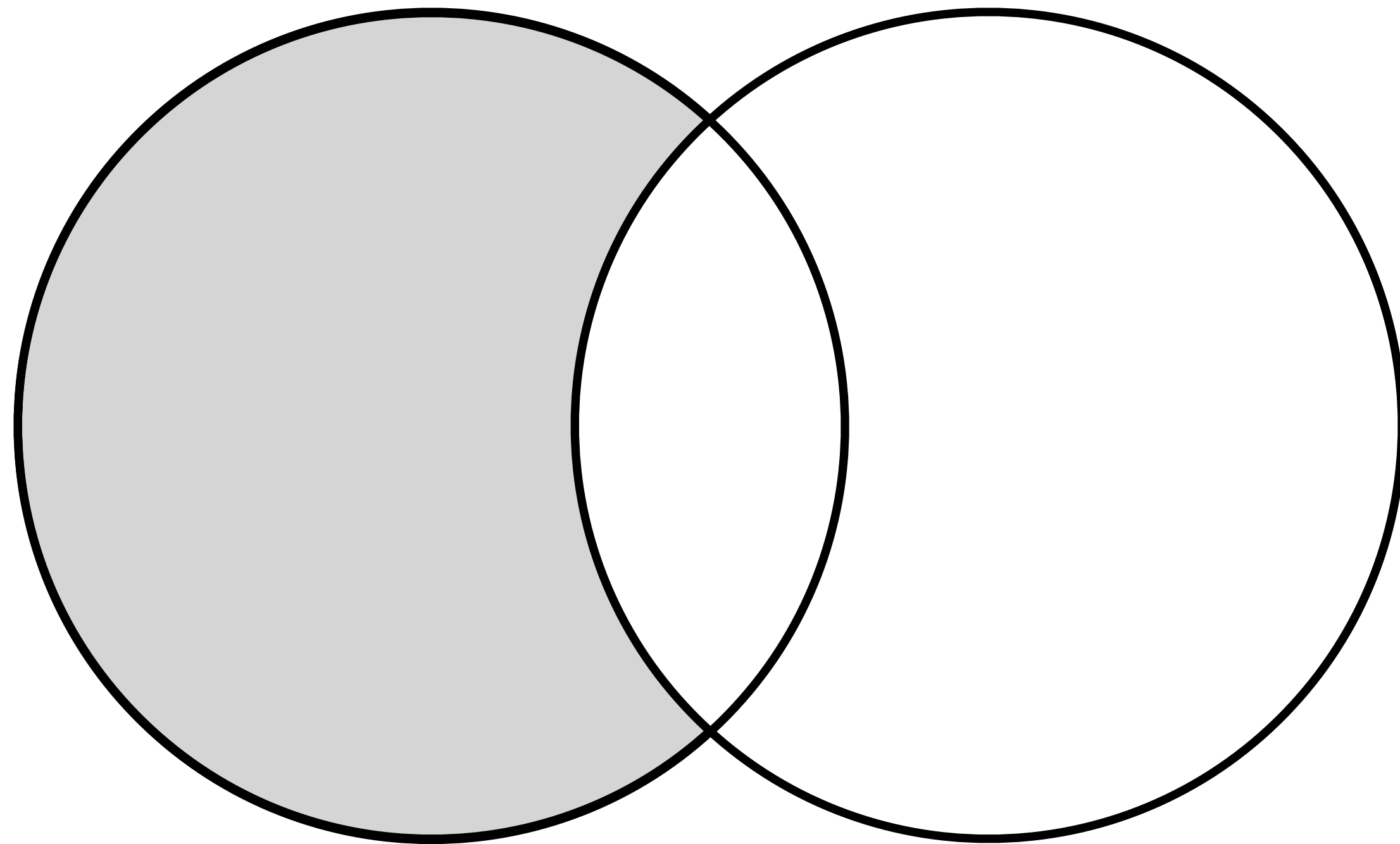


Mathematics is **sufficient** for critical thinking

**Model One  
(naive)**

Mathematics

Critical  
Thinking



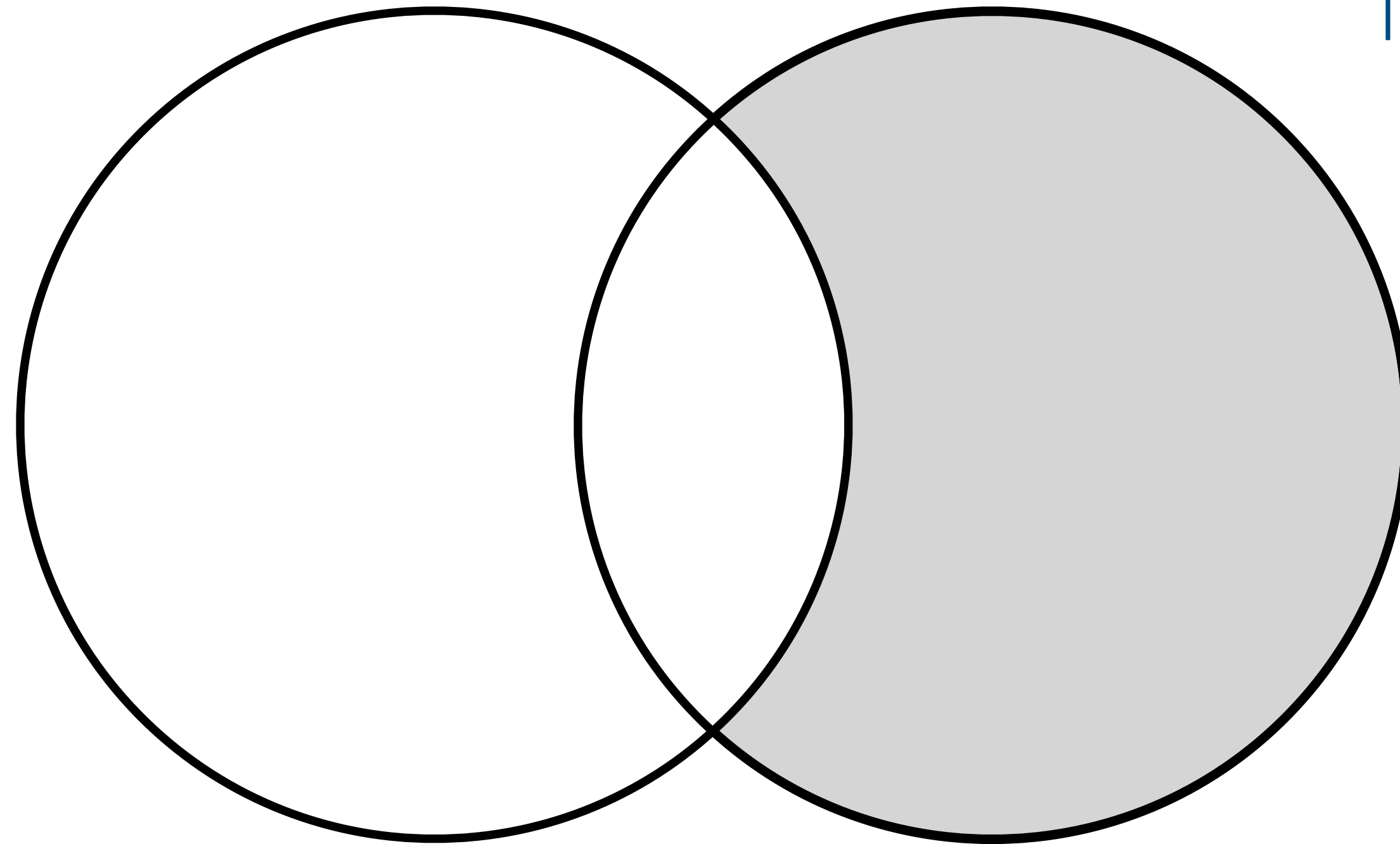
Mathematics is **sufficient** for critical thinking



**Model Two  
(absolutist)**

Mathematics

Critical  
Thinking

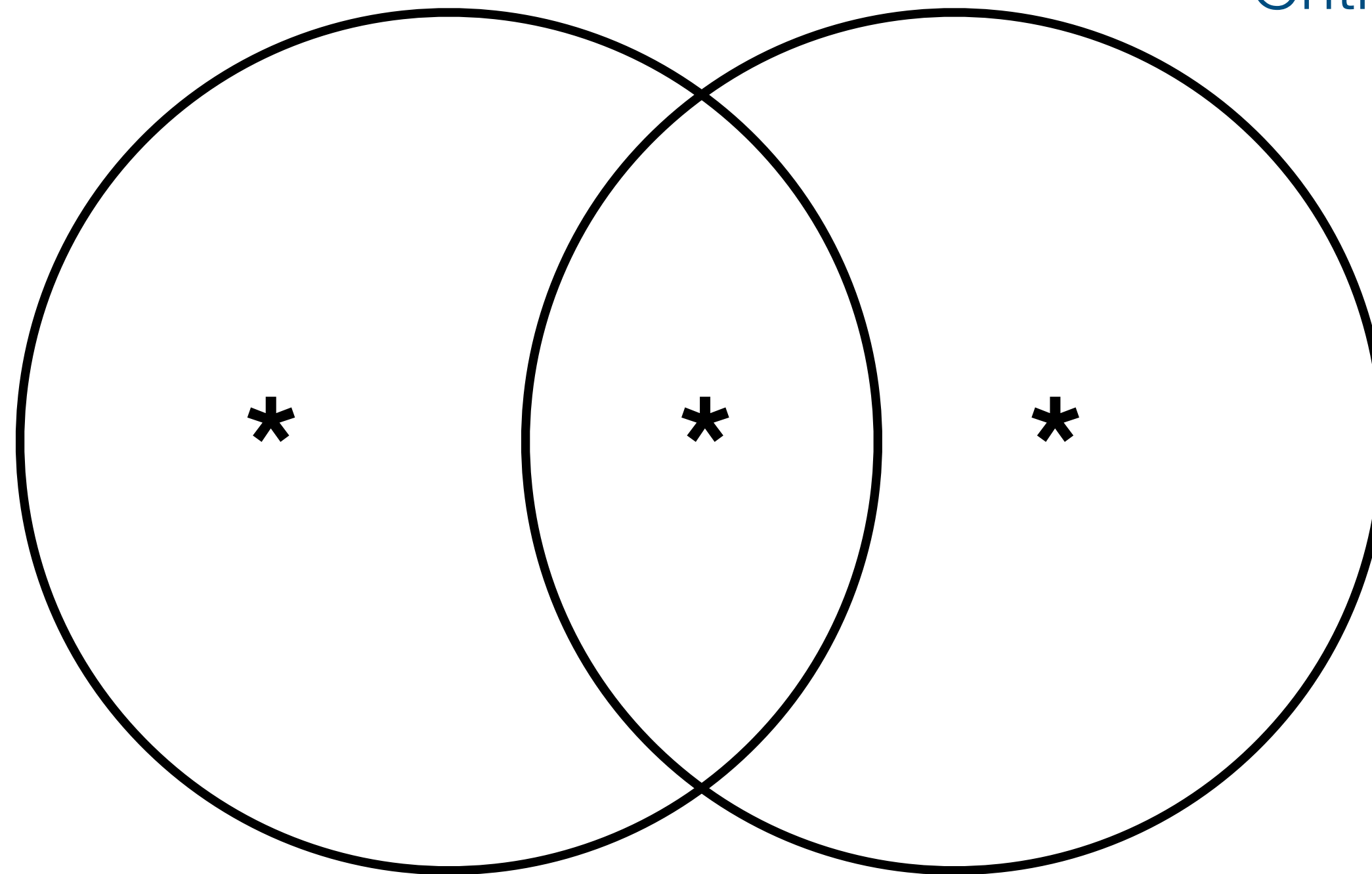


Mathematics is **necessary** for critical thinking

**Model Three  
(independence)**

Mathematics

Critical Thinking



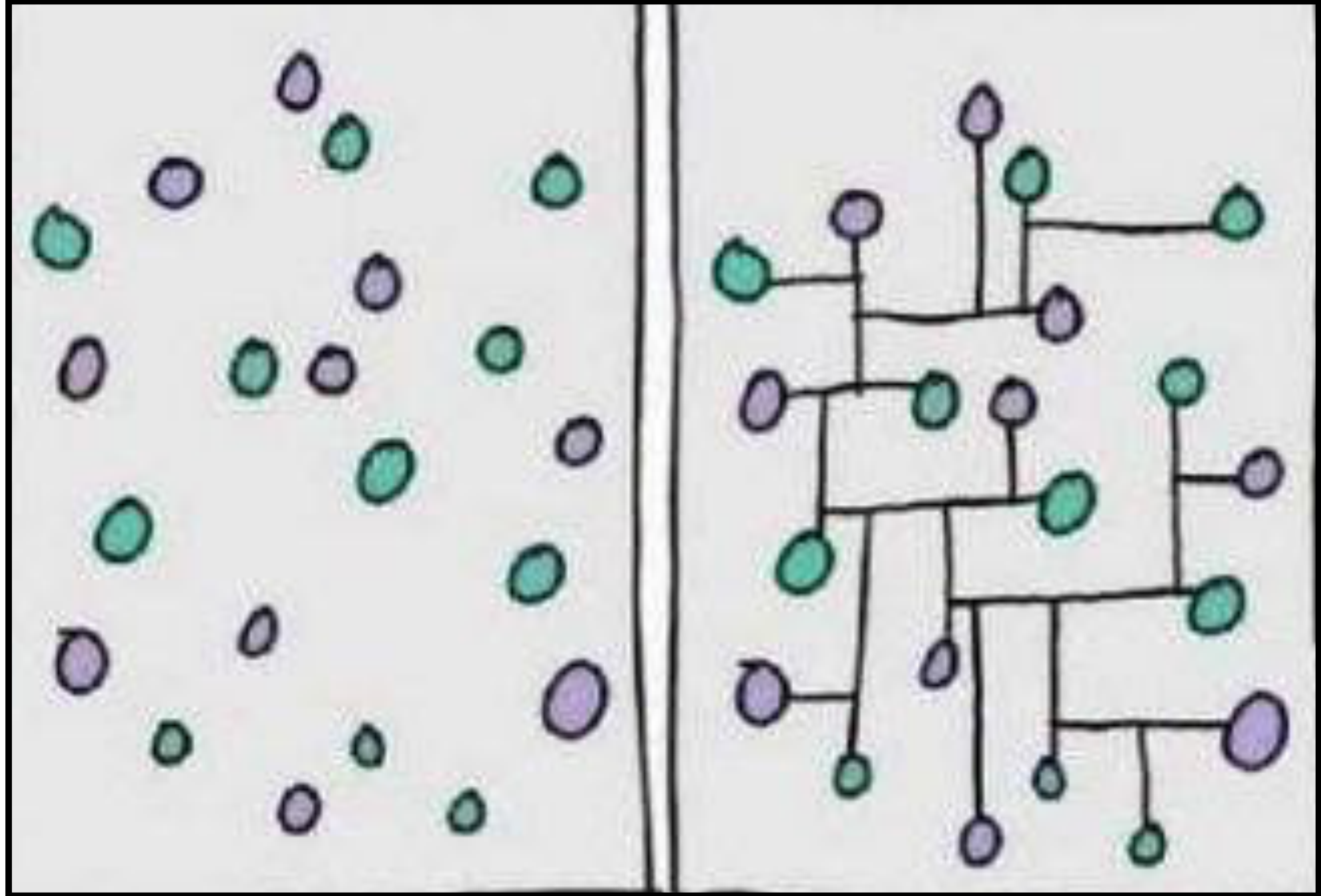
- Mathematics is neither **necessary** nor **sufficient** for critical thinking
- Critical thinking is neither **necessary** nor **sufficient** for mathematics

some initial comments

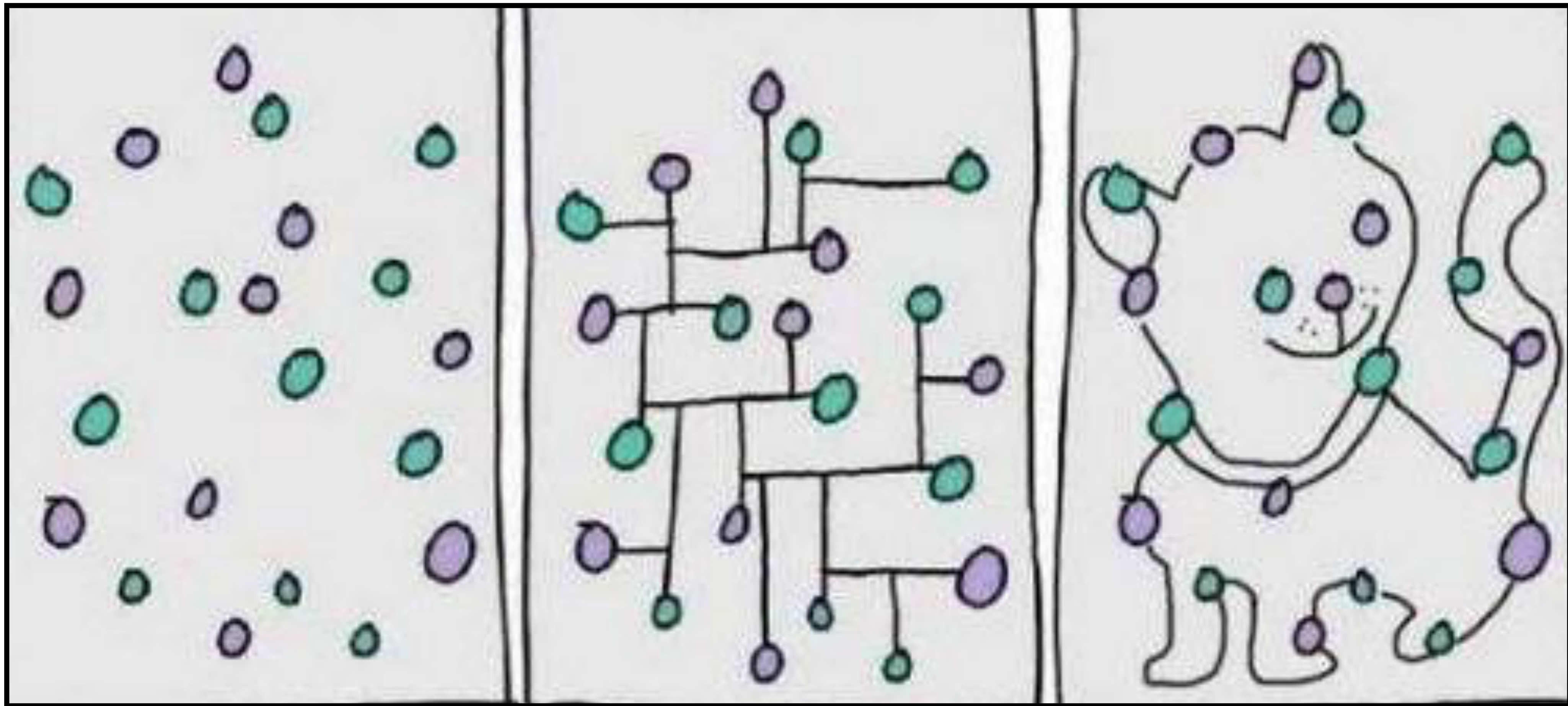


analytical thinking  $\neq$  critical thinking









“When students think critically in mathematics, they make **reasoned decisions or judgments** about what to do and think. In other words, students consider the **criteria** or grounds for a thoughtful decision and do not simply guess or apply a rule without **assessing its relevance.**”

*Critical Thinking Consortium [CA]*

What makes this definition unique to mathematics?

“When students think critically in mathematics, they make **reasoned decisions or judgments** about what to do and think. In other words, students consider the **criteria** or grounds for a thoughtful decision and do not simply guess or apply a rule without **assessing its relevance**



# Introductory activity — a mathematical scenario

Did the dingo go around the goanna?



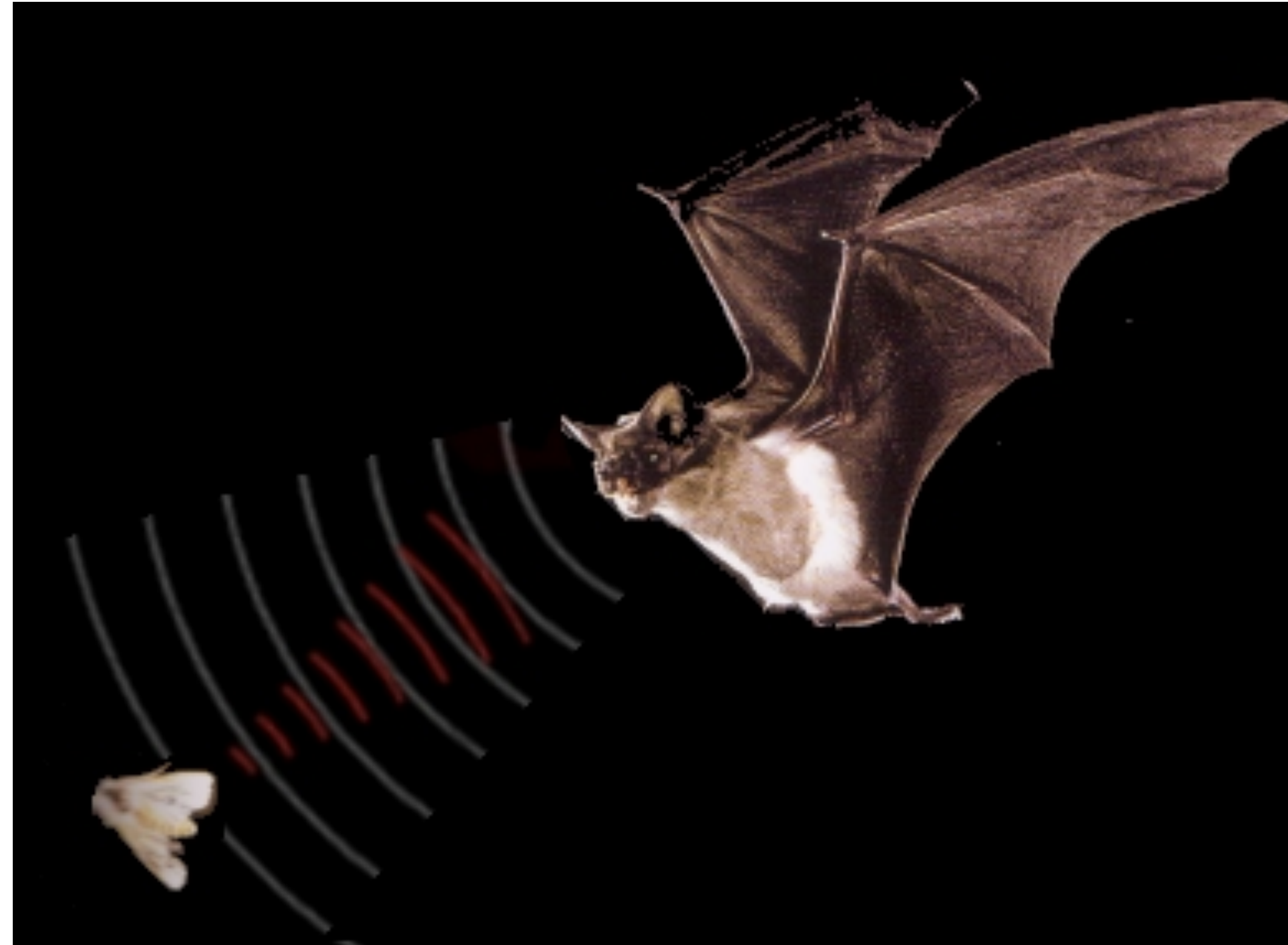


# Content

- organises work plans
- structures textbooks
- wrongly associated with 'rigour'
- informs assessment

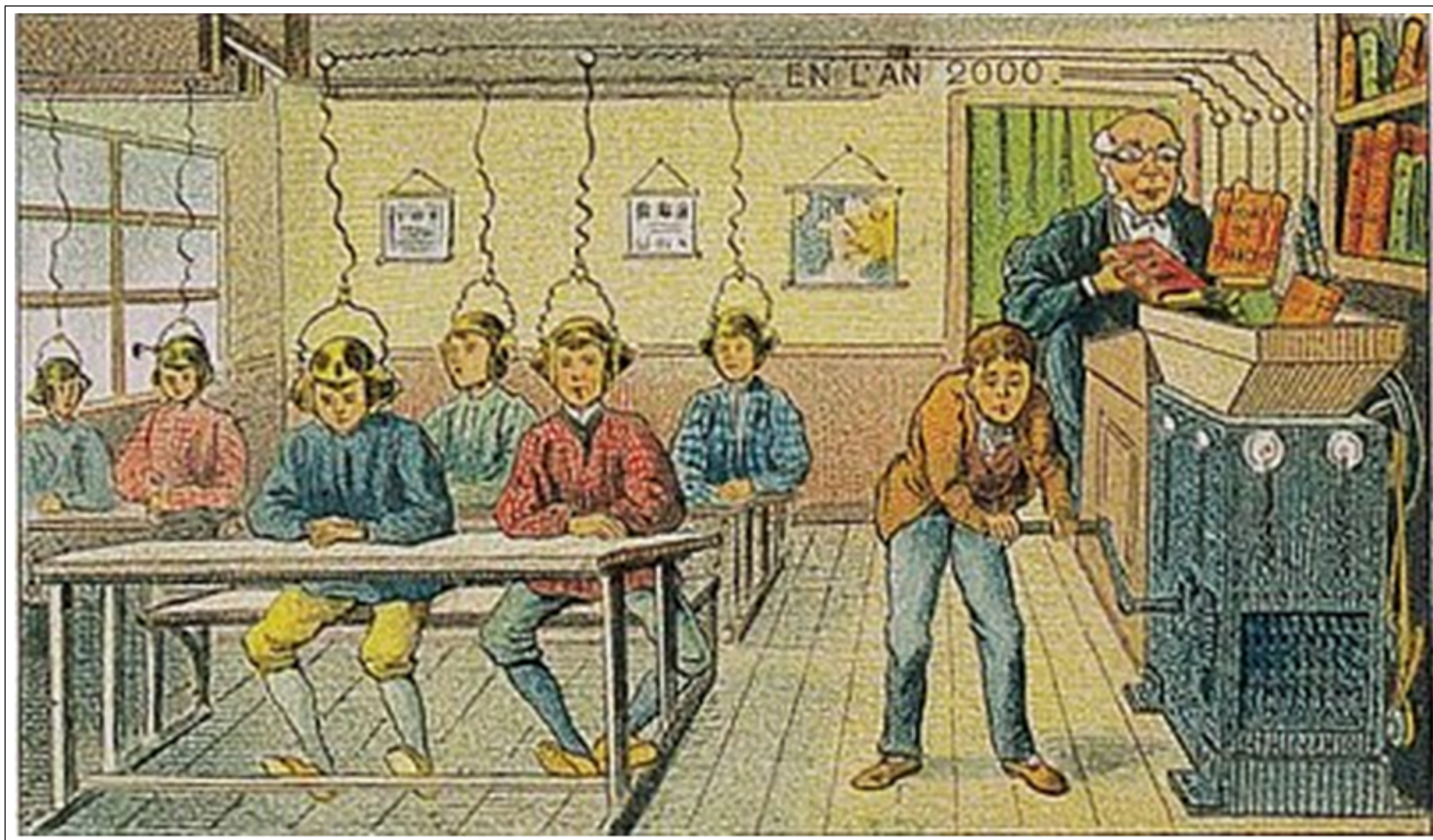
$F=ma$

PEEL in paragraphs  
Socrates taught Plato  
the Romans were ok  
 $y=mx+c$

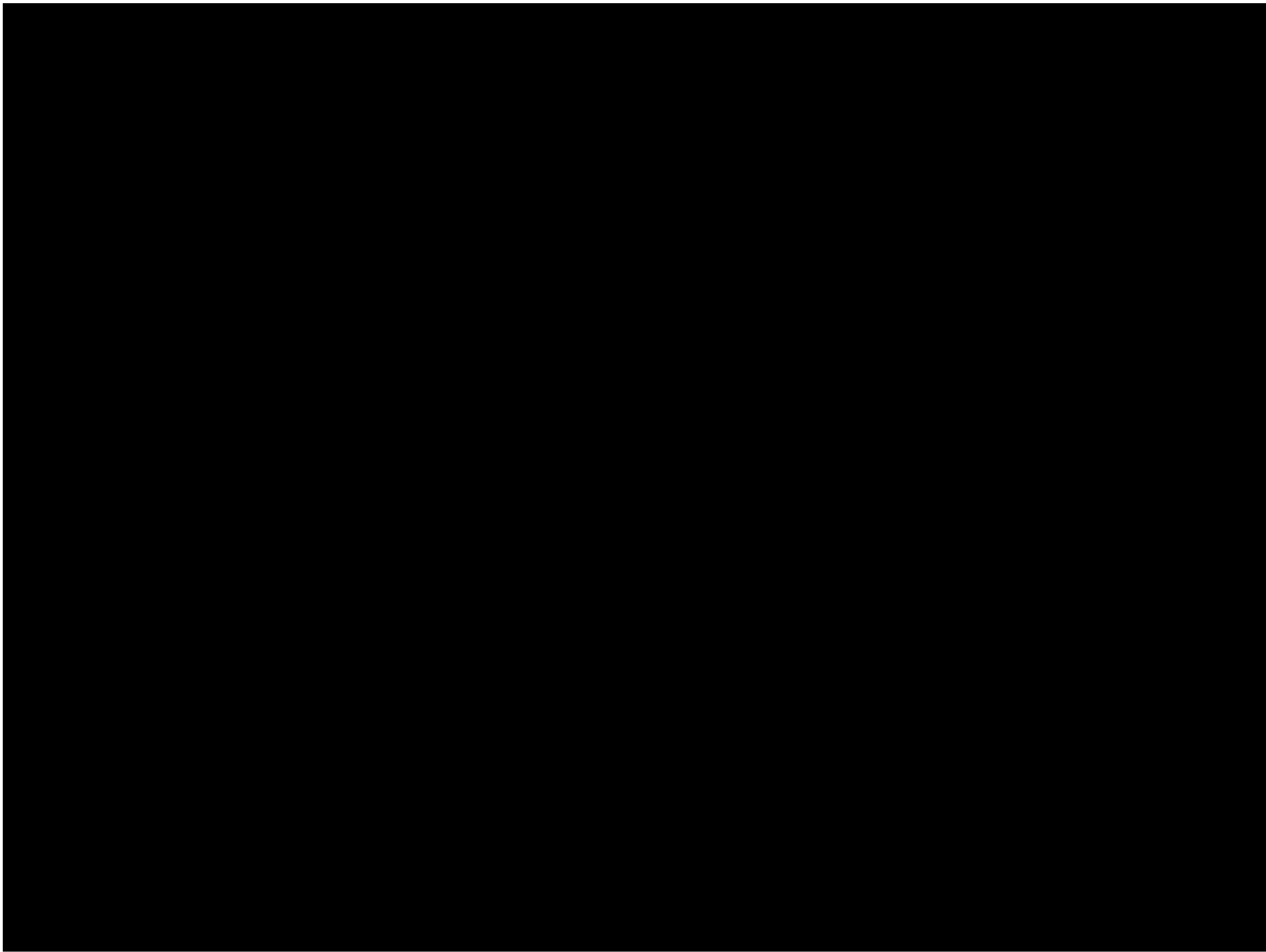


Teaching by echo-location





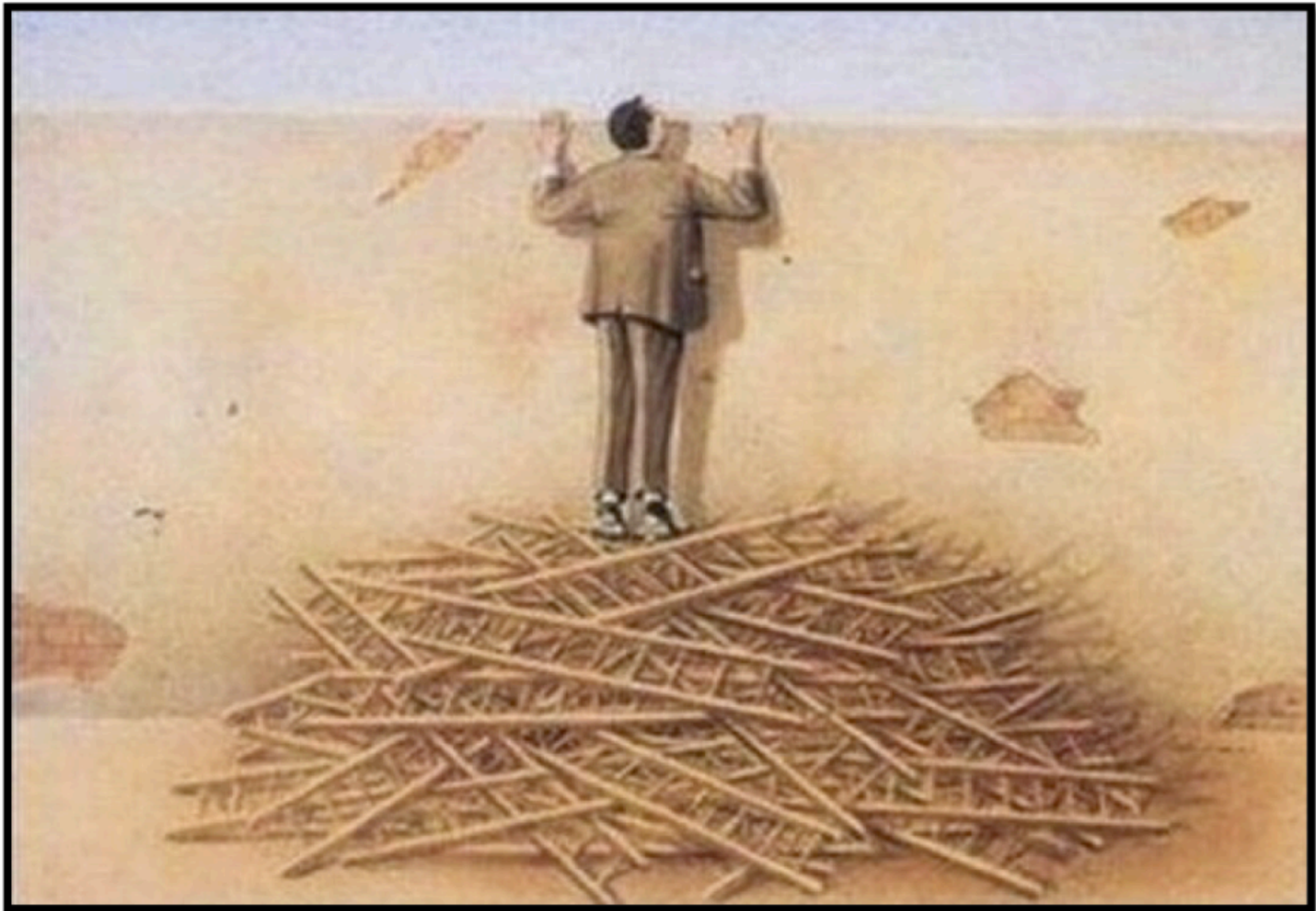




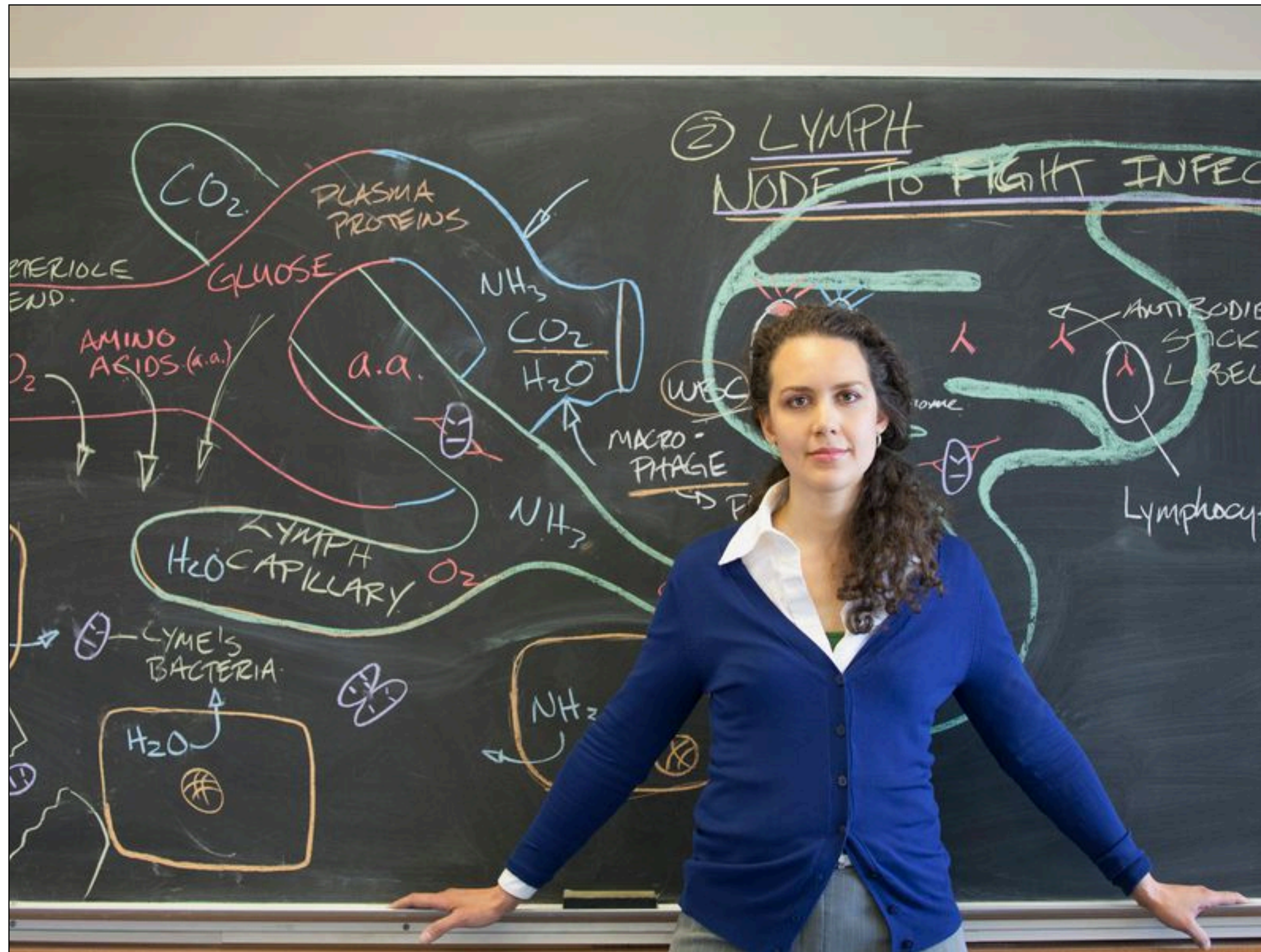
BBC TWO





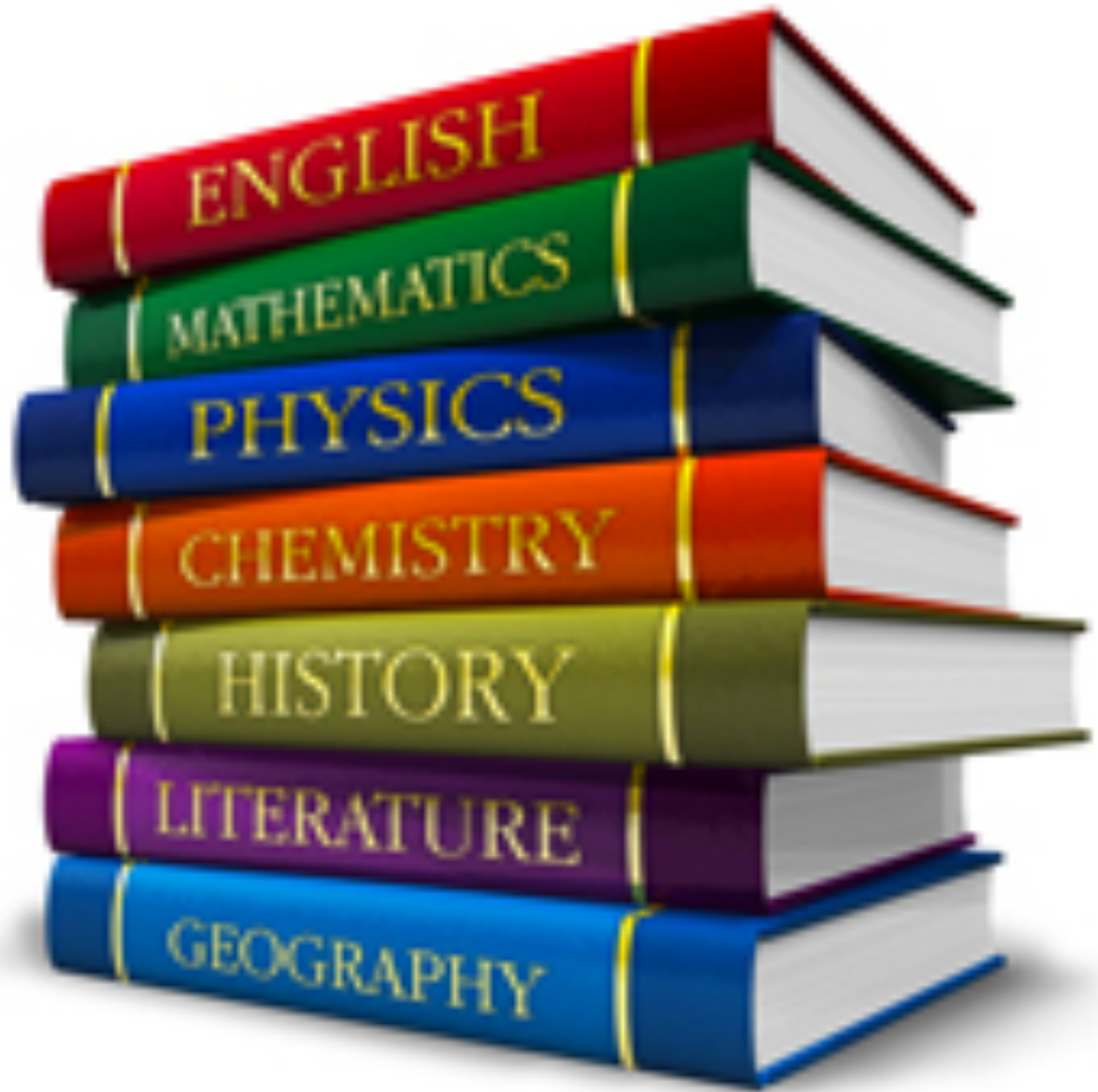






The good teacher





- ▶ Take from the canon of discipline knowledge that which is developmentally appropriate for students.
- ▶ Logically and engaging structure and sequence work.
- ▶ Test for understanding along the way

We often ask questions that require cognitive skills without showing why those skills are necessary or how they are used.

# Cognitive skills

(cognitive verbs)

- things we do with knowledge
- how we make new knowledge

evaluate  
analyse  
interpret  
synthesise  
identify

...

What's it like to be a bat?







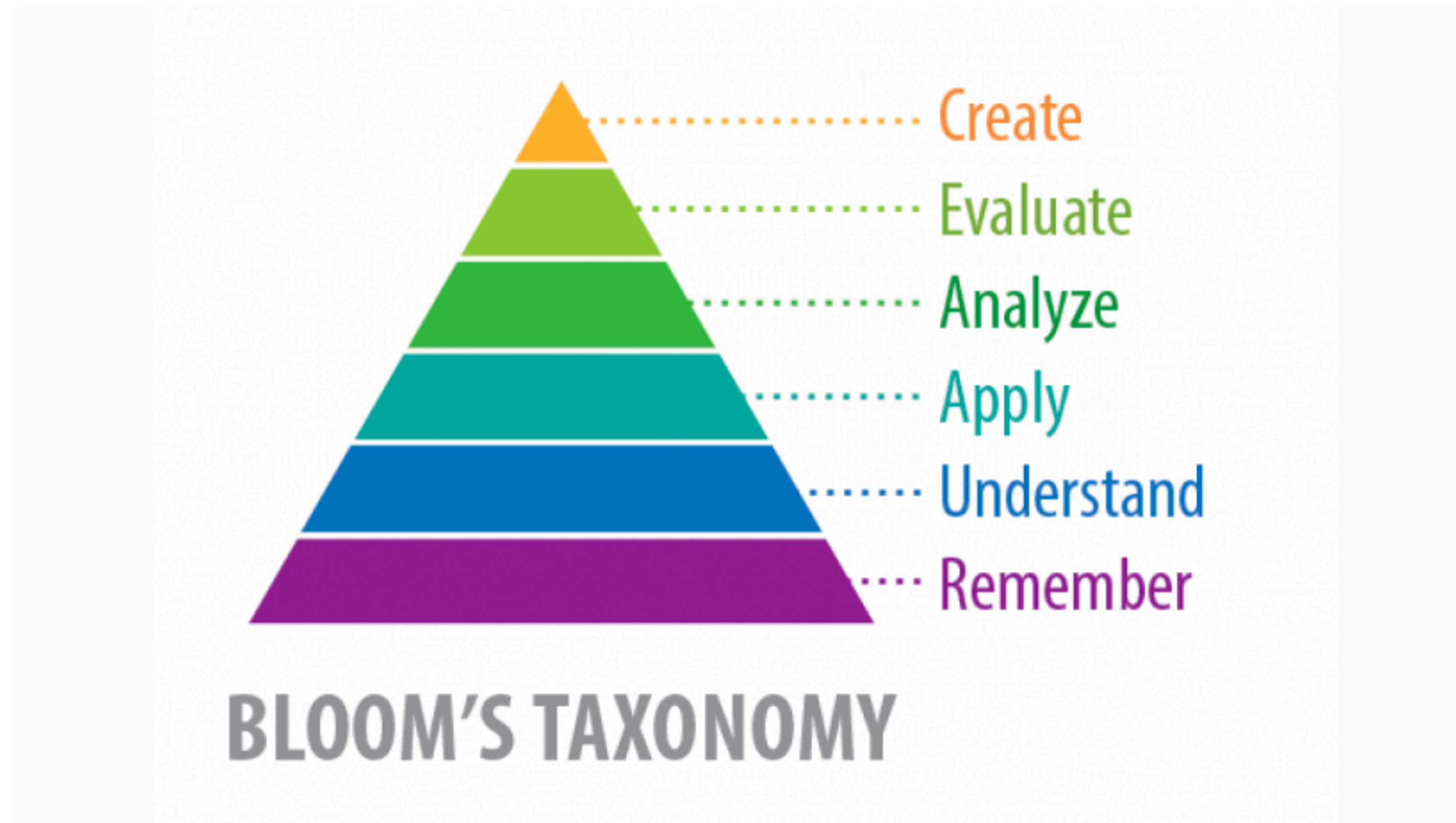
Non-propositional knowledge



Inquiry is the opportunity to use and develop cognitive skills

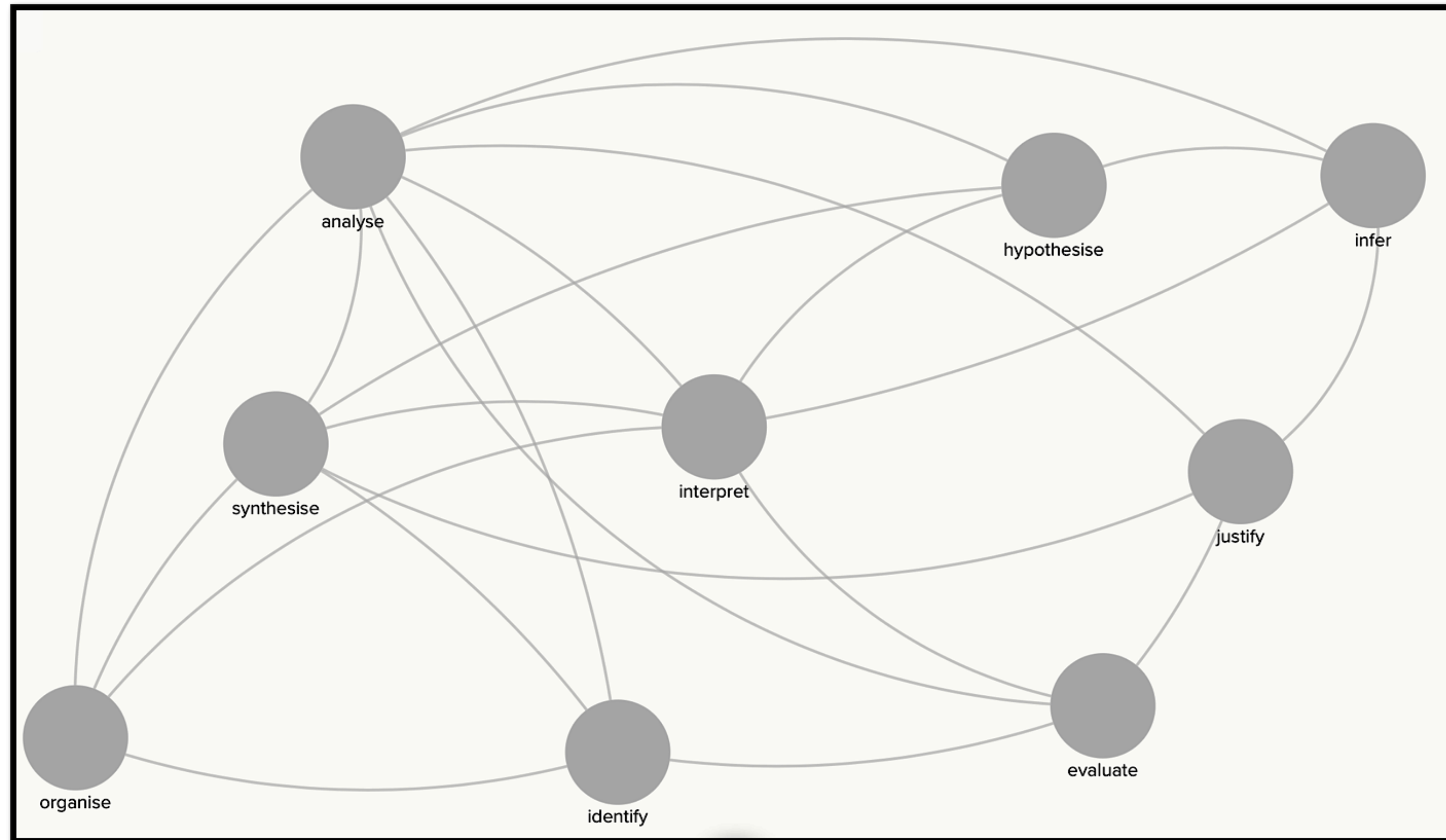


# A cognitive hierarchy





# A cognitive web



Evaluating our thinking?

thinking  $\equiv$  inquiry

# THE Q MATRIX

	Event <i>what</i>	Situation <i>where/when</i>	Alternatives <i>which</i>	people <i>who</i>	Reasons <i>why</i>	Means <i>how</i>
Present <i>is</i>						
Past <i>did/was</i>						
Possibility <i>can</i>						
Probability <i>would</i>						
Prediction <i>will</i>						
Imagination <i>might</i>						
Decision/Choice <i>should</i>						



# Values

- applied discerningly
- at the core of professional practice
- provide feedback on cognitive skill use
- broad application across disciplines

clarity  
precision  
accuracy  
coherence  
relevance  
cogency  
...

# Critical thinking as a values education

## Values of Inquiry—supporting questions



Values of inquiry modified from Elder, L. and R. Paul (2001). "Critical Thinking: Thinking with Concepts." *Journal of Developmental Education* 24(3).  
2011-2016, Attribution-NonCommercial-ShareAlike 2.5 Australia (CC BY-NC-SA 2.5 AU)  
Peter Ellerton University of Queensland, Australia



# Critical thinking as a values education

3 things to understand about values:

1. That something is a value
2. Why it is a value
3. How to apply the value

# Critical thinking as a values education





# The Critical Thinking Matrix

A high-resolution reference source for mapping critical thinking skills

Peter Ellerton, University of Queensland, Australia

*I think*

© UQCTP University of Queensland Critical Thinking Project, Peter Ellerton, University of Queensland		Values of Inquiry					
Cognitive Skills		Clarity (Intelligibility)	Accuracy	Precision	Depth (Complexity, relevance and significance)	Coherence	Breadth (Alternatives, perspectives, collaboration)
Interpretation	Categorising	The criteria for categorising are unambiguous and the common characteristics of elements within the category are explicitly stated.	Categorical distinctions are drawn from accurate representations or generalisations of characteristics. Hasty generalisations are avoided.	Categorical distinctions are based on quantifiable data, specific characteristics or clear logical definitions.	Categorisations are made using relevant and significant characteristics rather than superficial resemblances. Logical and causal relationships between categories are identified.	Logical distinctions between categories are appropriate and coherent. The logical relationships within and between categories is evident.	Alternative perspectives and criteria for categorising are explored. Preferring one framework over another is justified. Potential taxonomies are considered.
	Decoding	Terms are disambiguated and literal and intended meanings are distinguished when necessary. Implied meaning and social contexts are identified. Symbolic representations are identified and explained.	Intended or implied meaning is preserved in decoding. Literal and intended meanings are distinguished. Accurate use of symbols is evident.	Key terms are appropriately used to describe the information content. Correct procedures for working with quantitative or symbolic data are followed. Symbolic representations are used effectively.	Specific information is identified and foregrounded. Meaning is preserved by maintaining logical or causal relationships. Mastery of symbolic representation includes understanding the meaning of complex operations.	The logical content of propositions, phrases or terms is made clear and placed in context. The relationships between elements are understood.	Alternative meanings resulting from other cultural or cognitive perspectives are explored. Different interpretations of the situation are considered.
	Clarifying meaning	Key terms and technical terms are identified and explained. Literal and intended meanings are distinguished as necessary. Clarity is preserved as information moves between formats.	Statements are appropriately qualified. Limitations of understanding and representation are acknowledged. Intended or implied meaning is preserved. Paraphrasing and elucidation retain meaning.	Vagueness and ambiguity of terms and meaning identified. Key and technical terms identified and examined for appropriate use.	Nature and complexity of the problem understood and represented. Analogies or relevant similarities and illustrations used to elucidate and explain. Language examined for 'spin'.	Logical structures identified and logical coherency determined.	Language and visualisations reflect the need to cater for a diverse audience holding alternative views, approaches or perspectives.
Analysis	Examining ideas	Procedures of investigation are made explicit. Key concepts and structures are identified and named. Technical terms are used.	Faithful reproduction of information. Inaccuracies or contradictory information identified. Inferential relationships identified.	Detail preserved and reported. Vagueness and ambiguity eliminated or addressed. Technical terms are used appropriately and effectively.	Relevant and significant information is identified and foregrounded. Areas of focus are established. Problematic aspects are identified. Information necessary to frame and address the problem is identified. Ideas are compared and contrasted.	Causal and logical relationships are identified. Evidence is presented and evidential and inferential relationships are tested. General logical structure is identified and examined. Ideas are tested against existing knowledge.	Ideas are analysed within a transdisciplinary or collaborative approach, and through a variety of perspectives, including social, political, cultural and disciplinary.
	Identifying arguments	Premises and conclusions are made explicit. Argument structure is identified and discussed. Inferential pathways are articulated.	Argument types and structures are identified and named. Ambiguity is identified and addressed.	Nature of evidential material made clear. Procedures and algorithmic processes articulated in detail. Propositional content of premises and conclusions is identified and articulated.	The point at issue is identified. Relevant and significant information pertinent to the formation of premises is identified. Hidden premises are identified and discussed.	Logical relationships examined to determine the nature and form of argument. Claims are extracted from text and evidential relationships identified. Argument is tested for validity.	Arguments framed in various ways are recognised as potentially representing different perspectives. Recognition that the acceptance of evidence may depend on personal context, experience and perspective.
	Argument deconstruction	Correct use of terms. Identification of key components of arguments. Supporting evidence made clear. Diagrams or mapping used to make argumentation clear.	Premises, conclusions and inferential relationships are accurately presented.	Correct use of terms, including 'valid' and 'sound'. Representations are explicit and accurate.	Problematic aspects of argument structure/complexity are explored. Relevant and significant information affecting the reasoning process is identified and its role explained.	Cogency of argument is noted. Evidential and inferential links are examined for logical consistency. Hidden premises and unstated assumptions identified. Cognitive biases identified or postulated. Logical fallacies identified.	Relationships between unstated assumptions or elements, such as beliefs, are identified, and the effect this may have on the reasoning process is explored. Recognising limitations of a single discipline approach or of a single methodology.
Evaluation	Assessing claims	Evidence is presented in context. Direct links between evidence and claims are made explicit.	Claims are faithfully reproduced. Supporting evidence is accurately represented.	Detail of claims is preserved, including quantifiable aspects.	Direct links between evidence and claims are made explicit. Claims and conclusions are connected to the nature of the problem and of the evidence. Cognitive and social biases are explored. Assess the contextual relevance of questions, information, principles, rules or procedural directions.	Claims examined/assessed for logical coherence with each other and with evidence and methodology.	Recognising various levels of credibility that might be associated with varying perspectives about the claim. Understanding the nature of claims as a function of discipline or methodological approaches.
	Assessing arguments	Premises, conclusions and evidential relationships are articulated.	Strengths and weakness inherent in argument types, including inductive and deductive arguments, are identified in context.	Key terms are used correctly and amounts quantified where appropriate or necessary. The tools and processes of evaluation of inferences are explicitly stated.	Suitability of evidential relationships examined with regard to the nature of the problem. Proposed causal and logical relationships identified and examined for weaknesses and strengths.	Causal and logical connections tested. Inductive arguments are analysed for strength and weakness, including the use of analogies and generalisations. Deductive arguments are examined for validity and soundness. Logical fallacies identified and their effect on the argument assessed.	Additional information that may be necessary to strengthen the argument identified. Argument tested using alternative standards of various disciplines or methodological approaches.
	Synthesising claims	The synthesis is clearly derived from the constituent claims, with links made explicit.	Intended and implied meaning is preserved and generalisations and categorisations accurately represent the constituent claims.	Similarities and differences of positions are made clear, and quantified where appropriate or necessary, including how these affect the synthesis.	Relevant and significant information retained and highlighted in the synthesis. Inclusion and exclusion of material in synthesis explained. Common features identified from specific cases, both explicit and implicit.	Effective inductive generalisations made. Synthesis is coherent with the logical content of the constituent claims. Purpose and meaning are developed.	Awareness of the variety of beliefs and perspectives that may be compatible with a particular claim. Synthesis considered from various framings and axioms.
Inference	Querying evidence	Nature of evidence is clear and evidential relationships are articulated.	Evidence is faithfully reproduced and represented with honesty and charity.	Detail is sought and presented. Information is quantified where appropriate or necessary. Exact nature and role of evidence made clear.	Premises requiring evidential support are identified and strategies for seeking significant and relevant information that might inform or test hypotheses are determined.	Logical connections between matters of fact and the point at issue or problem to be solved are made clear. Implications of evidential material made clear.	Inquiry encompasses or takes into account various methodologies (e.g. transdisciplinary approach).
	Conjecturing alternatives	Possible inferential pathways (paths of reasoning) articulated based upon varying use of evidence and argumentation. Alternative hypothesis and potential conclusions are clearly expressed.	Inquiry and the exploration of alternative reasoning are sensitive to maintaining the integrity of evidence and information.	Alternatives supported by calculation or other algorithmic process.	Alternative hypotheses maintain the emphasis on significant and relevant information, as well as a focus on solving the problem. Complexity is managed and problematic causal and evidential relationships are addressed across possible outcomes.	Alternatives are logically coherent with the given information and their logical implications explored.	Alternative framing of problem explored. Collaborative or multidisciplinary reasoning employed.
	Concluding	Clear articulation of pathways from premises to conclusions, including use of evidence and argumentation.	Proper and correct use of algorithms or procedures to arrive at conclusions. Correctly identify evidential and inferential relationships and show how these lead to conclusions.	Conclusions contain specific and detailed information, quantified where appropriate or necessary.	Modes of reasoning used and conclusion reached appropriate to the nature of the problem.	Logical connections between premises and conclusions evident and explained. Inferences well-supported. Cogent approach taken (i.e. appeal to reason).	Conclusions reached using a variety of reasoning modes, such as mathematical, dialectic, scientific, inductive and deductive.
Explanation	Stating results	Correct use of terminology, unambiguous use of language and effective and clear categorical distinctions made. Explicit representation and explanation.	Statements, descriptions, diagrams and other representations maintain the integrity of information.	Detail preserved and presented. Information quantified. Correct use of terms. Vagueness and ambiguity eliminated or addressed.	Information that is significant and relevant is highlighted. Problematic aspects are outlined.	Logical connections made explicit, showing links to evidence and conclusions. Implications made clear.	Presentation of statements, descriptions, diagrams and other representations are sensitive to interpretations other than those of the author.
	Justifying procedures	Effective use of examples and illustrations. Inferential pathways made explicit. Standards of evaluation explained and presented.	Inquiry and investigations are presented faithfully and not modified to suit the nature of the conclusions.	Process and conceptual development recorded. Calculations used to provide quantified data.	Strategies explored and evaluated. Nature of inquiry appropriate to the problem.	Methodologies, algorithms and other procedures supported by logical analysis. Reasons given for choosing areas of focus and minimising other information. Standards of evaluation explained and presented.	Evidential, conceptual, methodological, criterionological and contextual considerations are made with reference to the nature of justification as a function of alternative perspectives, beliefs and assumptions.
	Presenting arguments	Argumentative prose, diagrams, charts, graphs and graphics convey a clear meaning, adhering to convention. Points at issue clearly defined and stated.	Evidence faithfully reproduced and counter-arguments and criticisms engaged with honesty and charity.	Quantitative data included. Unnecessary information is minimised.	Identify and address counter-arguments. Causal and logical relationships that relate to the situation or problem are identified and their role made explicit. Problematic aspects identified and solutions explained.	Logical structure and coherence evident. Well-supported inferences with implications explicitly represented.	Cogent presentation but with due consideration of various reasoning modes and how alternative perspectives may influence the acceptance or definition of evidence.
Self regulation	Metacognition	Reflective practice is evident and cognitive development across issues is clearly reported.	Authentic representation of students' own mental processes and cognitive development.	Reflection targeted to specific processes and outcomes.	Reflections show personal engagement with significant and relevant issues. Threshold (key) ideas and concepts are identified. Deficiencies in personal knowledge that may impact rational or objective analysis acknowledged and managed.	Logical analysis of own thoughts comparable in scope and rigour to analysis of others'.	Recognition of bias, erroneous thinking or fallacious reasoning. Collaboration sought for the purpose of testing own thoughts.
	Self-correction	Recognition of bias, erroneous thinking or fallacious reasoning is recognised and reported.	Self-criticism and redirection is authentic and resembles the criticism that would be made of third persons.	Reflection leads to specific and detailed changed or specific courses of action are articulated.	Revisions geared to improve outcomes and examined for consequences to original position, findings, or opinions.	Recognition and acceptance of logical errors in preliminary thinking. Rational conclusions contrasted with personal preferences or bias.	Willingness to modify thinking through collaborative inquiry. Self-correction seen as progress.

Cognitive skills modified from Facione, P. A. (1990). Critical Thinking: A Statement of Expert Consensus for Purposes of Educational Assessment and Instruction. Values of inquiry concept from Kuhn, T. S. (1970). The Structure of Scientific Revolutions. International Encyclopedia of Unified Science. Chicago, University of Chicago Press. 2. Values of inquiry modified from Eider, L. and R. Paul (2001). "Critical Thinking: Thinking with Concepts." Journal of Developmental Education 24(3).

CT only taught in a discipline context?



What is the least valuable thing you could put in this cup?



# Virtues

(habits/dispositions)

- come from mastery of values
- characteristic of critical thinkers
- create knowledge producers

resilience  
integrity  
open-mindedness  
curiosity  
honesty  
persistence

...

Knowledge makers of the future.







“When students think critically in mathematics, they make **reasoned decisions or judgments** about what to do and think. In other words, students consider the **criteria** or grounds for a thoughtful decision and do not simply guess or apply a rule without **assessing its relevance.**”

*Critical Thinking Consortium [CA]*

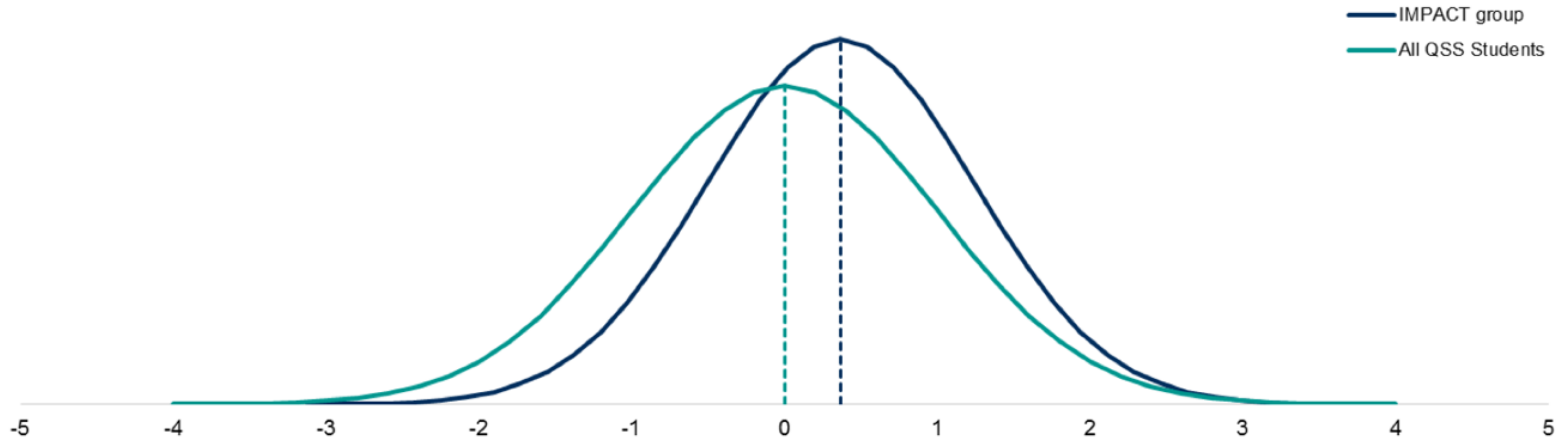
it works



## UNIFY - Critical Thinking - Year 4-5-6

*Attending more than 10 weeks*

*NAPLAN 2014-16 Year 3-5 Numeracy Relative Gain*



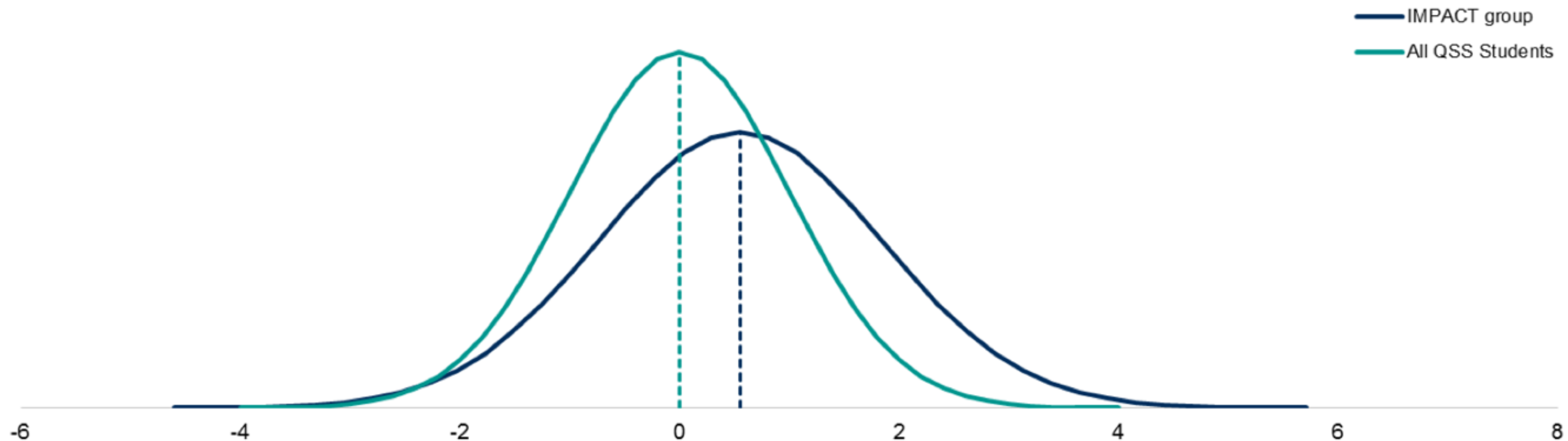
These distributions are normalised approximations based on the z-score mean and standard deviation for each group.

Year Interval	Strand	Attendance Group	IMPACT Program	zScore Mean	zScore StDev	N	Difference to QSS
3-5	Numeracy	More than 10 Weeks	UNIFY - Critical Thinking - Year 4-5-6	0.37	0.87	77	0.37
3-5	Numeracy	Not in Impact Program	-	0.00	1.00	35254	-

## UNIFY - Critical Thinking - Year 7-8-9-10

*Attending more than 10 weeks*

*NAPLAN 2014-16 Year 5-7 Numeracy Relative Gain*



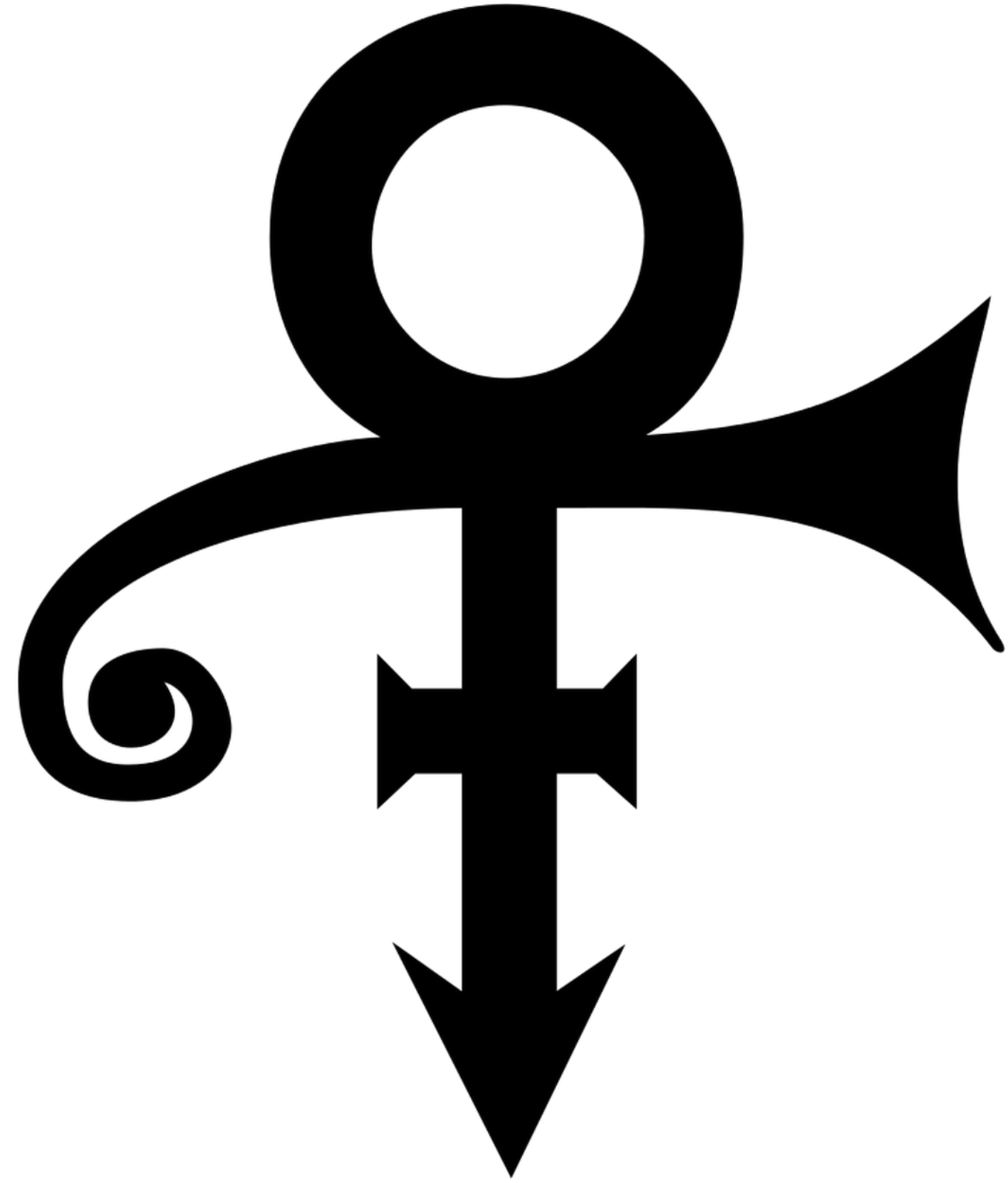
These distributions are normalised approximations based on the z-score mean and standard deviation for each group.

Year Interval	Strand	Attendance Group	IMPACT Program	zScore Mean	zScore StDev	N	Difference to QSS
5-7	Numeracy	More than 10 Weeks	UNIFY - Critical Thinking - Year 7-8-9-10	0.55	1.29	47	0.55
5-7	Numeracy	Not in Impact Program	-	0.00	1.00	27201	-

teaching value







ITEM 6 [\*\*\*\*]



The Greek flag is shown above. All its blue and white horizontal stripes are equal in width and its canton is square. The vertical spar on the Swiss cross has the same width as the horizontal spar. The proportion is 2:3.

What **exact** fraction of the Greek flag is blue?

Show clearly how you arrived at your answer.



# Solid Pathways



<http://indigenous.education.qld.gov.au/school/Pages/solid-pathways.aspx>

# Aspiring Thinkers Network



<https://impact.edu.au/professional-networks/aspiring-thinkers>

[https://  
theconversation.com/  
profiles/peter-  
ellerton-8574/articles](https://theconversation.com/profiles/peter-ellerton-8574/articles)



**META101x Philosophy  
and Critical Thinking**

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Create change